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1 Introduction

For a compact subset X of $\mathbb{R}^d$, satisfying certain regularity conditions, the classical Crofton formula relates integrals of intrinsic volumes defined on $j-$dimensional affine subspaces to intrinsic volumes of X,

$$\int_{\mathcal{F}_j^d} V_k(X \cap F_j) \mathrm{d}F_j^d = c_{d,j,k} V_{d-j+k}(X),$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$. Here, $\mathcal{F}_j^d$ is the set of $j-$dimensional affine subspaces and $\mathrm{d}F_j^d$ is the element of the motion invariant measure on $j-$dimensional affine subspaces in $\mathbb{R}^d$. Furthermore, $V_k(X)$, $k = 0, 1, \ldots, d$, are the intrinsic volumes of X. Finally, $c_{d,j,k}$ is a known constant.

Motivated by applications in local stereology, a rotational version of the Crofton formula has recently been derived, cf. [7]. This formula shows how rotational averages of intrinsic volumes measured on sections passing through fixed points are related to the geometry of the sectioned object. More specifically, for a compact subset $X \subset \mathbb{R}^d$ of positive reach, the functionals $\beta_{j,k}$, satisfying

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j) \mathrm{d}L_j^d = \beta_{j,k}(X),$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$, have been determined in [7]. For $k = j$, $\beta_{j,j}(X)$ is a simple integral while in the case $k < j$, $\beta_{j,k}(X)$ is a complicated integral over the unit normal bundle of X, involving principal curvatures and hypergeometric functions.

In the present paper, we address the 'opposite' problem of finding functionals $\alpha_{j,k}$, satisfying the following rotational integral equation

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j) \mathrm{d}L_j^d = V_{d-j+k}(X), \tag{1}$$

$j = 0, 1, \ldots, d$ and $k = 0, 1, \ldots, j$. The solution of the problem is inspired by some recent work reported in [3] and [4].

2 The general solution

The main tools for deriving solutions to (1) are the classical Crofton formula and a well-known geometric measure decomposition from integral geometry.

The motion invariant measure on $j-$dimensional affine subspaces can be decomposed as follows. For $F_j = x + L_j$, where L_j is a $j-$dimensional linear subspace and $x \in L_j^{\perp}$, we have $\mathrm{d}F_j^d = \mathrm{d}x^{d-j}\mathrm{d}L_j^d$ where $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$, the set of $j-$dimensional linear subspaces and, for given $L_j \in \mathcal{L}_j^d$, $\mathrm{d}x^{d-j}$ is the element of the Lebesgue measure in $L_j^{\perp}$. The total mass of $\mathrm{d}L_j^d$ is chosen to be

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j},$$

where

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1} \tag{2}$$

and $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere in $\mathbb{R}^k$. With this choice, the constant in the classical Crofton formula becomes

$$c_{d,j,k} = c_{d,j} \cdot \frac{\Gamma(\frac{j+1}{2})\Gamma(\frac{d+k-j+1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})}. \tag{3}$$

The geometric measure decomposition used in the derivation of solutions to (1) concerns the motion invariant measure on $r-$dimensional affine subpaces in $\mathbb{R}^d$. According to Gual-Arnau and Cruz-Orive [4], we have for $r = 0, 1, \ldots, d-1$ that

$$\mathrm{d}F_r^d = d(O, F_r)^{d-r-1}\mathrm{d}F_r^{r+1}\mathrm{d}L_{r+1}^d, \tag{4}$$

where $d(O, F_r)$ denotes the distance from F_r to the origin O. Note that for $r = 0$, (4) reduces to the standard polar decomposition of Lebesgue measure

$$\mathrm{d}x^d = |x|^{d-1}\mathrm{d}x^1\mathrm{d}L_1^d.$$

We formulate the main result of this paper in the proposition below.

Proposition 1. *Let Y be a compact subset of $\mathbb{R}^j$ of positive reach. For $j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$, the functional*

$$\alpha_{j,k}(Y) = \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{k-1}(Y \cap F_{j-1})\mathrm{d}F_{j-1}^j \tag{5}$$

is a solution to (1).

Proof. Using the Crofton formula and the measure decomposition (4), we find that

$$\begin{aligned}
&\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{k-1}(X \cap L_j \cap F_{j-1})\mathrm{d}F_{j-1}^j \mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-(j-1)-1} V_{k-1}(X \cap F_{j-1})\mathrm{d}F_{j-1}^j \mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^d} V_{k-1}(X \cap F_{j-1})\mathrm{d}F_{j-1}^d \\
&= V_{d-j+k}(X).
\end{aligned}$$

□

3 The case $k = j$

For $k = j$, Proposition 1 provides a functional with rotational average equal to the volume $V_d(X)$. This functional can be simplified considerably, as shown in the proposition below. We use here and in the following the notation $p(x|L_r)$ for the orthogonal projection of $x \in \mathbb{R}^d$ onto $L_r \in \mathcal{L}_r^d$.

Proposition 2. *Let the situation be as in Proposition 1 and suppose that $k = j$. Then,*

$$\alpha_{j,j}(Y) = \frac{1}{c_{d-1,j-1}} \int_Y |z|^{d-j} \mathrm{d}z^j.$$

Proof. Using that $F_{j-1} = L_{j-1} + x$, where $x \in L_{j-1}^\perp$, we find

$$\begin{aligned}
\alpha_{j,j}(Y) &= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{j-1}(Y \cap F_{j-1})\mathrm{d}F_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^\perp} |x|^{d-j} V_{j-1}(Y \cap (L_{j-1} + x))\mathrm{d}x^1 \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^\perp} \int_{Y \cap (L_{j-1}+x)} |x|^{d-j} \mathrm{d}y^{j-1} \mathrm{d}x^1 \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_Y |p(z|L_{j-1}^\perp)|^{d-j} \mathrm{d}z^j \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\int_{\mathcal{L}_{j-1}^j} \frac{|p(z|L_{j-1}^\perp)|^{d-j}}{|z|^{d-j}} \mathrm{d}L_{j-1}^j \right) \mathrm{d}z^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\frac{c_{j,j-1}}{B(\frac{1}{2}, \frac{j-1}{2})} \int_0^1 y^{\frac{d-j-1}{2}} (1-y)^{\frac{j-3}{2}} \mathrm{d}y \right) \mathrm{d}z^j.
\end{aligned}$$

At the last equality sign, we have used [6, Proposition 3.9]. The result now follows immediately, using (2) and (3). □

4 The case $k < j$

It is also possible to make the expression of the functional $\alpha_{j,k}$ more explicit for $k < j$. We will concentrate on the case where ∂X is a $(d-1)-$dimensional manifold of class C^2. For $k = 0, 1, \ldots, d-1$, the kth intrinsic volume has the following integral representation

$$V_k(X) = \frac{1}{\sigma_{d-k}} \int_{\partial X} \sum_{|I|=d-1-k} \prod_{i \in I} \kappa_i(x) \mathcal{H}^{d-1}(\mathrm{d}x), \tag{6}$$

where $\kappa_i(x)$, $i = 1, \ldots, d-1$, are the principal curvatures of ∂X at $x \in \partial X$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. Since ∂X is a $(d-1)$-dimensional manifold of class C^2, $\partial X \cap F_j$ is a $(j-1)$-dimensional manifold of class C^2 for almost all $F_j \in \mathcal{F}_j^d$. The principal curvatures of $\partial X \cap F_j$ at $x \in \partial X \cap F_j$ is denoted by $\kappa_{F_j,i}(x)$, $i = 1, \ldots, j-1$.

The proposition below gives a more explicit expression for $\alpha_{j,k}$ for $k < j$ than the one given in (5).

Proposition 3. *Let the situation be as in Proposition 1. Suppose $k < j$. Suppose that $Y \subset \mathbb{R}^j$ has a boundary ∂Y which is a $(j-1)-$dimensional manifold of class C^2. For $z \in \partial Y$, let $n(z)$ be the unit normal vector of ∂Y at z. Then,*

$$\begin{aligned} &c_{d,j-1,k-1}\ \sigma_{j-k}\ \alpha_{j,k}(Y) \\ &\quad = \int_{\partial Y} \int_{\mathcal{L}_{j-1}^j} \kappa(z; L_{j-1} + z)\, |p(n(z)|L_{j-1})|\, |p(z|L_{j-1}^\perp)|^{d-j} \mathrm{d}L_{j-1}^j \mathcal{H}^{j-1}(\mathrm{d}z), \end{aligned}$$

where for $F_{j-1} \in \mathcal{F}_{j-1}^d$ and $z \in \partial Y \cap F_{j-1}$

$$\kappa(z; F_{j-1}) = \begin{cases} 1 & \text{if } k = j-1 \\ \sum_{|I|=j-k-1} \prod_{i \in I} \kappa_{F_{j-1},i}(z) & \text{if } k < j-1. \end{cases}$$

Proof. According to (5), we have

$$\begin{aligned} \alpha_{j,k}(Y) &= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^j} \mathrm{d}(O, F_{j-1})^{d-j} V_{k-1}(Y \cap F_{j-1}) \mathrm{d}F_{j-1}^j \\ &= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^\perp} |x|^{d-j} V_{k-1}(Y \cap (L_{j-1} + x)) \mathrm{d}x^1 \mathrm{d}L_{j-1}^j. \end{aligned}$$

Using the integral representation (6) of intrinsic volumes, the expression above becomes

$$\begin{aligned} &c_{d,j-1,k-1} \alpha_{j,k}(Y) \\ &= \frac{1}{\sigma_{(j-1)-(k-1)}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^\perp} |x|^{d-j} \int_{\partial Y \cap (L_{j-1}+x)} \kappa(y; L_{j-1} + x) \mathcal{H}^{(j-1)-1}(\mathrm{d}y) \mathrm{d}x^1 \mathrm{d}L_{j-1}^j \\ &= \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^\perp} \int_{\partial Y \cap (L_{j-1}+x)} |p(y|L_{j-1}^\perp)|^{d-j} \kappa(y; L_{j-1} + y) \mathcal{H}^{j-2}(\mathrm{d}y) \mathrm{d}x^1 \mathrm{d}L_{j-1}^j. \end{aligned}$$

At the first equality sign we have used that $\partial(Y \cap F_{j-1}) = \partial Y \cap F_{j-1}$ for almost all F_{j-1}. Using [6, Propositions 2.10 and 5.2] and Fubini, we finally get

$$\begin{aligned} & c_{d,j-1,k-1}\alpha_{j,k}(Y) \\ & \quad = \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}^j_{j-1}} \int_{\partial Y} |p(n(z)|L_{j-1})| \ \ |p(z|L^{\perp}_{j-1})|^{d-j} \kappa(z, L_{j-1}+z) \mathcal{H}^{j-1}(\mathrm{d}z) \mathrm{d}L^j_{j-1} \\ & \quad = \frac{1}{\sigma_{j-k}} \int_{\partial Y} \int_{\mathcal{L}^j_{j-1}} \kappa(z, L_{j-1}+z) |p(n(z)|L_{j-1})| \ \ |p(z|L^{\perp}_{j-1})|^{d-j} \mathrm{d}L^j_{j-1} \mathcal{H}^{j-1}(\mathrm{d}z). \end{aligned}$$

□

For $k = j-1$, the expression for $\alpha_{j,k}(Y)$ given in Proposition 3 can be further simplified thanks to the following proposition. The proof is referred to the Appendix.

Proposition 4. *Let $L_j \in \mathcal{L}^d_j$, $j = 1, \ldots, d$. Let x and y be unit vectors in L_j. Then, for all $m, n \in \mathbb{N}$,*

$$\begin{aligned} & \int_{\mathcal{L}^j_{j-1}} |p(x|L_{j-1})|^m \ |p(y|L^{\perp}_{j-1})|^n \mathrm{d}L^j_{j-1} \\ & \quad = \frac{\sigma_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{j-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{j-1}{2}, \sin^2 \angle(x,y)\right). \end{aligned}$$

□

As a consequence of Proposition 4, we get for $m = 1$ and $n = d-j$

$$\alpha_{j,j-1}(Y) = \frac{1}{2c_{d-1,j-1}} \int_{\partial Y} |z|^{d-j} F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; \sin^2 \angle(n(z), z)\right) \mathcal{H}^{j-1}(\mathrm{d}z).$$

Appendix

In this appendix, we will prove Proposition 4. Without loss of generality, we assume that $x \cdot y > 0$. For simplicity, we write $\mathrm{d}z^j$ instead of $\mathcal{H}^j(\mathrm{d}z)$.

The Gauss *hypergeometric series* or *hypergeometric function* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0 \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \leq 0. \end{cases}$$

Note that $(x)_k = 0$ whenever $x \in \{0, -1, -2, \ldots\}$ and $k > -x$.

An application of [6, Propositions 3.2 and 3.3] gives

$$\begin{aligned}\int_{\mathcal{L}^j_{j-1}} & |p(x|L_{j-1})|^m \quad |p\left(y|L_{j-1}^\perp\right)|^n \mathrm{d}L^j_{j-1} \\ &= \int_{\mathcal{L}^j_1} |p(x|L_1^\perp)|^m \quad |p\left(y|L_1\right)|^n \mathrm{d}L^j_1 \\ &= \frac{1}{2}\int_{S^{j-1}} |p(x|\mathrm{span}\{\omega\}^\perp)|^m \quad |p\left(y|\mathrm{span}\{\omega\}\right)|^n \mathrm{d}\omega^{j-1} \\ &= \frac{1}{2}\int_{S^{j-1}} \sqrt{1-(x\cdot\omega)^2}^m \quad |y\cdot\omega|^n \mathrm{d}\omega^{j-1} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\binom{\frac{m}{2}}{k}(-1)^k\int_{S^{j-1}} |x\cdot\omega|^{2k}|y\cdot\omega|^n \mathrm{d}\omega^{j-1}. \end{aligned} \tag{7}$$

Now note that

$$\begin{aligned}\int_{S^{j-1}} & |x\cdot\omega|^{2k}|y\cdot\omega|^n \mathrm{d}\omega^{j-1} \\ &= \int_{S^{j-1}} |p(p(\omega|x\oplus y)|x)|^{2k}|p(p(\omega|x\oplus y)|y)|^n \mathrm{d}\omega^{j-1}. \end{aligned} \tag{8}$$

In order to compute (8), we will use the following lemma.

Lemma 1. *Let $B_p \in \mathcal{L}^d_p$. Then, for any non-negative measurable function $g : \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{S^{d-1}} g(p(x|B_p))\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2}\int_{S^{p-1}(B_p)}\int_0^1 g\left(t^{\frac{1}{2}}x_0\right)t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\mathrm{d}t\mathrm{d}x_0^{p-1},$$

where $S^{p-1}(B_p)$ is the unit sphere in B_p.

Proof. First, we use the co-area formula with

$$\begin{aligned}\psi \ : \ & S^{d-1} \to S^{p-1}(B_p) \\ & x \to \pi(x|B_p) := p(x|B_p)/|p(x|B_p)|.\end{aligned}$$

The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{d-1}) = |p(x|B_p)|^{-(p-1)}.$$

Hence, the co-area formula yields

$$\begin{aligned}\int_{S^{d-1}} g(p(x|B_p))\mathrm{d}x^{d-1} &= \int_{S^{d-1}} g\left(|p(x|B_p)|\pi(x|B_p)\right)\mathrm{d}x^{d-1} \\ &= \int_{S^{p-1}(B_p)}\int_{\psi^{-1}(x_0)} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\mathrm{d}x^{d-p}\mathrm{d}x_0^{p-1}. \end{aligned} \tag{9}$$

Next, let $x_0 \in S^{p-1}(B_p)$ be fixed and apply the area formula with

$$\begin{aligned}\xi \ : \ & B_p^\perp \to \psi^{-1}(x_0) \\ & \omega \mapsto \frac{\omega + x_0}{|\omega + x_0|}.\end{aligned}$$

The $(d-p)$-dimensional Jacobian of ξ is

$$J_{d-p}\xi(\omega, S^{d-1}) = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{d-p+1}{2}}.$$

Hence, since ξ maps $B_p^\perp$ bijectively onto $\psi^{-1}(x_0)$ and $|p(\xi(\omega)|B_p)| = \frac{1}{|\omega+x_0|} = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{1}{2}}$, we have

$$\begin{aligned}
&\int_{\psi^{-1}(x_0)} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\mathrm{d}x^{d-p} \\
&\quad= \int_{\psi^{-1}(x_0)} \sum_{\omega\in\xi^{-1}(x)} g(|p(\xi(\omega)|B_p)|x_0)|p(\xi(\omega)|B_p)|^{p-1}\mathrm{d}x^{d-p} \\
&\quad= \int_{\psi^{-1}(x_0)} \sum_{\omega\in\xi^{-1}(x)} g\left(\left(\frac{1}{1+|\omega|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|\omega|^2}\right)^{\frac{p-1}{2}} \mathrm{d}x^{d-p} \\
&\quad= \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{p-1}{2}}\left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}} \mathrm{d}x^{d-p} \\
&\quad= \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}} \mathrm{d}x^{d-p}.
\end{aligned}$$

Using [6, Proposition 2.8], we get

$$\begin{aligned}
&\int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}} \mathrm{d}x^{d-p} \\
&\qquad= \sigma_{d-p}\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\mathrm{d}t. \qquad (10)
\end{aligned}$$

Substitution with $s = \frac{1}{1+t^2}$ yields

$$\begin{aligned}
&\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\mathrm{d}t \\
&\qquad= \frac{1}{2}\int_0^1 g\left(s^{\frac{1}{2}}x_0\right) s^{\frac{p-2}{2}}(1-s)^{\frac{d-p-2}{2}}\mathrm{d}s
\end{aligned}$$

The last equation combined with (9) and (10) implies

$$\int_{S^{d-1}} g(p(x|B_p))\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2}\int_{S^{p-1}(B_p)}\int_0^1 g\left(t^{\frac{1}{2}}x_0\right) t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\mathrm{d}t\mathrm{d}x_0^{p-1}.$$

□

Applying Lemma 1 with $B = \operatorname{span}\{x, y\}$, we get

$$\begin{aligned}
&\int_{S^{j-1}} |p(p(\omega|x \oplus y)|x)|^{2k} |p(p(\omega|x \oplus y)|y)|^n \mathrm{d}\omega^{j-1} \\
&\quad = \frac{\sigma_{j-2}}{2} \int_{S^1(B)} \int_0^1 t^k |p(\omega_0|x)|^{2k} t^{n/2} |p(\omega_0|y)|^n t^{\frac{2-2}{2}} (1-t)^{\frac{j-2-2}{2}} \mathrm{d}t \mathrm{d}\omega_0^1 \\
&\quad = \frac{\sigma_{j-2}}{2} \int_{S^1(B)} |p(\omega_0|y)|^n |p(\omega_0|x)|^{2k} \mathrm{d}\omega_0^1 \int_0^1 t^{\frac{n+2k}{2}} (1-t)^{\frac{j-4}{2}} \mathrm{d}t \\
&\quad = \frac{\sigma_{j-2} B\left(\frac{n}{2}+k+1, \frac{j-2}{2}\right)}{2} \int_{S^1(B)} |p(\omega_0|y)|^n |p(\omega_0|x)|^{2k} \mathrm{d}\omega_0^1. \qquad (11)
\end{aligned}$$

Successive application of [6, Proposition 3.2] and [5, Corollary 4.2] yield

$$\begin{aligned}
&\int_{S^1(B)} |p(\omega_0|y)|^n |p(\omega_0|x)|^{2k} \mathrm{d}\omega_0^1 = 2 \int_{\mathcal{L}_1^2(B)} |p(x|L_1)|^{2k} |p(y|L_1)|^n \mathrm{d}L_1^2 \\
&= 2 \int_{-1}^1 \int_{S^1 \cap y^\perp} (1-t^2)^{\frac{2-1-2}{2}} |p(x|tx + \sqrt{1-t^2}\omega)|^{2k} |p(y|tx + \sqrt{1-t^2}\omega)|^n \mathrm{d}\omega \mathrm{d}t \\
&= 2 \int_{-1}^1 \int_{S^1 \cap y^\perp} (1-t^2)^{\frac{2-1-2}{2}} |t|^n |t(y \cdot x) + \sqrt{1-t^2}(x \cdot \omega)|^{2k} \mathrm{d}\omega \mathrm{d}t \\
&= 2 \int_{-1}^1 (1-t^2)^{\frac{2-1-2}{2}} |t|^n \Big(|t(y \cdot x) + \sqrt{1-t^2}\sqrt{1-(y \cdot x)^2}|^{2k} \\
&\qquad\qquad + |t(y \cdot x) - \sqrt{1-t^2}\sqrt{1-(y \cdot x)^2}|^{2k} \Big) \mathrm{d}t.
\end{aligned}$$

Using the binomial formula, the last expression becomes

$$\begin{aligned}
&2 \sum_{l=0}^{2k} \binom{2k}{l} \int_{-1}^1 \Big((1-t^2)^{\frac{2-1-2}{2}} |t|^n t^l (y \cdot x)^l (1-t^2)^{\frac{2k-l}{2}} \sqrt{1-(x \cdot y)^2}^{2k-l} \\
&\qquad + (-1)^l (1-t^2)^{\frac{2-1-2}{2}} |t|^n t^l (y \cdot x)^l (1-t^2)^{\frac{2k-l}{2}} \sqrt{1-(x \cdot y)^2}^{2k-l} \Big) \mathrm{d}t \\
&\quad = 4 \sum_{l=0}^{k} (x \cdot y)^{2l} (1-(x \cdot y)^2)^{k-l} \binom{2k}{2l} \int_0^1 (1-t^2)^{k-l-\frac{1}{2}} t^{n+2l} \mathrm{d}t \\
&\quad = 2 \sum_{l=0}^{k} (x \cdot y)^{2l} (1-(x \cdot y)^2)^{k-l} \binom{2k}{2l} B\left(\frac{n}{2}+l+\frac{1}{2}, k-l+\frac{1}{2}\right).
\end{aligned}$$

By applying the duplication formula for the Gamma function,

$$\Gamma(2z) = \Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\pi^{-\frac{1}{2}} 2^{2z-1},$$

we obtain

$$\begin{aligned}
&2 \sin^{2k} \angle(x,y) B\left(\frac{n}{2}+\frac{1}{2}, k+\frac{1}{2}\right) \sum_{l=0}^{k} \frac{(-k)_l \left(\frac{n}{2}+\frac{1}{2}\right)_l}{\left(\frac{1}{2}\right)_l} \frac{\left(-\tan^{-2}\angle(x,y)\right)^l}{l!} \\
&\quad = 2 \sin^{2k} \angle(x,y) B\left(\tfrac{n}{2}+\tfrac{1}{2}, k+\tfrac{1}{2}\right) F\left(-k, \tfrac{n}{2}+\tfrac{1}{2}; \tfrac{1}{2}; -\tan^{-2}\angle(x,y)\right).
\end{aligned}$$

According to [1, (15.3.4)] with $z=\cos^2\angle(x,y)$,

$$\sin^{2k}\angle(x,y)F\left(-k,\tfrac{n}{2}+\tfrac{1}{2};\tfrac{1}{2};-\tan^{-2}\angle(x,y)\right)=F\left(-k,-\tfrac{n}{2};\tfrac{1}{2};\cos^2\angle(x,y)\right).$$

By insertion in (11), we get

$$\int_{S^{j-1}}|x\cdot\omega|^{2k}|y\cdot\omega|^n \mathrm{d}\omega^{j-1}$$
$$=\sigma_{j-2}B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(k+\frac{1}{2},\frac{n}{2}+\frac{1}{2}\right)F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).$$

Hence, (7) becomes

$$\int_{\mathcal{L}^j_{j-1}}|p(x|L_{j-1})|^m\ \ |p\left(y|L^{\perp}_{j-1}\right)|^n \mathrm{d}L^j_{j-1}$$
$$=\frac{\sigma_{j-2}}{2}\sum_{k=0}^{\infty}\binom{\frac{m}{2}}{k}(-1)^k B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(\frac{n+1}{2},k+\frac{1}{2}\right)$$
$$\cdot F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).$$

Since

$$\frac{\sigma_{j-2}}{2}\binom{\frac{m}{2}}{k}(-1)^k B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(\frac{n+1}{2},k+\frac{1}{2}\right)$$
$$=\frac{\sigma_{j-1}}{2}B\left(\frac{j-1}{2},\frac{n+1}{2}\right)\frac{\left(-\frac{m}{2}\right)_k}{k!}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n+j}{2}\right)_k},$$

we now have

$$\int_{\mathcal{L}^j_{j-1}}|p(x|L_{j-1})|^m\ \ |p\left(y|L^{\perp}_{j-1}\right)|^n \mathrm{d}L^j_{j-1}$$
$$=\frac{\sigma_{j-1}}{2}B\left(\frac{j-1}{2},\frac{n+1}{2}\right)\sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1}{2}\right)_k}{\left(\frac{n+j}{2}\right)_k}\frac{F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right)}{k!}.$$

Using the power series expansion of the hypergeometric function, then expanding $(1-\sin^2\angle(x,y))^k$ and applying the identities

$$\frac{\binom{k+l}{l}}{(k+l)!}=\frac{1}{l!}\frac{1}{k!}\qquad\text{and}\qquad (a)_{k+l}=(a)_l(a+l)_k,$$

it is straightforward to prove that the last expression equals

$$\frac{\sigma_{j-1}}{2}B\left(\frac{n+1}{2},\frac{m}{2}+\frac{j-1}{2}\right)F\left(-\frac{m}{2},-\frac{n}{2};\frac{j-1}{2};\sin^2\angle(x,y)\right).$$

The proof is complete.

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