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Abstract

The present paper is concerned with the local times of a Lévy process reflected at two barriers 0 and $K > 0$. The reflected process is decomposed into the original process plus local times at 0 and K and a starting condition, and we study ℓ^K, the mean rate of increase of the local time at K when the reflected process is started in stationarity. We derive asymptotics ($K \to \infty$) for ℓ^K when the Lévy process has mean zero. The precise form of the asymptotics depends on the existence or non-existence of a finite second moment, paralleling the difference between the normal and the stable central limit theorem. To achieve the asymptotic results, we prove a uniform integrability criterion for Lévy processes and a continuity result for ℓ^K, which are of independent interest.

Keywords continuity of the local time, finite buffer, Lévy process, reflection, loss rate, Skorokhod problem, stable central limit theorem, stable distribution, uniform integrability.

1 Introduction

A *Lévy process* $S = \{S_t\}_{t\geq 0}$ is a real-valued stochastic process on $\mathbb{R}$ with stationary independent increments which is continuous in probability and has $X_0 = 0$ a.s. We reflect the Lévy process at barriers 0 and $K > 0$. The reflected process $V^K = \{V_t^K\}_{t\geq 0}$ can be constructed as part of the solution to a two-sided Skorokhod problem, which yields a representation:

$$V_t^K = y + S_t + L_t^0 - L_t^K \tag{1.1}$$

of the reflected process started at $y \in [0, K]$, where $L^0 = \{L_t^0\}$ and $L^K = \{L_t^K\}$ are the local times at 0, K respectively. More precisely, $\left(V^K, L^0, L^K\right)$ is a triplet of processes such that $V_t^K \in [0, K]$ and

$$\int_0^T V_t^K \,\mathrm{d}L_t^0 = 0 \quad \forall T \quad \text{and} \quad \int_0^T (K - V_t^K) \,\mathrm{d}L_t^K = 0 \quad \forall T\,. \tag{1.2}$$

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The process V^K is regenerative (as a cycle, take e.g. an excursion from 0 to K followed by an excursion from K to 0). Also such a cycle clearly has an absolutely continuous distribution, and it follows by general theory (Asmussen [3] VI.1) that there exists a unique stationary distribution π^K such that the distribution of V_t^K converges to π^K weakly and in total variation. The object of the present paper is asymptotic properties as $K \to \infty$ of the stationary rate of growth $\ell^K := \mathbb{E}_{\pi^K} L_1^K$ of the local time

Besides its intrinsic probabilistic interest, this problem has a long applied motivation. Two-sided reflected processes may be used to model waiting time processes in queues with finite capacity (Bekker and Zwart [5], Cohen [9], Cooper et al. [10], Daley [11]), or a finite dam or fluid model (Asmussen [3] , Moran [23], Stadje [28]). Furthermore, they are used in models of network traffic or telecommunications systems involving a finite buffer (Jelenković [15], Kim and Shroff [17] , Zwart [31]), and in this context the loss rate can be interpreted as the bit loss rate in a finite data buffer.

In view of this applied literature, we shall henceforth refer to ℓ^K as the *loss rate* (at the upper barrier K). In the Lévy process context, it is the object of study of the recent papers Asmussen and Pihlsgård [4] and Andersen [1]. In [4], an explicit expression for ℓ^K in terms of the characteristic triplet of the Lévy process is provided and used to derive the asymptotic behavior of ℓ^K as K tends to infinity in the case where the Lévy process is light-tailed and the mean is either strictly positive or strictly negative. Furthermore, in [4] the loss rate of a strictly stable Lévy process is explicitly calculated. The case of negative mean and heavy tails case is treated in Andersen [1]. In this paper we derive loss rate asymptotics when the mean is zero, i.e. $\mathbb{E}S_1 = 0$.

The main contribution of this paper is Theorem 2 which provides an asymptotic expression as $K \to \infty$ for the loss rate in the zero-mean case. The basic intuition behind this is simple: $\mathbb{E}S_1 = 0$ implies that the Lévy process after appropriate scaling and time change has a limit which is Brownian motion in the case of finite variance and (subject to a condition on regular variation) is stable in the case of infinite variance. For these limits, explicit expressions for the asymptotic loss rate have been derived in Asmussen and Pihlsgård [4], so the main technical problems becomes to establish continuity of $\ell^K = \ell^K(S)$ as function of S. This is of some of independent interest and is formulated in Theorem 3. A uniform integrability property is required, and conditions for this are given as Theorem 4.

The paper is organized as follows: In Section 2, we give some background on Lévy processes, the Skorokhod problem, and the stationary distribution. In Section 3 we state the main results of the paper, and the proofs are given in Sections 4, 5 and 6 .

2 Preliminaries

To every Lévy process $S = \{S_t\}_{t\geq 0}$ is associated a unique *characteristic triplet* (θ, σ, ν), where $\theta \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure (*the Lévy measure*) which satisfies $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ and $\nu(\{0\}) = 0$. The *Lévy exponent* is defined by

$$\kappa(s) := \theta s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{\infty} \left[e^{sx} - 1 - sI(|x| \leq 1)\right] \nu(\mathrm{d}x)$$

and is defined for s in $\Theta := \{s \in \mathbb{C} \mid \mathbb{E}e^{\Re(s)S_1} < \infty\}$. The Lévy exponent is the unique function satisfying $\mathbb{E}e^{sS_t} = e^{t\kappa(s)}$ and $\kappa(0) = 0$, and we have

$$\mathbb{E}S_1 = \kappa'(0) = \theta + \int_{|y|>1} y\,\nu(\mathrm{d}y) \tag{2.1}$$

(the mean is assumed to be well-defined and finite for all Lévy processes encountered in the paper). We use the cadlag version of $\{S_t\}$, which exists because of stochastic continuity. Standard references for Lévy processes are Bertoin [6], Kyprianou [21] and Sato [26].

We will also need weak convergence properties:

Proposition 1. *Let $S^0, S^1, S^2, \ldots$ such that S_n has characteristic triplet $(\theta_n, \sigma_n, \nu_n)$. Then the following properties are equivalent:*
(i) $S_t^n \xrightarrow{\mathcal{D}} S_t^0$ *for some* $t > 0$;
(ii) $S_t^n \xrightarrow{\mathcal{D}} S_t^0$ *for all* t;
(iii) $\{S_t^n\} \xrightarrow{\mathcal{D}} \{S_t^0\}$ *in* $D[0, \infty)$;
(iv) $\tilde{\nu}_n \to \tilde{\nu}_0$ *weakly, where* $\tilde{\nu}_n$ *is the bounded measure*

$$\tilde{\nu}_n(\mathrm{d}y) := \sigma_n \delta_0(\mathrm{d}y) + \frac{y^2}{1+y^2}\nu_n(\mathrm{d}y) \tag{2.2}$$

and $c_n \to c_0$ *where*

$$c_n := \theta_n + \int \left(\frac{y}{1+y^2} - yI(|y| \leq 1)\right) \nu_n(\mathrm{d}y)$$

See e.g. Kallenberg [16] pp. 244–248, in particular Lemma 13.15 and 13.17. If one of (i)–(iv) hold, we write simply $S^n \xrightarrow{\mathcal{D}} S^0$.

The existence and uniqueness of a solution to the Skorokhod problem is proved in Tanaka [29] and in a more pragmatic manner in Asmussen [3] XIV.3. Verbally, the condition (1.2) states that $\{L_t^0\}$ can only increase when $V_t = 0$ and $\{L_t^K\}$ can only increase when $V_t = K$, which supports our interpretation of $\ell^K = \mathbb{E}_{\pi^K} L_1^K$ as a loss rate in a system where the "free traffic" is modeled by $\{S_t\}$.

The stationary distribution has the representation

$$\overline{\pi}_K(y) = \pi^K[y, K] = \mathbb{P}\big(S_{\tau[y-K,y)} \geq y\big), \quad 0 \leq y \leq K, \tag{2.3}$$

where $\tau[u, v) = \inf\big\{t > 0 \mid S_t \notin [u, v)\big\}$, see Asmussen [3] pp. 393-394 as well as Lindley [22] and Siegmund [27]. This implies that the Laplace transform of π^K can be found in closed from whenever the scale function of S is explicitly available. For examples of this, see Hubalek and Kyprianou [14].

From Theorem 3.6 in Asmussen and Pihlsgård [4], we have the following expression for the loss rate, in terms of the characteristic triplet of the Lévy process and the stationary distribution:

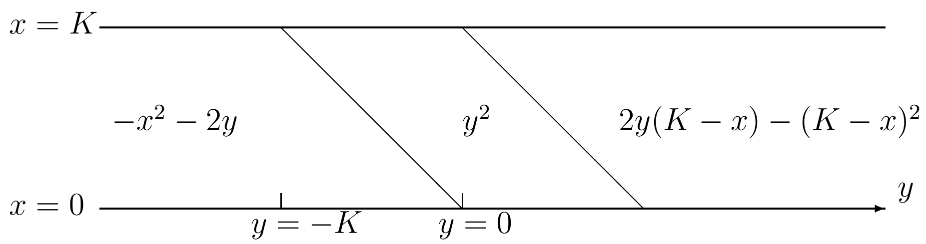
$$\ell^K = \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_K(x)\,\mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi^K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y), \tag{2.4}$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y \le -x \\ y^2 & \text{if } -x < y < K-x \\ 2y(K-x)-(K-x)^2 & \text{if } y \ge K-x \end{cases} \tag{2.5}$$

For a graphical illustration, see Fig. 1 that depicts $\varphi(x,y)$ in the region $(x,y) \in [0,K] \times \mathbb{R}$ relevant for (2.4) (note that y is on the horizontal axis and x on the vertical).

Figure 1: The function $\varphi(x,y)$



One should note that various explicit expression for L_t^0 and L_t^K have been derived (in part independently) by a number of authors, see Andersen and Mandjes [2], Borovkov [8], Cooper et al. [10], Kruk et al. [18] and Kruk et al. [19]. However, they all have a form that is so complicated that they do not appear to be of use neither for deriving (2.4), (2.5) nor for the present purposes.

3 Main results

Our main result provides the asymptotics in the case $\mathbb{E}S_1 = 0$ of zero drift.

Theorem 2.
(a) *Let $\{S_t\}$ be a Lévy process with characteristic triplet (θ,σ,ν) and $E[S_1] = 0$, $\int_{-\infty}^{\infty} x^2\,\nu(dx) < \infty$. Then*

$$\ell^K \sim \frac{1}{2K}\int_{-\infty}^{\infty} y^2\nu(dy) + \frac{\sigma^2}{2K}, \qquad K \to \infty. \tag{3.1}$$

(b) *Let $\{S_t\}$ be an Lévy process with characteristic triplet (θ,σ,ν). Assume $\mathbb{E}S_1 = 0$ and that for some $1 < \alpha < 2$, there exists slowly varying function $L_0(x), L_1(x)$ and*

$L_2(x)$ *such that for* $L(x) := L_1(x) + L_2(x)$ *we have*

$$\overline{\nu}(x) = x^{-\alpha} L_1(x), \qquad \underline{\nu}(x) = |x|^{-\alpha} L_2(x), \tag{3.2}$$

$$\lim_{x\to\infty} \frac{L_2(x)}{L(x)} = \frac{\beta+1}{2}, \qquad \lim_{x\to\infty} L_0(x)^\alpha L(x) = 1. \tag{3.3}$$

Then, setting $\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2))$, $d = (\beta+1)/2$ *and* $c = (1-\beta)/2$ *we have* $\ell^K \sim \gamma/(K^{\alpha-1} L_0^\alpha(K))$ *where*

$$\gamma = \frac{cB(2-\alpha\rho, \alpha\rho) + dB(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))(\alpha-1)(2-\alpha)}$$

The parameter ρ defined in Theorem 2 is known as the *positivity parameter* as it satisfies $\rho = \mathbb{P}(S_t > 0)$ when S is a strictly α-stable Lévy process, see Zolotarev [30].

We note incidentally that Theorem 2 also gives the asymptotics of $\ell^0 = \mathbb{E}_{\pi^K} L_1^0$ because a balance argument together with (1.1) gives $0 = \mathbb{E}S_1 + \ell^0 - \ell^K$ so that $\ell^0 = \ell^K$ in the mean zero case $\mathbb{E}S_1 = 0$.

To prove Theorem 2, we will use the fact that by properly scaling our Lévy process we may construct a sequence of Lévy processes which converges weakly to either a Brownian Motion or a stable process. Since ℓ^K has been calculated for both Brownian Motion and stable processes in Asmussen and Pihlsgård [4], we may use this convergence to obtain loss rate asymptotics in the case of zero drift, provided that the loss rate is continuous in the sense that weak convergence (in the sense of Proposition 1) of the involved processes implies convergence of the associated loss rates. To state our result:

Theorem 3. *Let* $\{S^n\}_{n=0,1,\ldots}$ *be a sequence of Lévy processes with associated loss rates* $\ell^{K,n}$. *Suppose* $S^n \xrightarrow{\mathcal{D}} S^0$ *and that the family* $(S_1^n)_{n=1}^\infty$ *is uniformly integrable. Then* $\ell^{K,n} \to \ell^{K,0}$ *as* $n \to \infty$.

We shall also need:

Theorem 4. *Let* $\{X_n\}_{n=1,2,\ldots}$ *be a sequence of weakly convergent infinitely divisible random variables, with characteristic triplets* $(\theta_n, \sigma_n, \nu_n)$. *Then for* $\alpha > 0$:

$$\lim_{a\to\infty} \sup_n \int_{[a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0 \;\Leftrightarrow\; \{|X_n|^\alpha \mid n \geq 1\} \textit{ is uniformly integrable}$$

The result is certainly not unexpected, but does not appear to be in the literature; the closest we could find is Theorem 25.3 in Sato [26].

4 Proof of Theorem 3

We consider a sequence of Lévy process $\{S^n\}$ such that $S^n \xrightarrow{\mathcal{D}} S^0$ and use obvious notation like $\ell^{K,n}, \pi^{K,n}$ etc. Furthermore, we let $\tau^n(A)$ denote the first exit time of S^n from A. Here A will always be an interval.

We first show that weak convergence of S_1^n implies weak convergence of the stationary distributions.

Proposition 5. $S^n \xrightarrow{\mathcal{D}} S^0 \Rightarrow \pi^{K,n} \xrightarrow{\mathcal{D}} \pi^{K,0}$.

Proof. According to Theorem 13.17 in Kallenberg [16] we may assume $\Delta_{n,t} := \sup_{v\le t} |S^n(v) - S^0(v)| \xrightarrow{\mathbb{P}} 0$. Then

$$\begin{aligned}\mathbb{P}\big(S^0_{\tau^0[y+\epsilon-K,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-K,y+\epsilon) \le t\big) \\ \le \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) + \mathbb{P}(\Delta_{n,t} > \epsilon) \\ \le \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,.\end{aligned}$$

Letting first $n \to \infty$ gives

$$\liminf_{n\to\infty} \overline{\pi}^{K,n}(y) \;\ge\; \mathbb{P}\big(S^0_{\tau^0[y+\epsilon-K,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-K,y+\epsilon) \le t\big)\,,$$

and letting next $t \to \infty$, we obtain

$$\liminf_{n\to\infty} \overline{\pi}^{K,n} \;\ge\; \overline{\pi}^{K,0}(y+\epsilon)\,. \tag{4.1}$$

Similarly,

$$\begin{aligned}\mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) \le \mathbb{P}\big(S^0_{\tau^0[y-\epsilon-K,y-\epsilon)} \ge y\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,, \\ \limsup_{n\to\infty} \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) \le \overline{\pi}^{K,0}(y-\epsilon)\,.\end{aligned} \tag{4.2}$$

However,

$$\mathbb{P}\big(\tau^n[y-K,y) > t\big) \;\le\; \mathbb{P}\big(\tau^0[y-\epsilon-K,y+\epsilon) > t\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,,$$

so that

$$\limsup_{n\to\infty} \mathbb{P}\big(\tau^n[y-K,y) > t\big) \;\le\; \mathbb{P}\big(\tau^0[y-\epsilon-K,y+\epsilon) > t\big)\,.$$

Since the r.h.s. can be chosen arbitrarily small, it follows by combining with (4.2) that

$$\limsup_{n\to\infty} \overline{\pi}^{K,n}(y) \;=\; \limsup_{n\to\infty} \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y\big) \;\le\; \overline{\pi}^{K,0}(y-\epsilon)\,.$$

Combining with (4.1) shows that $\overline{\pi}^{K,n}(y) \to \overline{\pi}^{K,0}(y)$ at each continuity point y of $\overline{\pi}^{K,0}$, which implies convergence in distribution. □

We will need the following lemma.

Lemma 6. *The function $\varphi(x,y)$ in continuous in the region $(x,y) \in [0,K] \times \mathbb{R}$ and satisfies $0 \le \varphi(x,y) \le 2y^2 \wedge 2K|y|$.*

Proof. By elementary calculus. For continuity, check that the expressions for $\varphi(x,y)$ on the regions $x+y \le 0$ and $x+y \ge K$ equal y^2 on the lines $x+y=0$ and $x+y=K$. The claimed inequality is clear for $0 \le x+y \le K$. Consider $x+y<0$. Then $\varphi(x,y) \le -2xy \le 2y^2$ and $\varphi(x,y) \le -2xy \le 2K|y|$. Similarly for $x+y>K$, we have $\varphi(x,y) \le 2y(K-x)$ which yields $\varphi(x,y) \le 2y^2$ and $\varphi(x,y) \le -2xy \le 2Ky$. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Recall the definition (2.2) of the bounded measure $\tilde{\nu}$ and let $\tilde{\varphi}_K(x,y) := \varphi_K(x,y)(1+y^2)/y^2$ for $y \neq 0$, $\tilde{\varphi}_K(x,0) = 1$. The continuity of φ implies $\varphi(x,y) \sim y^2$ as $y \to 0$ and it easily follows that $\tilde{\varphi}(x,y)$ is continuous jointly in x, y. We also get

$$\int_{-\infty}^{\infty} \tilde{\varphi}(x,y)\tilde{\nu}_n(\mathrm{d}y) = \sigma_n^2 + \int_{-\infty}^{\infty} \varphi(x,y)\nu_n(\mathrm{d}y)$$

so that

$$\begin{aligned} a_n := \sigma_n^2 + \int_0^K \pi^{K,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \varphi(x,y)\nu_n(\mathrm{d}y) \\ = \int_0^K \pi^{K,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \tilde{\varphi}(x,y)\tilde{\nu}_n(\mathrm{d}y)\,. \end{aligned}$$

Let $\tilde{\nu}_n^1, \tilde{\nu}_n^2$ denote the restrictions of $\tilde{\nu}_n$ to the sets $|y| \leq a$, resp. $|y| > a$. Then $0 \leq \varphi(x,y) \leq 2K|y|$, and uniform integrability (Theorem 4) imply that we can choose a such that

$$0 \leq \int_{[-a,a]^c} \tilde{\varphi}(x,y)\tilde{\nu}_n^2(\mathrm{d}y) < \epsilon$$

for all x and n (note that $\tilde{\nu}_n \leq \nu_n$ on $\mathbb{R}\setminus\{0\}$). We may also further assume that a and $-a$ are continuity points of ν_0 which implies $\tilde{\nu}_n^1 \to \tilde{\nu}_0$ weakly. In particular,

$$\sup_n \tilde{\nu}_n^1([-a,a]) < \infty. \tag{4.3}$$

Define

$$f_n(x) = \int_{-a}^{a} \varphi(x,y)\nu_n(\mathrm{d}y) + \sigma_n^2 = \int_{-a}^{a} \tilde{\varphi}(x,y)\tilde{\nu}_n^1 \mathrm{d}y$$

so that $f_n(x) \to f_0(x)$. Being continuous on the compact set $[0,K] \times [-a,a]$, $\tilde{\varphi}_K(x,y)$ is uniformly continuous. Together with (4.3) this implies that given ϵ_1, there exists ϵ_2 such that $|f_n(x') - f_n(x'')| < \epsilon_1$ for all n whenever $|x' - x''| < \epsilon_2$. I.e., the family $(f_n)_0^\infty$ is equicontinuous and uniformly bounded. In particular, the convergence$f_n(x) \to f_0(x)$ is uniform in $x \in [0,K]$. Together with $\int f_0 \,\mathrm{d}\pi^{K,n} \to \int f_0 \,\mathrm{d}\pi^{K,0}$ this implies $\int f_n \,\mathrm{d}\pi^{K,n} \to \int f_0 \,\mathrm{d}\pi^{K,0}$ (see also Pollard [24] Example 19 p. 73 for related arguments). Putting this together with the uniform integrability estimate above and letting $\epsilon \to 0$ gives $a_n \to a_0$.

By uniform integrability $\mathbb{E}S_1^n \to \mathbb{E}S_1^0$, and further $\pi^{K,n} \xrightarrow{\mathcal{D}} \pi^{K,0}$ implies $\int_0^K \overline{\pi}^{K,n} \to \int_0^K \overline{\pi}^{K,0}$. Remembering $a_n \to a_0$ and inspecting the expression (2.4) for the loss rate shows that indeed $\ell^{K,n} \to \ell^{K,0}$. □

5 Proof of Theorem 4

The result proposition is standard:

Proposition 7. *Let $p > 0$ and let $X_n \in L^p$, $n = 0, 1, \ldots$, such that $X_n \xrightarrow{\mathcal{D}} X_0$. Then $\mathbb{E}|X_n|^p \to \mathbb{E}|X_0|^p$ if and only if the family $\{|X_n|^p\}_{n\geq 1}$ is uniformly integrable.*

Theorem 4 is proved through several preliminary results. First, we prove Lemma 8 which essentially states we may disregard the behavior of the Lévy measures on the interval $[-1,1]$ in questions regarding uniform integrability. It is therefore sufficient to Theorem 4 for compound Poisson distributions, which is done in Proposition 10 and Proposition 11.

We start by examining the case where the Lévy measures have uniformly bounded support, i.e., there exists $A > 0$ such that $\nu_n([A,A]^c) = 0$ for all n. We know from Lemma 25.6 and Lemma 25.7 in Sato [26] that this implies the existence of finite exponential moments for X_n and therefore $\mathbb{E}X_n^m$ exists and is finite as well for all $n, m \in \mathbb{N}$.

Lemma 8. *Suppose* $X_n \xrightarrow{\mathcal{D}} X_0$ *and the Lévy measures have uniformly bounded support. Then* $\mathbb{E}X_n^m \to \mathbb{E}X_0^m$ *for* $m = 1, 2, \dots$ *In particular* (cf. Proposition 7) *the family* $\{|X_n|^\alpha\}_{n\geq 1}$ *is uniformly integrable for all* $\alpha > 0$.

Proof. By Lemma 25.6 of [26], the characteristic exponent $\kappa_n(s)$ of X_n is defined for all $s \in \mathbb{C}$, and we can work with the moment generating function $\mathbb{R} \ni t \to \mathbb{E}\mathrm{e}^{tX} \in \mathbb{R}$, which the by the Levy-Khinchine representation can be written as $\mathbb{E}\mathrm{e}^{tX_n} = e^{\kappa_n(t)}$ where

$$\kappa_n(t) = \theta_n t + \sigma_n^2 t^2/2 + \int_{-A}^{A} \big(\mathrm{e}^{ty} - 1 - tyI(|y| \leq 1)\big)\nu_n(\mathrm{d}y) \tag{5.1}$$

With the aim of applying Lemma 13.15 in Kallenberg [16], we rewrite (5.1) as

$$\kappa_n(t) = c_n t + \int_{-A}^{A} \left(\mathrm{e}^{ty} - 1 - \frac{ty}{1+y^2}\right) \frac{1+y^2}{y^2} \tilde{\nu}_n(\mathrm{d}y) \tag{5.2}$$

where $\tilde{\nu}_n$ is as above and

$$c_n = \theta_n + \int_{-A}^{A} \left(\frac{y}{1+y^2} - yI(|y| \leq 1)\right) \nu_n(\mathrm{d}y)$$

According to Lemma 13.15 in [16], the weak convergence of $\{X_n\}_{n\geq 1}$ implies $c_n \to c_0$ and $\tilde{\nu}_n \xrightarrow{\mathcal{D}} \tilde{\nu}$. Since the integrand in (5.2) is bounded and continuous, this implies that $\kappa_n(t) \to \kappa_0(t)$, which in turn implies that all exponential moments converge. In particular, the family $\{\mathrm{e}^{X_n} + \mathrm{e}^{-X_n}\}_{n\geq 1}$ is uniformly integrable, which implies that $\{|X_n|^\alpha\}_{n\geq 1}$ is so. □

Next, we express the condition of uniform integrability using the tail of the involved distributions. We will need the following lemma on weakly convergent compound Poisson distributions.

Lemma 9. *Let* $U_0, U_1, \dots$ *be a sequence of positive random variables such that* $U_n > 1$, *and let* $N_0, N_1, \dots$ *be Poisson random variables with rates* $\lambda_0, \lambda_1 \dots$ *Set* $X_n := \sum_1^{N_n} U_{i,n}$ (empty sum $= 0$) *with the* $U_{i,n}$ *being i.i.d for fixed* n *with* $U_{i,n} \overset{\mathcal{D}}{=} U_n$. *Then* $X_n \xrightarrow{\mathcal{D}} X_0$ *if and only if* $U_n \xrightarrow{\mathcal{D}} U_0$ *and* $\lambda_n \to \lambda_0$.

Proof. We use the continuity theorem for characteristic functions. The characteristic function of X_n is $\mathbb{E}^{\mathrm{i}sX_n} = \exp\{\lambda_n(\mathbb{E}^{\mathrm{i}sU_n} - 1\}$. From this the 'if' part is immediately clear. For the converse, we observe that $\exp(-\lambda_n) \to \exp(-\lambda_0) = \mathbb{P}(X_0 < 1/2)$

since $1/2$ is a continuity point of X_0 (note that $\mathbb{P}(X_0 \leq x) = \mathbb{P}(X_0 = 0)$ for all $x < 1$). Taking logs yields $\lambda_n \to \lambda_0$ and the necessity of $U_n \xrightarrow{\mathcal{D}} U_0$ then is obvious from the continuity theorem for characteristic functions. □

Using the previous result, we are ready to prove part of our main result for a class of compound Poisson distributions:

Proposition 10. *Let $U_0, U_1, \ldots,$ $N_0, N_1, \ldots,$ and $X_0, X_1, \ldots$ be as in Lemma 9. Assume $X_n \xrightarrow{\mathcal{D}} X_0$. Then for $\alpha > 0$.*

$$\lim_{a\to\infty} \sup_n \mathbb{E}\big[X_n^\alpha I(X_n > a)\big] = 0 \Rightarrow \lim_{a\to\infty} \sup_n \mathbb{E}\big[U_n^\alpha I(U_n > a)\big] = 0\,.$$

Proof. Let $G_n(x) = \mathbb{P}(X_n \leq x), F_n(x) = \mathbb{P}(U_n \leq x), \overline{F}_n(x) = 1 - F_n(x), \overline{G}_n(x) = 1 - G_n(x)$, and let $F_n^{*m}(x), G_n^{*m}(x)$ denote the m'th fold convolutions. Then

$$\overline{G}_n(x) = \sum_{m=1}^{\infty} \frac{\lambda_n^m}{m!} \mathrm{e}^{-\lambda_n} \overline{F}_n^{*m}(x) \qquad x > 0$$

which implies $\overline{G}_n(x) \geq \lambda_n \mathrm{e}^{-\lambda_n} \overline{F}_n(x)$. Letting $\beta = \sup_n \mathrm{e}^{\lambda_n}/\lambda_n$, which is finite by Lemma 9, we get: $\overline{F}_n(x) \leq \beta \overline{G}_n(x)$. Therefore:

$$\begin{aligned}
\mathbb{E}[U_n^\alpha I(U_n > a)] &= \int_0^\infty \alpha t^{\alpha-1} \mathbb{P}(U_n > a \vee t)\mathrm{d}t \\
&= a^\alpha \overline{F}_n(a) + \alpha \int_a^\infty t^{\alpha-1} \overline{F}_n(t)\mathrm{d}t \\
&\leq \beta a^\alpha \overline{G}_n(a) + \beta\alpha \int_a^\infty t^{\alpha-1} \overline{G}_n(t)\mathrm{d}t \\
&= \beta \mathbb{E}[X_n^\alpha I(X_n > a)]\,.
\end{aligned}$$

Taking supremum and limits completes the proof. □

Next, we prove the converse of Proposition 10.

Proposition 11. *Under the assumptions of Proposition 10 we have, for $\alpha > 0$:*

$$\lim_{a\to\infty} \sup_n \mathbb{E}\big[U_n^\alpha I(U_n > a)\big] = 0 \Rightarrow \lim_{a\to\infty} \sup_n \mathbb{E}\big[X_n^\alpha I(X_n > a)\big] = 0\,.$$

Proof. We use the notation of Proposition 10. By Lemma 9 we have $F_n^{*1} \xrightarrow{\mathcal{D}} F_0^{*1}$ and by the Portmanteau lemma $F_n^{*m} \xrightarrow{\mathcal{D}} F_0^{*m}$. We note that the assumption of uniform integrability of the U_n^α implies that $\mathbb{E}\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \to \mathbb{E}\big(\sum_{i=1}^m U_{i,0}\big)^\alpha$, since the $U_{i,n}$ are i.i.d in i and $U_{i,n} \stackrel{\mathcal{D}}{=} U_n$. Fix $m \in \mathbb{N}$. Since $\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \leq m^\alpha \sum_{i=1}^m U_{i,n}^\alpha$ and the family $\big(m^\alpha \sum_{i=1}^m U_{i,n}^\alpha\big)_{n\geq 1}$ is uniformly integrable, we have that also the family $\big(\sum_{i=1}^m U_{i,n}\big)^\alpha_{n\geq 1}$ is uniformly integrable. As noted above we have $\sum_{i=1}^m U_{i,n} \xrightarrow{\mathcal{D}} \sum_{i=1}^m U_{i,0}$, so Proposition 7 implies $\mathbb{E}\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \to \mathbb{E}\big(\sum_{i=1}^m U_{i,0}\big)^\alpha$.

We next show $\mathbb{E}X_n^\alpha \to \mathbb{E}X_0^\alpha$ and thereby the assertion of the proposition. We have:

$$\begin{aligned}\lim_n \mathbb{E}X_n^\alpha &= \lim_n \sum_{m=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \\ &= \sum_{m=0}^{\infty} \lim_n \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \\ &= \sum_{m=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,0}\Big)^\alpha \frac{\lambda_0^m}{m!}\mathrm{e}^{-\lambda_0} = \mathbb{E}X_0^\alpha ,\end{aligned}$$

where we used dominated convergence with the bound

$$\mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \le \gamma m^{\alpha+1}\beta^m/m!$$

with $\gamma = \sup_n \mathbb{E}U_n^\alpha$ and $\beta = \sup_n \lambda_n$. □

Proof of Theorem 4. Using the Lévy -Khinchine representation, we may write

$$X_n = X_n^{(1)} + X_n^{(2)} + X_n^{(3)} \tag{5.3}$$

where the $\big(X_n^{(i)}\big)_{n\ge1}$ are sequences of infinitely divisible distributions having characteristic triplets $(0,0,[\nu]_{\{y<-1\}})$, $(\theta_n,\sigma_n,[\nu_n]_{\{|y|\le1\}})$ and $(0,0,[\nu_n]_{\{y>1\}})$, respectively. Assume the family $\big(|X_n|^\alpha\big)_{n\ge1}$ is uniformly integrable. We wish to apply Proposition 10 to the family $\big((X_n^{(3)})^\alpha\big)$, and therefore we need to show that this family is uniformly integrable. First, we we rewrite (5.3) as $X_n - X_n^{(2)} = X_n^{(1)} + X_n^{(3)}$ and use Lemma 8 together with the inequality $|x-y|^\alpha \le 2^\alpha(|x|^\alpha + |y|^\alpha)$ to conclude that the family $\big(|X_n - X_n^{(2)}|^\alpha\big)_{n\ge1}$ is uniformly integrable, which in turn implies that the family $\big(|X_n^{(1)} + X_n^{(3)}|^\alpha\big)_{n\ge1}$ is uniformly integrable.

Assuming w.l.o.g. that 1 is a continuity point of ν_0, we have that $X_n^{(1)}$ is weakly convergent and therefore tight. This implies that there exists $r>0$ such that $\mathbb{P}\big(|X_n^{(1)}| \le r\big) \ge 1/2$ for all n, which implies that for all n and for all t so large that $(t^{1/\alpha}-r)^\alpha > t/2$, we have:

$$\begin{aligned}(1/2)\mathbb{P}\big((X_n^{(3)})^\alpha > t\big) &\le \mathbb{P}\big(|X_n^{(1)}| \le r\big)\mathbb{P}\big(X_n^{(3)} > t^{1/\alpha}\big) \\ \mathbb{P}\big(|X_n^{(1)}| \le r, X_n^{(3)} > t^{1/\alpha}\big) &\le \mathbb{P}\big(X_n^{(1)} + X_n^{(3)} > t^{1/\alpha} - r\big) \\ &= \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > (t^{1/\alpha}-r)^\alpha\big) \le \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > t/2\big) .\end{aligned}$$

This implies that $\big((X_n^{(3)})^\alpha\big)$ is uniformly integrable, since $\big(|X_n^{(1)} + X_n^{(3)}|^\alpha\big)$ is so. Applying Proposition 10 yields

$$\lim_a \sup_n \int_a^\infty y^\alpha \nu_n(\mathrm{d}y) = 0 \tag{5.4}$$

Together with a similar relation for $\int_{-\infty}^{-a}$ this gives

$$\lim_{a\to\infty} \sup_n \int_{[a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0\,.$$

For the converse, we assume $\lim_a \sup_n \int_{[a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0$, and return to our decomposition (5.3). As before, we apply Lemma 8 to obtain that the family $\big(X_n^{(2)}\big)$ is uniformly integrable. Furthermore, applying Proposition 11, we obtain that the families $\big(|X_n^{(1)}|^\alpha\big)$ and $\big(|X_n^{(3)}|^\alpha\big)$ are uniformly integrable, and since $|X_n| \le 3^\alpha\big(|X_n^1|^\alpha + |X_n^2|^\alpha + |X_n^3|^\alpha\big)$, the proof is complete. □

6 Proof of Theorem 2

First we note the effect that scaling and time-changing a Lévy process has on the loss rate:

Proposition 12. *Let $\beta,\delta > 0$ and define $S_t^{\beta,\delta} = S_{\delta t}/\beta$. Then the loss rate $\ell^{K/\beta}(S^{\beta,\delta})$ for $S^{\beta,\delta}$ equals δ/β times the loss rate $\ell^K(S) = \ell^K$ for S.*

Proof. It is clear that scaling by β results in the same scaling of the loss rate. For the effect of δ, note that the loss rate is the expected local time in stationarity per unit time and that one unit of time for $S^{\beta,\delta}$ corresponds to δ units of time for S. □

Proof of Theorem 2 (a). Define $S_t^K := S_{tK^2}/K$. Then by Proposition 12 we have

$$K\ell^K(S) = \ell^1(S^K)$$

By the central limit theorem we have $S_1^K \xrightarrow{\mathcal{D}} N(0,\psi^2)$ as $K \to \infty$, where

$$\psi^2 = \mathbb{V}\mathrm{ar}(S_1^1) = \sigma^2 + \int_{-\infty}^{\infty} y^2 \nu(\mathrm{d}y)\,.$$

By Proposition 1, this is equivalent to $S^K \xrightarrow{\mathcal{D}} \psi B$ where B is standard Brownian motion. We may apply Theorem 3, since

$$\mathbb{E}\big[(S_1^K)^2\big] = \mathbb{V}\mathrm{ar}(S_1^1)\,,$$

that is, $\{S_1^K\}_{K=1}^\infty$ is bounded in L^2 and therefore uniformly integrable, and we obtain $\lim_K K\ell^K(S) = \lim_K \ell^1\big(S^K\big) = \ell^1(\psi B) = \psi^2/2$, where the last equality follows directly from the expression for the loss rate given by (2.4). □

Proof of Theorem 2 (b). First we note that the stated conditions implies that the tails of ν are regularly varying, and therefore they are subexponential. Then by Embrechts et al. [12] we have that the tails of $P(S_1 < x)$ are equivalent to those of ν and hence we may write $P(S_1 > x) = x^{-\alpha}L_1(x)g_1(x)$, and $P(S_1 < -x) = x^{-\alpha}L_2(x)g_2(x)$ where $\lim_{x\to\infty} g_i(x) = 1$. $i = 1,2$. The next step is to show that the fact that tails of the distribution function is regularly varying allows us to apply the stable central limit theorem. Specifically, we show that the assumptions of Theorem 1.8.1 in Samorodnitsky and Taqqu [25] are fulfilled.

We notice that if we define $M(x) := L_1(x)g_1(x) + L_2(x)g_2(x)$ then $M(x)$ is slowly varying and

$$x^\alpha(P(S_1 < -x) + P(S_1 > x)) = M(x)\,. \tag{6.1}$$

Furthermore:

$$\frac{P(S_1 > x)}{P(S_1 < -x) + P(S_1 > x)} = L_2(x)g_2(x)/M(x) \sim L_2(x)/L(x) \to 1\,, \qquad x \to \infty \tag{6.2}$$

since $L_(x) \sim M(x)$. Let $L_0^\#(x)$ denote the de Bruin conjugate of L_0 (cf. Bingham et al. [7] p. 29) and set $f(n) := n^{(1/\alpha)} L_0^\#(n^{(1/\alpha)})$. Let $f^\leftarrow$ be the generalized inverse of f. By asymptotic inversion of regularly varying functions (p. 28-29 [7]) we have $f^\leftarrow(n) \sim (nL_0(n))^\alpha$ and using (3.3) we have

$$\frac{f^\leftarrow(n)L(n)}{n^\alpha} \sim \frac{(nL_0(n))^\alpha L(n)}{n^\alpha} = L_0(n)^\alpha L(n) \to 1$$

and since $f^\leftarrow(f(n)) \sim n$ we have

$$\frac{nM(f(n))}{f(n)^\alpha} \sim \frac{nL(f(n))}{f(n)^\alpha} \sim \frac{f^\leftarrow(f(n))L(f(n))}{f(n)^\alpha} \to 1 \tag{6.3}$$

and therefore, if we define $\sigma = (-\Gamma(1-\alpha)\cos(\alpha\pi/2))^{1/\alpha}$.

$$\frac{nM(\sigma^{-1}f(n))}{(\sigma^{-1}f(n))^\alpha} \sim \frac{nM(f(n))}{(\sigma^{-1}f(n))^\alpha} \to \sigma^\alpha \tag{6.4}$$

using slow variation of M. By combining (6.1), (6.2) and (6.4) we may apply the stable CLT Theorem 1.8.1 [25][1] to obtain $S_K/f(K) \xrightarrow{\mathcal{D}} X$ where X is a r.v. with c.h.f. φ, where

$$\varphi(t) = \exp(-|\sigma t|^\alpha(1 - \mathrm{i}\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2))$$

Recalling that κ is the characteristic exponent of S_1, this is equivalent to

$$\mathrm{e}^{\kappa(t/f(K))}K \to \varphi(t)$$

and therefore

$$\mathrm{e}^{\kappa(t/f(f^\leftarrow(K)))}(KL_0(K))^\alpha \sim \mathrm{e}^{\kappa(t/f(f^\leftarrow(K)))} f^\leftarrow(K) \to \varphi(t)$$

that is, for $S_t^K = S_{t(KL_0(K))^\alpha}/f(f^\leftarrow(K))$ we have $S_1^K \xrightarrow{\mathcal{D}} X$. Setting $d = (\beta+1)/2$ and $c = (\beta-1)/2$ we may use formula (3.37.13) in Hoffmann-Jørgensen [13] to obtain

$$\begin{aligned} &-|\sigma t|^\alpha(1 - \mathrm{i}\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2) && (6.5)\\ &\quad = -|\sigma t|^\alpha(1 + \mathrm{i}(d-c)\,\mathrm{sgn}(t)\tan(\alpha\pi/2) && (6.6)\\ &\quad = d\alpha \int_{-\infty}^0 (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)(-t)^{-\alpha-1}\,\mathrm{d}t && (6.7)\\ &\qquad + c\alpha \int_0^\infty (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)t^{-\alpha-1}\,\mathrm{d}t && (6.8) \end{aligned}$$

[1]Note that the constants there should be replaced by their inverses.

That is, the characteristic triplet of X is $(\tau, 0, \nu)$, where

$$\nu(\mathrm{d}t) = \begin{cases} \frac{\alpha c}{(-t)^{\alpha+1}}\,\mathrm{d}t & t < 0 \\ \frac{\alpha d}{t^{\alpha+1}}\,\mathrm{d}t & t > 0 \end{cases} \tag{6.9}$$

and τ is a centering constant.
We wish to use Theorem 3 and have to prove uniform integrability. Since $f(f^{\leftarrow}(K)) \sim K$ we have $K/2 < f(f^{\leftarrow}(K))$ for large enough K, and for these K and $1 < r < \alpha < 2$, we find, using Khinchine's Inequality (eg. (4.32.1) in [13])

$$\begin{aligned}
-|\mathbb{E}\left|S_1^K\right|^r &\leq \frac{2^r}{K^r}\mathbb{E}\left|\sum_{i=1}^{[(KL_0(K))^\alpha]} S_i - S_{i-1} + S_{(KL_0(K))^\alpha} - S_{[(KL_0(K))^\alpha]}\right|^r \\
&\leq \frac{2^{r+1}}{K^r}\mathbb{E}\left|\sum_{i=1}^{[(KL_0(K))^\alpha]} S_i - S_{i-1}\right|^r + 2^r\mathbb{E}\left|S_{(KL_0(K))^\alpha} - S_{[(KL_0(K))^\alpha]}\right|^r \\
&\leq \frac{2^{r+1}}{K^r}\mathbb{E}\left|\sum_{i=1}^{[(KL_0(K))^\alpha]} S_i - S_{i-1}\right|^r + 2^r\mathbb{E}\left|S_1\right|^r \\
&\leq \frac{2^{r+2}}{K^r}[(KL_0(K))^\alpha]^{\frac{r}{2}-1}\sum_{i=1}^{[(KL_0(K))^\alpha]} \mathbb{E}|S_i - S_{i-1}|^r + 2^r\mathbb{E}\left|S_1\right|^r \\
&= \frac{2^{r+2}}{K^r}[(KL_0(K))^\alpha]^{\frac{r}{2}}\mathbb{E}|S_1|^r + 2^r\mathbb{E}\left|S_1\right|^r \\
&\leq 2^{r+2}K^{r(\frac{\alpha}{2}-1)}L_0(K)^{\frac{\alpha r}{2}}\mathbb{E}|S_1|^r + 2^r\mathbb{E}\left|S_1\right|^r
\end{aligned}$$

so that $\{S_t^K\}_{K=1}^\infty$ is bounded i L^r. We may therefore apply Theorem 3 and Proposition 12 to obtain

$$K^{\alpha-1}L_0(K)^\alpha \ell^K(S) \sim K^{\alpha-1}L_0(K)^\alpha \ell^{f(f^{\leftarrow}(K))}(S) = \ell^1\left(S^K\right).$$

Letting $K \to \infty$ and using the expression for the loss rate of a stable distribution which is calculated in Example 3.2 in Asmussen and Pihlsgård [4] (see also Kyprianou [20]), yields the desired result. □

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