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Representation and properties of CGPII processes

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Abstract

It is shown that the class of conditionally Gaussian processes with independent increments is stable under marginalisation and conditioning. Moreover, in general such processes can be represented as integrals of a time changed Brownian motion where the time change and the integrand are jointly independent of the Brownian motion. Examples are given.

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1 Introduction

Two common ways of constructing mathematical models of random processes is by stochastic integration, as in stochastic volatility modelling, and by time change, as in subordination. Looking at the question from the other side, one may ask whether a given model can be represented in either of these two ways and, if so, how to determine the ingredients in the representations. More generally, we may consider the same type of questions but now allowing integration and time change simultaneously. The present paper discusses some aspects of the latter setting where the ingredients are a Brownian motion B and some other stochastic process(es) that are independent of B. The restriction imposed by the independence assumption is of course substantial, but it still allows considerable flexibility and encompasses many interesting and useful models.

We consider the class of conditionally Gaussian processes with independent increments (CGPII), i.e. processes $X = (X_t)_{t \geq 0}$ for which there exists some process $\Sigma = (\Sigma_t)_{t \geq 0}$, with values Σ_t in the cone S_+^d of nonnegative definite matrices, such that, given Σ, X has independent increments with $X_0 = 0$ and $X_t - X_s \sim N_d(0, \Sigma_t - \Sigma_s)$ for all $0 \leq s < t$. This class is closed under linear transformation, marginalisation

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and conditioning, and we show how, in wide generality, such a process X is representable in law as a stochastic integral with respect to a time changed Brownian motion. More precisely we show that X is a CGPII if and only if it is representable as

$$X \stackrel{\mathcal{D}}{=} \left(\int_0^t H_s \, \mathrm{d}(B \circ A)_s \right)_{t \geq 0} \tag{1.1}$$

where B is a Brownian motion, $A = (A_t)_{t \geq 0}$ is a time change (that is, a real-valued nondecreasing process) and the pair (A, H) is independent of B. Notice that the right-hand side of (1.1) with $A_t = t$ corresponds to the class of Stochastic Volatility models without drift, and that when A is deterministic, $B \circ A$ is a Gaussian process.

Part of the motivation for the present paper lies in the fact that the very fruitful idea of subordination of one-dimensional Brownian motion does not extend in general to multivariate settings. For the case of vector Lévy processes this has been discussed in detail in Barndorff-Nielsen et al. (2001). Another case of interest is whether it is possible to construct a d-dimensional S_+^d-parameter Brownian motion $B = (B_\sigma)_{\sigma \in S_+^d}$ (see Subsection 1.1 for the precise definition of this) and a process Σ, as above, such that $X = B \circ \Sigma$ is a Lévy process. This question was answered in the negative by Pedersen and Sato (2004). One aspect of the discussion below is that, nevertheless, the question does in a certain sense have a positive answer.

The paper is organised as follows. In the next subsection we discuss representation of general processes with conditionally independent increments; in particular we give conditions under which such a process can be represented as a time change of a Lévy process, and we discuss nonexistence of the S_+^d-parameter Brownian motion. Section 3 contains various fundamental properties of conditionally Gaussian processes with independent increments; in Section 4 we specialise to the case of integrated Gaussian processes, and finally Section 5 contains some examples.

1.1 Background on CPII's

The class of processes with conditionally independent increments (CPII's) is treated in detail in Jacod and Shiryaev (2003, II.6-II.7). In particular, they consider the semimartingale property, even in cases where one has so-called progressive CPII's, and they give a nice representation of progressive CPII's, see e.g. their Proposition II.7.12. The problem we shall address in the present paper is related to the latter in the sense that we discuss how to represent a CPII in a convenient way, in particular when the conditional increments are Gaussian. It should be noted, however, that our representation is not directly related to Jacod and Shiryaev (2003, Proposition II.7.12) since we do not treat the 'progressive' case. In addition, we focus on defining a representation in terms of the operations *time change* and *integration* of a process with independent increments.

The following definitions and results are based on Pedersen and Sato (2004) which we refer to for more information and further results.

Let $K \neq \{0\}$ be a cone in $\mathbb{R}^k$, i.e., it is a non-empty closed convex set closed under multiplication by nonnegative reals and containing no straight line through 0. Let $(\mu_s)_{s \in K}$ denote a K-parameter convolution semigroup on $\mathbb{R}^d$; that is, for all $s \in K$, μ_s is a probability measure on $\mathbb{R}^d$ and $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$ for all $s^1, s^2 \in K$, and $\mu_0 = \delta_0$, the Dirac measure at 0. Moreover, if $s^n \to s$ in K then $\mu_{s^n} \xrightarrow{\mathcal{D}} \mu_s$. Finally, let $\Sigma = (\Sigma_t)_{t\geq 0}$ denote a K-valued càdlàg nondecreasing process, where 'nondecreasing' means that for all $0 \leq t_1 \leq t_2$ we have $\Sigma_{t_2} - \Sigma_{t_1} \in K$.

Consider an $\mathbb{R}^d$-valued càdlàg process $X = (X_t)_{t\geq 0}$ which satisfies that, given Σ, X has independent increments with $\mathcal{L}(X_{t_2} - X_{t_1} \mid \Sigma) = \mu_{\Sigma_{t_2}-\Sigma_{t_1}}$ for all $0 \leq t_1 \leq t_2$. We say that X is a CPII associated with $(\mu_s)_{s\in K}$. Notice that since the conditional distribution of X given Σ is specified consistently, the pair (X, Σ) does exist, at least on an enlargement of the original probability space.

When seeking for a representation of X it turns out to be important to distinguish between the two cases where the convolution semigroup is generative resp. non-generative. Here we recall that $(\mu_s)_{s\in K}$ is generative if there exists an $\mathbb{R}^d$-valued K-parameter Lévy process $Z = (Z_s)_{s\in K}$ (by definition, this means that Z has stationary independent increments along K-increasing sequences and is K-càdlàg) such that $\mathcal{L}(X_s) = \mu_s$ for all $s \in K$. We say in this case that Z is associated with $(\mu_s)_{s\in K}$.

Most convolution semigroups are generative; indeed this is the case if one of the following conditions are satisfied.

(i) $K = \mathbb{R}_+$; in this case let $\mu = \mu_1$; then it is well-known that for all $s \in K = \mathbb{R}_+$ we have $\mu_s = \mu^s$.

(ii) $K = \mathbb{R}_+^k$ or, more generally, K has a strong basis. (See Pedersen and Sato (2004) for the definition of the latter.)

(iii) $d = 1$.

(iv) For all $s \in K$, μ_s does not have a Gaussian component.

In relation to (iv) we recall that since μ_s is infinitely divisible there exists a triplet (A_s, ν_s, γ_s), the so-called characteristic triplet, consisting of a nonnegative definite $d \times d$ matrix A_s, a Lévy measure ν_s on $\mathbb{R}^d$ and a constant $\gamma_s \in \mathbb{R}^d$, such that the characteristic function of μ_s is given by

$$\widehat{\mu}_s(z) = \exp\Big[-\tfrac{1}{2} z^* A_s z + \mathrm{i}\,\gamma_s^* z + \int_{\mathbb{R}^d} \big(e^{\mathrm{i}\,z^* x} - 1 - \mathrm{i}\,z^* x\, 1_D(x)\big)\,\nu_s(\mathrm{d}x)\Big], \quad z \in \mathbb{R}^d,$$

where $D = \{x \in \mathbb{R}^d \mid |x| \leq 1\}$. Then μ_s has no Gaussian component if $A_s = 0$.

We emphasize that generally the law of an $\mathbb{R}^d$-valued K-parameter Lévy process $Z = (Z_s)_{s\in K}$ associated with $(\mu_s)_{s\in K}$ is not uniquely determined by the semigroup unless we are in special cases such as in (i) above.

Assume $(\mu_s)_{s\in K}$ is generative and let $Z = (Z_s)_{s\in K}$ be a measurable K-parameter Lévy process associated with this semigroup which is independent of Σ. Then it is not hard to see that (X, Σ) has the representation

$$(X, \Sigma) \stackrel{\mathcal{D}}{=} (Z \circ \Sigma_t, \Sigma_t)_{t\geq 0}. \tag{1.2}$$

(Here $Z \circ \Sigma_t := Z_{\Sigma_t}$.) That is, X appears in law as a time change of Z. Notice that if $K = \mathbb{R}_+$ and Σ is a subordinator, then X appears in law by subordination of Z by Σ; similarly, when K is an arbitrary cone and $\Sigma = (\Sigma_t)_{t\geq 0}$ is a K-valued Lévy process, then X appears, in law, by subordination of Z by Σ, and in this case X is a Lévy process as well.

Although many convolution semigroups are generative, non-generative semigroups do exist; the most interesting example is the so-called *canonical S_+^d-parameter convolution semigroup*. In this case $K = S_+^d$ and μ_s is the d-dimensional normal distribution on $\mathbb{R}^d$ with mean 0 and variance s for all $s \in S_+^d$. It would be appropriate to call an $\mathbb{R}^d$-valued S_+^d-parameter Lévy process associated with the canonical S_+^d-parameter convolution semigroup an *S_+^d-parameter Brownian motion*. However, when $d \geq 2$, the canonical S_+^d-parameter convolution semigroup is non-generative, cf. Pedersen and Sato (2004, Theorem 4.1) and hence no such Brownian motion exists.

Notice that if X is a CGPII associated with the canonical S_+^d-parameter convolution semigroup, then $(X_{t_2} - X_{t_1} \mid \Sigma) \sim N_d(0, \Sigma_{t_2} - \Sigma_{t_1})$ for all $0 \leq t_1 < t_2$. Clearly, since this semigroup is non-generative when $d \geq 2$, there is no S_+^d-parameter Lévy process associated with it, and hence X is not representable as in (1.2). However, in the present paper we show that X has a convenient representation, which involves an integral as well as a time change. More generally we derive various fundamental properties of X.

In the remainder of the paper the only cone to be considered is $K = S_+^d$.

2 Preliminaries

Throughout vectors are column vectors unless otherwise stated; superscript $*$ denotes transposition. Let d and k be positive integers. Let $(\Omega, \mathcal{F}, P)$ denote a probability space on which all random variables in the following are defined. We use $\stackrel{\mathcal{D}}{=}$ to denote identity in law of random variables and stochastic processes. The law of a random variable X is denoted $\mathcal{L}(X)$.

When μ is a distribution on $\mathbb{R}^d$, $\widehat{\mu}$ denotes, as always, the characteristic function; $\widehat{\mu}(z) = \int_{\mathbb{R}^d} \exp(\mathrm{i}\, z \cdot x)\, \mu(\mathrm{d}x)$ for $z \in \mathbb{R}^d$.

A real-valued càdlàg process $A = (A_t)_{t \geq 0}$ is called an increasing (or a nondecreasing) process if $t \mapsto A_t(\omega)$ is nondecreasing with $A_0(\omega) = 0$ for all $\omega \in \Omega$. Denote the set of increasing processes by $\mathcal{A}$. Notice that $\mathcal{A}$ contains all deterministic real-valued càdlàg functions $a = (a_t)_{t \geq 0}$ for which $a_0 = 0$ and $t \mapsto a_t$ is nondecreasing.

Let $A \in \mathcal{A}$ and let $Y = (Y_t)_{t \geq 0}$ be a real-valued measurable process. We say that Y is locally integrable with respect to A (or locally A-integrable) if $\int_0^t |Y_s| \,\mathrm{d}A_s < \infty$ a.s. for all $t > 0$. (If Y and A are deterministic, Y is locally integrable with respect to A if $\int_0^t |Y_s| \,\mathrm{d}A_s < \infty$ for all $t > 0$.)

When $X = (X_t)_{t \geq 0}$ is a d-dimensional càdlàg process and $A \in \mathcal{A}$ we use the notation $X \circ A = (X \circ A_t)_{t \geq 0}$ to denote the d-dimensional time changed process $X \circ A_t := X_{A_t}$.

As above, let S_+^d denote the set of symmetric nonnegative definite $d \times d$ matrices. An S_+^d-valued function $\nu = (\nu_t)_{t \geq 0}$ is said to be nondecreasing if $\nu_t - \nu_s \in S_+^d$ for all $0 \leq s < t$.

Let Y be a random element from Ω into some space $\mathcal{Y}$. Occasionally we need a regular conditional probability of P given Y, $P(A \mid y)$, defined for $A \in \mathcal{F}$ and $y \in \mathcal{Y}$. We shall always assume that $P(\cdot \mid \cdot)$ exists and satisfies the 'useful rule' $P(Y = y \mid y) = 1$ for all y. When U is a random variable, $U \mid Y$ signifies that we work under the probability measure $P(\cdot \mid Y)$.

Let $Y = (Y_t)_{t\geq 0}$ be a càdlàg $\mathbb{R}^k$-valued stochastic process with $Y_0 = 0$. We say that Y has *independent increments* if, for all $n \geq 1$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the variables $Y_{t_1}, Y_{t_2} - Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}}$ are independent; Y has *stationary increments* if, for all $n \geq 1$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, we have

$$(Y^*_{t_2} - Y^*_{t_1}, \ldots, Y^*_{t_n} - Y^*_{t_{n-1}})^* \stackrel{\mathcal{D}}{=} (Y^*_{t_2 - t_1}, \ldots, Y^*_{t_n - t_1} - Y^*_{t_{n-1}-t_1})^*.$$

For $Y = (Y_t)_{t\geq 0}$, an $\mathbb{R}^d$ or S^d_+-valued càdlàg process or function, let

$$D(Y) := \{t > 0 \mid \Delta Y_t \neq 0\} \tag{2.1}$$

be the jump times of Y.

We shall consider S^d_+ as a closed subset of $\mathbb{R}^{d\times d}$ equipped with the inherited topology.

2.1 Representation of nondecreasing S^d_+-valued functions

Let ν denote a nondecreasing S^d_+-valued càdlàg function with $\nu_0 = 0 \in S^d_+$. Denote by ν^{ij}_t the (i, j)th entry of ν_t for $i, j = 1, \ldots, d$. It is then easily seen that for all i and j the mapping $t \mapsto \nu^{ij}_t$ is of bounded variation and that $t \mapsto \nu^{ii}_t$ is nondecreasing. Moreover, the (signed) measure induced by ν^{ij} is dominated by the measure induced by ν^{ii}. Thus, there exists a deterministic $a \in \mathcal{A}$ (take e.g. $a = \nu^{11} + \cdots + \nu^{dd}$) and $\phi_t \in S^d_+$ defined for $t \geq 0$ such that ϕ is locally integrable with respect to a (in the sense that all entries of ϕ are locally a-integrable) and

$$\nu_t = \int_0^t \phi_s \, \mathrm{d}a_s \quad \text{for all } t \geq 0. \tag{2.2}$$

Notice that in this equation we can choose a as the Lebesgue measure if and only if $\nu^{11}, \ldots, \nu^{dd}$ are absolutely continuous with respect to the Lebesgue measure.

2.2 Time changed Brownian motion

Let $B = (B_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion; that is, B is given as $B = (B^1, \ldots, B^d)$ where the coordinate processes $B^1, \ldots, B^d$ are independent and B^i, for $i = 1, \ldots, d$, is a one-dimensional Brownian motion , i.e. a continuous process with stationary independent increments such that $B^i(t) \sim N(0, t)$ for all $t \geq 0$. Moreover, let $a \in \mathcal{A}$ be deterministic.

The process $B \circ a$ is a deterministic time change of B, so in particular it is a k-dimensional càdlàg Gaussian process with independent increments and thus a k-dimensional square integrable martingale in its own filtration.

Let $c : \mathbb{R}_+ \to \mathbb{R}^d$ be measurable and assume $\|c\|^2$ is locally a-integrable. Then the integral

$$\int_0^t c^*_s \, \mathrm{d}(B \circ a)_s, \quad t \geq 0,$$

exists and is càdlàg as a function of t; moreover, the independent increments are inherited from $B \circ a$ to the integral since c is deterministic, and

$$\int_0^t c_s^* \,\mathrm{d}(B \circ a)_s \sim N(0, \int_0^t \|c_s\|^2 \,\mathrm{d}a_s) \quad \text{for all } t \geq 0. \tag{2.3}$$

Generalising this, let $C : \mathbb{R}_+ \to \mathbb{R}^{d \times k}$ be measurable and let C^*C be locally a-integrable. The k-dimensional integral $\int_0^t C_s^* \,\mathrm{d}(B \circ a)_s$, which is defined coordinate-by-coordinate, then exists, and is càdlàg with independent increments such that

$$\int_0^t C_s^* \,\mathrm{d}(B \circ a)_s \sim N_k(0, \int_0^t C_s^* C_s \,\mathrm{d}a_s) \quad \text{for all } t \geq 0. \tag{2.4}$$

2.3 Representation of càdlàg Gaussian processes with independent increments

A d-dimensional process $G = (G_t)_{t \geq 0}$ is called a *Gaussian process with independent increments* (in short: a GPII$_d$) if $G_0 = 0$, G is càdlàg and has independent increments, and $G_t - G_s$ follows a d-dimensional zero-mean Gaussian distribution for all $0 \leq s \leq t$. GPII's are treated in detail in, e.g., Jacod and Shiryaev (2003, II.4d).

Let G denote a GPII$_d$ and define $\nu_t := \mathrm{Var}(G_t)$, the $d \times d$ variance matrix of G_t. We say that G is GPII$_d$-ν in order to emphasize the variance matrix. Notice that the law of a GPII$_d$-ν is uniquely determined by ν, and $\nu_0 = 0$ (where the right-hand side denotes the $d \times d$ null matrix). It is readily verified that $t \mapsto \nu_t$ is càdlàg and $\nu_t - \nu_s$ is nonnegative definite for all $0 \leq s < t$ since $\nu_t - \nu_s = \mathrm{Var}(G_t - G_s)$; that is, ν is nondecreasing in S_+^d.

The following result shows that the class of Gaussian process with independent increments corresponds to integrals of a time changed Brownian motion, where the integrand as well as the time change are deterministic. More precisely, we can summarise the findings of the preceding subsections as follows.

As always, $[X]$ denotes the quadratic variation of a semimartingale X and $\langle Y \rangle$ is the sharp bracket of a locally square integrable martingale Y, cf. Jacod and Shiryaev (2003, I.4).

Proposition 2.1. *Let B denote a d-dimensional standard Brownian motion.*

(i) Let $C : \mathbb{R}_+ \to \mathbb{R}^{d \times k}$ be measurable and $a \in \mathcal{A}$ be deterministic. Assume that C^*C is locally a-integrable. Then the k-dimensional process

$$\int_0^t C_s^* \,\mathrm{d}(B \circ a)_s, \quad t \geq 0,$$

is a GPII$_k$-ν process with ν given by $\nu_t = \int_0^t C_s^* C_s \,\mathrm{d}a_s$ for $t \geq 0$.

(ii) Let $\nu = (\nu_t)_{t \geq 0}$ be an S_+^d-valued nondecreasing càdlàg function with $\nu_0 = 0$. Decompose ν as in (2.2), that is $\mathrm{d}\nu_t = \phi_t \mathrm{d}a_t$, where ϕ is locally a-integrable, and let $\phi_t^{1/2} \in S_+^d$ denote the $d \times d$ nonnegative definite square root of ϕ_t. The d-dimensional process

$$\int_0^t \phi_s^{1/2} \,\mathrm{d}(B \circ a)_s, \quad t \geq 0,$$

is then a GPII$_d$-ν.

(iii) Let $\nu = (\nu_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg function with $\nu_0 = 0$. Let $G = (G_t)_{t\geq 0}$ be a GPII$_d$-ν. Then G is a square integrable $((\mathcal{F}_t^G), P)$-martingale, and

$$\begin{aligned} D(G) &= D(\nu) \quad P-a.s. \\ \langle G\rangle_t &= \nu_t \\ [G]_t &= \nu_t^c + \sum_{0<s\leq t} (\Delta G_s)(\Delta G_s)^*, \end{aligned}$$

where $\nu_t^c := \nu_t - \sum_{0<s\leq t}\Delta\nu_s$ is the continuous part of ν, and $D(G)$ and $D(\nu)$ are defined in (2.1).

Proof. (i) Follows immediately from Subsection 2.2, and (ii) follows from (i) and (2.2) since $\phi_t^{1/2}\phi_t^{1/2} = \phi_t$. The final part follows Jacod and Shiryaev (2003, II.4.36) (alternatively, use the representation of G in (ii)). □

Let G denote a GPII$_d$-ν and represent G as in (ii) above; that is,

$$G_t = \int_0^t \phi_s^{1/2}\,\mathrm{d}(B\circ a)_s \quad \text{for all } t\geq 0, \tag{2.5}$$

where B is a d-dimensional standard Brownian motion and (a,ϕ) are defined from ν by (2.2). When $C:\mathbb{R}_+\to\mathbb{R}^{d\times k}$ is measurable and $C^*\phi C$ is locally a-integrable, the integral

$$\int_0^t C_s^*\,\mathrm{d}G_s = \int_0^t C_s^*\phi_s^{1/2}\,\mathrm{d}(B\circ a)_s \tag{2.6}$$

exists, is càdlàg with independent increments, and

$$\int_0^t C_s^*\,\mathrm{d}G_s \sim N(0, \int_0^t C_s^*\phi_s C_s\,\mathrm{d}a_s) \quad \text{for all } t\geq 0.$$

That is, (2.6) is a GPII$_k$.

3 Conditionally Gaussian processes with independent increments

Let $\Sigma = (\Sigma_t)_{t\geq 0}$. be an S_+^d-valued càdlàg nondecreasing process with $\Sigma_0 = 0$. A d-dimensional càdlàg process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ is called a *conditionally Gaussian process with independent increments and conditional variance* Σ (short: A CGPII$_d$-Σ) if $X \mid \Sigma$ is a GPII$_d$-Σ.

The next result gives a few fundamental properties but first we need some notation. For $\Gamma\in S_+^d$ let μ_Γ denote the d-dimensional Gaussian distribution with mean 0 and variance Γ on $\mathbb{R}^d$; that is, μ_Γ is the probability measure on $\mathbb{R}^d$ with

$$\widehat{\mu}_\Gamma(z) = \exp\left(-\tfrac{1}{2}z^*\Gamma z\right), \quad \text{for } z\in\mathbb{R}^d.$$

The set $\{\mu_\Gamma \mid \Gamma \in S_+^d\}$ is the canonical S_+^d-parameter convolution semigroup, cf. Pedersen and Sato (2004) and Subsection 1.1, and a d-dimensional càdlàg process X with $X_0 = 0$ is a CGPII$_d$-Σ if and only if $X \mid \Sigma$ has independent increments with $\mathcal{L}(X_t - X_s \mid \Sigma) = \mu_{\Sigma_t - \Sigma_s}$ for all $0 \le s < t$.

Let $M_b(\mathbb{R}^d)$ resp. $M_b(S_+^d)$ denote the set of bounded and measurable functions from $\mathbb{R}^d$ resp. S_+^d to $\mathbb{R}$. For $f \in M_b(\mathbb{R}^d)$ define $\bar{f} : S_+^d \to \mathbb{R}$ by

$$\bar{f}(\Gamma) := \int_{\mathbb{R}^d} f(z)\,\mu_\Gamma(\mathrm{d}z), \quad \text{for } \Gamma \in S_+^d.$$

For f continuous, $\bar{f}$ is continuous as well since $\Gamma_n \to \Gamma$ implies $\mu_{\Gamma_n} \xrightarrow{\mathcal{D}} \mu_\Gamma$. By the Monotone Class Lemma (see e.g. Rogers and Williams (2000, II.3)) it follows that $\bar{f} \in M_b(S_+^d)$ for all $f \in M_b(\mathbb{R}^d)$.

Proposition 3.1. *Let $\Sigma = (\Sigma_t)_{t\ge 0}$ denote a nondecreasing càdlàg S_+^d-valued process with $\Sigma_0 = 0$. Let $X = (X_t)_{t\ge 0}$ denote an $\mathbb{R}^d$-valued càdlàg process with $X_0 = 0$.*

(i) X is a CGPII$_d$-Σ if and only if

$$\begin{cases} \text{for all } n \ge 1, 0 = t_0 < t_1 < \ldots < t_n, \\ f_j \in M_b(\mathbb{R}^d) \text{ and } g_j \in M_b(S_+^d), j = 1, \ldots, n, \text{ we have} \\ E\left[\prod_{j=1}^n f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\right] \\ = E\left[\prod_{j=1}^n \bar{f}_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\right]. \end{cases} \tag{3.1}$$

(ii) Let X be a CGPII$_d$-Σ. If Σ has stationary increments then so has (X, Σ).

(iii) Let X be a CGPII$_d$-Σ. If Σ has independent increments then so has (X, Σ).

Proof. (i) Assume X is a CGPII$_d$-Σ. Notice that since $X \mid \Sigma$ is a GPII$_d$-Σ we have, for n, t_j, f_j and g_j as in (3.1),

$$\begin{aligned} &E\Big[\prod_{j=1}^n f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) \mid \Sigma\Big] \\ &\quad = \prod_{j=1}^n g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) E\Big[\prod_{j=1}^n f_j(X_{t_j} - X_{t_{j-1}}) \mid \Sigma\Big] \\ &\quad = \prod_{j=1}^n g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) \bar{f}_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}). \end{aligned}$$

Therefore,

$$\begin{aligned} &E\Big[\prod_{j=1}^n f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big] \\ &\quad = E\Big[E\Big[\prod_{j=1}^n f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) \mid \Sigma\Big]\Big] \\ &\quad = E\Big[\prod_{j=1}^n \bar{f}_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big], \end{aligned}$$

which is the only if conclusion in (3.1).

Assume conversely that (3.1) is satisfied. The finite dimensional marginals of $(X, \Sigma) = (X_t, \Sigma_t)_{t\geq 0}$ are uniquely determined by this equation and hence so is the law of $(X, \Sigma) = (X_t, \Sigma_t)_{t\geq 0}$. The first part of the proof shows that (X, Σ) has the same law as $(\widetilde{X}, \Sigma)$, where $\widetilde{X}$ is a CGPII$_d$-Σ. Since the joint distribution determines the conditional distribution uniquely almost surely, it follows that $\mathcal{L}(X \mid \Sigma) = \mathcal{L}(\widetilde{X} \mid \Sigma)$, that is, $X \mid \Sigma$ is a GPII$_d$-Σ; by definition, X is therefore a CGPII$_d$-Σ.

(ii) Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. For $f_j \in M_b(\mathbb{R}^d)$ and $g_j \in M_b(S_+^d)$, $j = 1, \ldots, n$, we then have, by (i),

$$
\begin{aligned}
E\Big[\prod_{j=1}^{n} & f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big] \\
&= E\Big[\prod_{j=1}^{n} \bar{f}_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big] \\
&= E\Big[\prod_{j=1}^{n} \bar{f}_j(\Sigma_{t_j - t_1} - \Sigma_{t_{j-1} - t_1}) g_j(\Sigma_{t_j - t_1} - \Sigma_{t_{j-1} - t_1})\Big] \\
&= E\Big[\prod_{j=1}^{n} f_j(X_{t_j - t_1} - X_{t_{j-1} - t_1}) g_j(\Sigma_{t_j - t_1} - \Sigma_{t_{j-1} - t_1})\Big],
\end{aligned}
$$

where the second equality is due to the stationary increments of Σ.

(iii) Let $n \geq 1$ and $0 = t_0 < t_1 < \ldots < t_n$. For $f_j \in M_b(\mathbb{R}^d)$ and $g_j \in M_b(S_+^d)$, $j = 1, \ldots, n$, we then have, by (i) and the independent increments in Σ, that

$$
\begin{aligned}
E\Big[\prod_{j=1}^{n} & f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big] \\
&= \prod_{i=1}^{n} E\Big[\bar{f}_j(\Sigma_{t_j} - \Sigma_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big] \\
&= \prod_{i=1}^{n} E\Big[f_j(X_{t_j} - X_{t_{j-1}}) g_j(\Sigma_{t_j} - \Sigma_{t_{j-1}})\Big]
\end{aligned}
$$

and hence the result. □

The (semi)martingale property of CGPII's is treated in section II.6 of Jacod and Shiryaev (2003). Basically, the idea is that the martingale property of X under P is implied by martingale property under $P(\cdot \mid \Sigma)$ (at least if X is suitably bounded or integrable); in particular, their Lemma II.6.14 gives the following.

Theorem 3.2. *Let Σ be as above and X denote a CGPII$_d$-Σ. Let $\mathcal{H}_t := \sigma(\Sigma) \vee \mathcal{F}_t^X$ for $t \geq 0$, where $\sigma(\Sigma)$ is the σ-algebra generated by the process Σ.*

(i) The process X is a d-dimensional locally square integrable $((\mathcal{H}_t), P)$-martingale.

(ii) We have

$$\begin{aligned} D(X) &= D(\Sigma) \quad P-a.s. \\ \langle X\rangle_t &= \Sigma_t \\ [X]_t &= \Sigma_t^c + \sum_{0<s\le t} (\Delta X_s)(\Delta X_s)^* \end{aligned}$$

where $D(X)$ and $D(\nu)$ are defined in (2.1), $\Sigma_t^c := \Sigma_t - \sum_{0<s\le t}\Delta\Sigma_s$ is the continuous component of Σ, and $\langle X\rangle$ is calculated with respect to $(\mathcal{H}_t)_{t\ge 0}$.

(iii) If Σ is a continuous process then X is a continuous d-dimensional local $((\mathcal{F}_t^X), P)$-martingale.

(iv) If Σ is an integrable process (by which we mean that $E[\Sigma_t^{11}+\cdots+\Sigma_t^{dd}]<\infty$ for all $t>0$) then X is a d-dimensional square integrable $((\mathcal{F}_t^X), P)$-martingale.

Proof. (i) and (ii): First we prove (i) and that $\langle X\rangle = \Sigma$.

Notice that since Σ is $\mathcal{H}_0$-measurable and càdlàg it follows that this process is $(\mathcal{H}_t)$-predictable. Let $(\tau_n)_{n\ge 1}$ denote a localising sequence with respect to $(\mathcal{H}_t)_{t\ge 0}$ which is chosen such that

$$E[\Sigma_{t\wedge\tau_n}^{11} + \cdots + \Sigma_{t\wedge\tau_n}^{dd}] < \infty \text{ for all } t \ge 0 \text{ and } n \ge 1. \tag{3.2}$$

(Notice that such a sequence exists since Σ is $\mathcal{H}_0$-measurable.) Define the $d\times d$ matrix process $U=(U_t)_{t\ge 0}$ by

$$U := (XX^*) - \Sigma.$$

It then suffices to show that the stopped processes X^{τ_n} and U^{τ_n} are, respectively, a d-dimensional and a $d\times d$-dimensional $((\mathcal{H}_t), P)$-martingale for all n.

Let $P(\cdot \mid \cdot)$ denote a regular conditional probability of P given the process Σ. Since X is a GPII$_d$-Γ under $P(\cdot\mid\Gamma)$ for all nondecreasing S_+^d-valued càdlàg functions $\Gamma = (\Gamma_t)_{t\ge 0}$ with $\Gamma_0 = 0$, Proposition 2.1 (iii) implies that X is a d-dimensional square integrable $((\mathcal{F}_t^X), P(\cdot\mid\Gamma))$-martingale and $XX^*-\Gamma$ is a $d\times d$-dimensional $((\mathcal{F}_t^X), P(\cdot\mid\Gamma))$-martingale. Since $\Sigma = \Gamma$ $P(\cdot\mid\Gamma)$-almost surely it follows that X is a d-dimensional square integrable $((\mathcal{H}_t), P(\cdot\mid\Gamma))$-martingale and U is a $d\times d$-dimensional $((\mathcal{H}_t), P(\cdot\mid\Gamma))$-martingale. In particular, the stopped processes X^{τ_n} and U^{τ_n} are respectively a d-dimensional square integrable $((\mathcal{H}_t), P(\cdot\mid\Gamma))$-martingale and a $d\times d$-dimensional $((\mathcal{H}_t), P(\cdot\mid\Gamma))$-martingale. Thus,

$$E[\|X_t^{\tau_n}\|^2 \mid \Sigma] = \Sigma_{t\wedge\tau_n}^{11} + \cdots + \Sigma_{t\wedge\tau_n}^{dd} \text{ for all } t\ge 0 \text{ and } n\ge 1,$$

which, by (3.2), implies

$$E[\|X_t^{\tau_n}\|^2] < \infty \text{ for all } t \ge 0 \text{ and } n \ge 1.$$

As the stopped processes X^{τ_n} and U^{τ_n} are martingales under $P(\cdot\mid\Gamma)$ and integrable under P, a slight extension of Lemma II.6.14 in Jacod and Shiryaev (2003) shows that X^{τ_n} is a d-dimensional $((\mathcal{H}_t), P)$-martingale and U^{τ_n} a $d\times d$-dimensional $((\mathcal{H}_t), P)$- martingale.

As $X \mid \Sigma$ is a GPII_d-Σ, the first and the third statement in (ii) follow from Proposition 2.1 (iii).

(iii) If Σ is a continuous process, then so is X by (ii). In this case it is well-known that since X is a local $((\mathcal{H}_t), P)$-martingale, it is also a local martingale in a smaller filtration to which it is adapted.

(iii) In this case we can take $\tau_n = \infty$ in (3.2). Since X is a square integrable $((\mathcal{H}_t), P)$-martingale adapted to $(\mathcal{F}_t^X)$, X is a square integrable $((\mathcal{F}_t^X), P)$-martingale. □

Remark 3.3. Notice that by (i) above it follows that a CGPII_d-Σ X is an $((\mathcal{F}_t^X), P)$-semimartingale; interestingly, however, it seems that generally X is *not* a d-dimensional local $((\mathcal{F}_t^X), P)$-martingale.

Before stating the next result we consider integrals with respect to a time changed Brownian motion in the case when the time change is stochastic. Thus, let B denote a d-dimensional Brownian motion and $A \in \mathcal{A}$. Let $H = (H_t)_{t\geq 0}$ be an $\mathbb{R}^{d\times k}$-valued measurable process such that H^*H locally integrable with respect to A (in the sense that all entries are locally A-integrable); define

$$\Sigma_t := \int_0^t H_s^* H_s \,\mathrm{d}A_s, \quad t \geq 0, \tag{3.3}$$

and assume that (A, H) is independent of B. The integral

$$\int_0^t H_s^* \,\mathrm{d}(B \circ A)_s, \quad t \geq 0,$$

then exists and by Proposition 2.1, $(\int_0^t H_s \,\mathrm{d}(B \circ A)_s)_{t\geq 0} \mid (A, H)$ is a GPII_k-Σ.

Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be an arbitrary S_+^d-valued nondecreasing càdlàg process with $\Sigma_0 = 0$ (not necessarily given by (3.3)). We need to decompose Σ as in (2.2) which motivates the following. A pair (A, Φ) consisting of an increasing process $A \in \mathcal{A}$ and a measurable S_+^d-valued process $\Phi = (\Phi_t)_{t\geq 0}$ is a *decomposition* of Σ if $\sigma(A, \Phi) = \sigma(\Sigma)$, and Φ is locally A-integrable with

$$\Sigma_t = \int_0^t \Phi_s \,\mathrm{d}A_s \quad \text{for all } t \geq 0 \quad \text{almost surely.} \tag{3.4}$$

Notice that (a non-unique) decomposition of Σ always exists.

The following result characterises the class CGPII as the class of integrals with respect to a time changed Brownian motion where the integrand and time change are simultaneously independent of the Brownian motion.

Proposition 3.4. *We have the following.*

(i) Let B denote a d-dimensional standard Brownian and $A \in \mathcal{A}$; also let $H = (H_t)_{t\geq 0}$ be an $\mathbb{R}^{d\times k}$-valued measurable process for which H^*H is locally A-integrable; assume (A, H) is independent of B. Then, the k-dimensional integral

$$\int_0^t H_s^* \,\mathrm{d}(B \circ A)_s, \quad t \geq 0,$$

is a $CGPII_k$-Σ with $\Sigma_t = \int_0^t H_s^* H_s \,\mathrm{d}A_s$.

(ii) *Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg process with $\Sigma_0 = 0$. Let (A,Φ) be a decomposition of Σ and B denote a d-dimensional Brownian motion which is independent of Σ. Let $\Phi^{1/2} \in S_+^d$ denote the nonnegative definite square root of Φ. Then the d-dimensional integral*

$$\int_0^t \Phi_s^{1/2}\, \mathrm{d}(B\circ A)_s, \quad t\geq 0,$$

is a CGPII$_d$-Σ.

Proof. (i) As seen above $(\int_0^t H_s\, \mathrm{d}(B\circ A)_s)_{t\geq 0} \mid (A,H)$ is a GPII$_k$-Σ. Since the conditional distribution of $(\int_0^t H_s\, \mathrm{d}(B\circ A)_s)_{t\geq 0}$ given (A,H) depends on Σ only it follows that $(\int_0^t H_s\, \mathrm{d}(B\circ A)_s)_{t\geq 0} \mid \Sigma$ is a GPII$_k$-Σ as well. Thus, by definition, $(\int_0^t H_s\, \mathrm{d}(B\circ A)_s)_{t\geq 0}$ is a CGPII$_k$-Σ.

(ii) Since Σ is independent of B and (A,Φ) is $\sigma(\Sigma)$-measurable, (A,Φ) is independent of B; thus the result follows from (i) since $\Phi^{1/2}$ is symmetric with $\Phi = \Phi^{1/2}\Phi^{1/2}$. □

Remark 3.5. Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg process with $\Sigma_0 = 0$. Let us consider some cases where a CGPII$_d$-Σ $X = (X_t)_{t\geq 0}$ can be represented solely in terms of either an integrated or a time changed Brownian motion. Let $B = (B^1,\dots,B^d)$ be a d-dimensional Brownian motion independent of Σ.

(i) If there exists a representation (A,Φ) of Σ with $A_t = t$ (that is, all entries in Σ are absolutely continuous with respect to the Lebesgue measure), we have $B\circ A = B$, and in this case

$$X_t := \int_0^t \Phi_s^{1/2}\, \mathrm{d}B_s, \quad t\geq 0,$$

is a CGPII$_d$-Σ. Thus, no time change of the Brownian motion is needed to represent X.

(ii) When $d = 1$, Σ is in fact just a real-valued nondecreasing process; that is, $\Sigma \in \mathcal{A}$ and B is a one-dimensional Brownian motion. In this case we can take $\Phi_t = 1$ and $A_t = \Sigma_t$ as a decomposition of Σ, and then $X := B\circ\Sigma$ is a CGPII$_1$-Σ. That is, no integration is needed.

(iii) In certain special cases one can represent X in terms of time changes alone, even when $d > 1$. For example, assume that Σ is a diagonal matrix,

$$\Sigma_t = \mathrm{diag}(\Sigma_t^{11},\dots,\Sigma_t^{dd}), \quad t\geq 0. \tag{3.5}$$

Then the process $X := (B^1\circ\Sigma^{11},\dots,B^d\circ\Sigma^{dd})^*$ is a CGPII$_d$-Σ.

(iv) Motivated by (iii) it would be tempting to try and represent an arbitrary CGPII$_d$-Σ process as B_{Σ_t} where $B = \{B_s \mid s\in S_d^+\}$ should be an S_+^d-parameter Brownian motion. However, as shown by Pedersen and Sato (2004), see also subsection 1.1, such a process does not exist.

Proposition 3.6. *(Marginalisation and conditioning). Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg process with $\Sigma_0 = 0$ and suppose X is a CGPII$_d$-Σ. Let (A, Φ) be a decomposition of Σ.*

(i) Let $C : S_+^d \to \mathbb{R}^{d\times k}$ be measurable and assume that the process $C(\Sigma)^*\Phi C(\Sigma) := (C(\Sigma_t)^*\Phi_t C(\Sigma_t))_{t\geq 0}$ is locally A-integrable. Define the k-dimensional process $\widetilde{X}$ by

$$\widetilde{X}_t := \int_0^t C(\Sigma_s)^* \,\mathrm{d}X_s \quad \textit{for } t \geq 0$$

and let $\widetilde{\Sigma}$ be given by

$$\widetilde{\Sigma}_t := \int_0^t C(\Sigma_s)^*\Phi_s C(\Sigma_s)\,\mathrm{d}A_s \quad \textit{for } t \geq 0.$$

Then $\widetilde{X}$ is a CGPII$_k$-$\widetilde{\Sigma}$.

(ii) If $C \in \mathbb{R}^{d\times k}$ is a fixed $d \times k$ matrix, then C^*X is a CGPII$_k$-$\widetilde{\Sigma}$ with $\widetilde{\Sigma} = C^*\Sigma C$.

(iii) Decompose X as $X^* = ((X^1)^*, (X^2)^*)$, where $X^i = (X_t^i)_{t\geq 0}$ is d_i-dimensional for $i = 1, 2$, with $d_1 + d_2 = d$. Similarly, decompose Σ_t as

$$\Sigma_t = \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{21} & \Sigma_t^{22} \end{pmatrix}, \tag{3.6}$$

where, e.g., Σ^{11} is $d_1 \times d_1$. Then, X^1 is a CGPII$_{d_1}$-Σ^{11}.

(iv) Decompose X as in (iii) and Φ as

$$\Phi_t = \begin{pmatrix} \Phi_t^{11} & \Phi_t^{12} \\ \Phi_t^{21} & \Phi_t^{22} \end{pmatrix}, \tag{3.7}$$

where, e.g., Φ^{11} is $d_1 \times d_1$. Assume Φ^{22} is invertible. Define the d_1-dimensional process $X^{1\cdot 2}$ by

$$X_t^{1\cdot 2} := X_t^1 - \int_0^t \Phi_s^{12}(\Phi_s^{22})^{-1}\,\mathrm{d}X_s^2 \quad \textit{for } t \geq 0,$$

and let $\Sigma^{11\cdot 2}$ be given by

$$\Sigma_t^{11\cdot 2} := \int_0^t \Phi_s^{11} - \Phi_s^{12}(\Phi_s^{22})^{-1}\Phi_s^{21}\,\mathrm{d}A_s \quad \textit{for } t \geq 0. \tag{3.8}$$

Under $P(\cdot \mid \Sigma)$, the processes $X^{1\cdot 2}$ and X^2 are independent, and $X^{1\cdot 2} \mid \Sigma$ is a GPII$_{d_1}$-$\Sigma^{11\cdot 2}$. In particular $X^{1\cdot 2}$ is a CGPII$_{d_1}$-$\Sigma^{11\cdot 2}$ and so is $X^{1\cdot 2} \mid X^2$.

Proof. (i): As we know from Proposition 3.1 that the joint law of a CGPII$_d$-Σ and Σ is uniquely determined by Σ we may, by Proposition 3.4 (ii), assume that X is given by

$$X_t = \int_0^t \Phi_s^{1/2}\,\mathrm{d}(B \circ A)_s \quad \text{for } t \geq 0,$$

where B is a d-dimensional standard Brownian motion independent of Σ. We then have

$$\widetilde{X}_t = \int_0^t (\Phi_s^{1/2} C(\Sigma_s))^* \, \mathrm{d}(B \circ A)_s \quad \text{for } t \geq 0.$$

Since Σ is independent of B, and (A, Φ) is $\sigma(\Sigma)$-measurable, $(A, \Phi^{1/2} C(\Sigma))$ is independent of B so the result follows from Proposition 3.4 (i).

(ii) and (iii): (ii) is immediate from (i), and (iii) follows from (ii) with C being the matrix that picks out X^1 of X.

(iv): Represent X as in the proof of (i). Notice that

$$\begin{aligned}
\begin{pmatrix} X_t^{1\cdot 2} \\ X_t^2 \end{pmatrix} &= \int_0^t D_s \, \mathrm{d}X_s \\
&= \int_0^t D_s^* \Phi_s^{1/2} \, \mathrm{d}(B \circ A)_s \\
&= \int_0^t (\Phi_s^{1/2} D_s)^* \, \mathrm{d}(B \circ A)_s,
\end{aligned}$$

where

$$D_s^* = \begin{pmatrix} I_{d_1} & -\Phi_s^{12} (\Phi_s^{22})^{-1} \\ 0 & I_{d_2} \end{pmatrix},$$

0 denotes the null matrix of an appropriate dimension, and I_k is the $k \times k$ identity matrix. Leaving out details for the convenience of the reader it follows that

$$D_s^* \Phi_s D_s = \begin{pmatrix} \Phi_s^{11} - \Phi_s^{12} (\Phi_s^{22})^{-1} \Phi_s^{21} & 0 \\ 0 & \Phi_s^{22} \end{pmatrix}.$$

Due to Proposition 2.1 (i) we see that, under $P(\cdot \mid \Sigma)$, $X^{1\cdot 2}$ and X^2 are independent and are, respectively, a GPII_{d_1}-$\Sigma^{11\cdot 2}$ and a GPII_{d_2}-Σ^{22}. Thus, $X^{1\cdot 2} \mid (X^2, \Sigma)$ is a GPII_{d_1}-$\Sigma^{11\cdot 2}$ as well. Since this distribution depends on $\Sigma^{11\cdot 2}$ only, $X^{1\cdot 2} \mid (X^2, \Sigma^{11\cdot 2})$ is a GPII_{d_1}-$\Sigma^{11\cdot 2}$, that is, $X^{1\cdot 2} \mid X^2$ is a CGPII_{d_1}-$\Sigma^{11\cdot 2}$. Similarly, $X^{1\cdot 2} \mid \Sigma^{11\cdot 2}$ is a GPII_{d_1}-$\Sigma^{11\cdot 2}$ and hence $X^{1\cdot 2}$ is a CGPII_{d_1}-$\Sigma^{11\cdot 2}$. □

Example 3.7. Let $B^i = (B_t^i)_{t \geq 0}, i = 1, \ldots, 3$, denote three independent one-dimensional standard Brownian motions. Let $b \in \mathcal{A}$ be deterministic and have no continuous component; that is, $b_t = \sum_{0 < s \leq t} \Delta b_s$ for all $t \geq 0$. Finally, let $\Psi^i : [0, \infty[\to \mathbb{R}$ be a continuous and bounded function for $i = 1, 2$; assume $\Psi^2 \neq 0$.

Define the two-dimensional process $X = (X_t^1, X_t^2)_{t \geq 0}$ as

$$\begin{aligned}
X_t^1 &= B_t^1 + \int_0^t \Psi_s^1 \, \mathrm{d}(B^3 \circ b)_s \\
X_t^2 &= B_t^2 + \int_0^t \Psi_s^2 \, \mathrm{d}(B^3 \circ b)_s.
\end{aligned}$$

Then X is a GPII_2-Σ with Σ given by

$$\Sigma_t = \begin{pmatrix} t + \int_0^t (\Psi_s^1)^2 \, \mathrm{d}b_s & \int_0^t \Psi_s^1 \Psi_s^2 \, \mathrm{d}b_s \\ \int_0^t \Psi_s^1 \Psi_s^2 \, \mathrm{d}b_s & t + \int_0^t (\Psi_s^2)^2 \, \mathrm{d}b_s \end{pmatrix}, \quad t \geq 0.$$

In this case Σ has the decomposition (a, Φ) with $a_t = t + b_t$ and, with $D(b)$ defined in (2.1),

$$\Phi_t = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } t \notin D(b) \\ \begin{pmatrix} (\Psi_t^1)^2 & \Psi_t^1\Psi_t^2 \\ \Psi_t^1\Psi_t^2 & (\Psi_t^2)^2 \end{pmatrix} & \text{for } t \in D(b). \end{cases}$$

Thus,

$$\begin{aligned} \Sigma_t^{11\cdot 2} &= \int_0^t \Phi_s^{11} - \Phi_s^{12}(\Phi_s^{22})^{-1}\Phi_s^{21}\,\mathrm{d}a_s \\ &= \int_{]0,t]\cap D(b)} \Phi_s^{11} - \Phi_s^{12}(\Phi_s^{22})^{-1}\Phi_s^{21}\,\mathrm{d}a_s + \int_{]0,t]\cap D(b)^c} \Phi_s^{11} - \Phi_s^{12}(\Phi_s^{22})^{-1}\Phi_s^{21}\,\mathrm{d}a_s \\ &= \int_{]0,t]\cap D(b)} 0\,\mathrm{d}b_s + \int_{]0,t]\cap D(b)^c} 1\,\mathrm{d}s \\ &= t \quad \text{for } t \geq 0. \end{aligned}$$

Similar calculations show that $X_t^{1\cdot 2} = B_t^1$. From this follows that $X^{1\cdot 2} \mid X^2$ is a one-dimensional standard Brownian motion. Notice that even though Σ and X are discontinuous processes, $X^{1\cdot 2}$ is a continuous process.

Remark 3.8. Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg process with $\Sigma_0 = 0$ and $D = (D_t)_{t\geq 0}$ denote a d-dimensional càdlàg process. Extending the definition of CGPII$_d$'s to allow a 'drift' as well we consider the following. An $\mathbb{R}^d$ valued càdlàg process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ is called a CGPII$_d$-(D, Σ) if, given (D, Σ), the process $\widetilde{X} := X - D$ is a GPII$_d$-Σ. That is, given (D, Σ), X is a GPII plus a 'drift' D.

(i) The conclusion in Proposition 3.6 (iv) can be rephrased as follows with the notation and assumptions stated on page 13: The process $X^1 \mid X^2$ is a CGPII$_{d_1}$-$(D, \Sigma^{11\cdot 2})$ with D defined by

$$D_t = \int_0^t \Phi_s^{12}(\Phi_s^{22})^{-1}\,\mathrm{d}X_s^2, \quad t \geq 0.$$

(ii) More generally, many of the results in the present section have a counterpart in the 'drift' case as well.

4 Integrated GPII's

Let $\nu = (\nu_t)_{t\geq 0}$ be an S_+^d-valued nondecreasing càdlàg function with $\nu_0 = 0$ and G denote a GPII$_d$-ν. Let (a, ϕ) be a deterministic decomposition of ν. Let $H = (H_t)_{t\geq 0}$ be an $\mathbb{R}^{d\times k}$-valued measurable process independent of G satisfying that $H^*\phi H$ is locally a-integrable.

The k-dimensional process

$$X_t := \int_0^t H_s^* \, \mathrm{d}G_s \quad t \geq 0, \tag{4.1}$$

appears as an integral with respect to a Gaussian process with independent increments and is hence referred to as an *integrated GPII*, or an IGPII for short. By Proposition 2.1 we have

$$G \overset{\mathcal{D}}{=} \Big(\int_0^t \phi_s^{1/2} \, \mathrm{d}(B \circ a_s) \Big)_{t \geq 0} \tag{4.2}$$

where B is a d-dimensional standard Brownian motion independent of H; thus

$$X \overset{\mathcal{D}}{=} \Big(\int_0^t (\phi_s^{1/2} H_s)^* \, \mathrm{d}(B \circ a)_s \Big)_{t \geq 0}$$

which shows that an IGPII corresponds to an integral of a time changed standard Brownian motion, where the time change is deterministic. In particular Proposition 3.4 shows that X is a CGPII$_k$-Σ, with

$$\Sigma_t = \int_0^t H_s^* \phi_s H_s, \, \mathrm{d}a_s. \tag{4.3}$$

Notice that Σ has the decomposition $(a, H^* \phi H)$ where a is deterministic.

The latter property, that a is deterministic, characterises in fact the class of IGPII; to see this, let $\Sigma = (\Sigma_t)_{t \geq 0}$ be an arbitrary S_+^k-valued nondecreasing càdlàg process with $\Sigma_0 = 0$ and assume that Σ has a decomposition (a, Φ) with a deterministic. Let B denote a k-dimensional standard Brownian independent of Φ. By Proposition 3.4 it follows that the IGPII given by

$$\int_0^t \Phi_s^{1/2} \, \mathrm{d}(B \circ a)_s, \quad t \geq 0, \tag{4.4}$$

is a CGPII$_k$-Σ.

Let us summarise these findings.

Proposition 4.1. *We have the following.*

(i) Let X denote the IGPII (4.1) *with G an GPII$_d$-ν, where ν has decomposition (a, ϕ), and H is measurable and $\mathbb{R}^{d \times k}$-valued and independent of G. Then X is a CGPII$_k$-Σ with Σ given by* (4.3).

(ii) Let $\Sigma = (\Sigma_t)_{t \geq 0}$ be an arbitrary S_+^k-valued nondecreasing càdlàg process with $\Sigma_0 = 0$, and assume that Σ has a decomposition (a, Φ) with a deterministic. Then the IGPII (4.4)*, with B a k-dimensional standard Brownian motion independent of Φ, is a CGPII$_k$-Σ.*

When Σ has a decomposition (a, Φ) with a deterministic, we shall refer to a CGPII$_k$-Σ X as an IGPII$_k$-Σ, since in this case X can be represented as an integral of a GPII by the above result. Notice that in (4.1) the dimension, k, of the IGPII is not necessarily equal to d, the dimension of the background GPII. However, by (ii) we see that it is possible to find a representation with $k = d$.

Remark 4.2. Let us summarise how to find the marginal and conditional distributions in IGPII's.

(i) Let X be an IGPII$_k$-Σ. Decompose Σ as

$$\Sigma_t = \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{21} & \Sigma_t^{22} \end{pmatrix},$$

where, e.g., Σ^{11} is $k_1 \times k_1$. Then X^1 is an IGPII$_{k_1}$-Σ^{11}.

(ii) Let X be an IGPII-Σ and (a, Φ) be a decomposition of Σ with a deterministic. Decompose Φ as

$$\Phi_t = \begin{pmatrix} \Phi_t^{11} & \Phi_t^{12} \\ \Phi_t^{21} & \Phi_t^{22} \end{pmatrix},$$

where, e.g., Φ^{11} is $k_1 \times k_1$. Assume Φ^{22} is invertible. Define the k_1-dimensional process $X^{1\cdot 2}$ by

$$X_t^{1\cdot 2} := X_t^1 - \int_0^t \Phi_s^{12}(\Phi_s^{22})^{-1}\,\mathrm{d}X_s^2 \quad \text{for } t \geq 0,$$

and let $\Sigma^{11\cdot 2}$ be given by

$$\Sigma_t^{11\cdot 2} := \int_0^t \Phi_s^{11} - \Phi_s^{12}(\Phi_s^{22})^{-1}\Phi_s^{21}\,\mathrm{d}a_s \quad \text{for } t \geq 0.$$

According to Proposition 3.6 (iv), $X^{1\cdot 2} \mid X^2$ is an IGPII$_{k_1}$-$\Sigma^{11\cdot 2}$.

5 Examples

Example 5.1. Let $\Sigma_t = \int_0^t \Phi_s\,\mathrm{d}s$, where Φ is an S_+^d-valued measurable process. Then we say that an IGPII$_d$-Σ is a Stochastic Volatility Model with (squared) volatility Φ. Of particular interest is the case where Φ is a matrix OU process, as introduced in Barndorff-Nielsen and Stelzer (2007) and applied to multivariate stochastic volatility modelling by Pigorsch and Stelzer (2008b,a).

Notice that if X is a volatility model with squared volatility Φ then it can be represented as

$$X \overset{\mathcal{D}}{=} \left(\int_0^t H_s^*\,\mathrm{d}B\right)_{t\geq 0} \tag{5.1}$$

whenever $H = (H_t)_{t\geq 0}$ is a measurable $\mathbb{R}^{k\times d}$ process with $H^*H = \Phi$ independent of B, which is a k-dimensional standard Brownian motion. In particular we have

$$X \overset{\mathcal{D}}{=} \left(\int_0^t \Phi_s^{1/2}, \mathrm{d}\widetilde{B}_s\right)_{t\geq 0} \tag{5.2}$$

where $\Phi^{1/2}$ is the $d \times d$ nonnegative definite square root of Φ independent of $\widetilde{B}$ which is a d-dimensional standard Brownian motion.

The marginal distributions in SV models are given in Remark 4.2 (i) and the conditional distributions in Remark 4.2 (ii) with $a_t = t$.

Example 5.2. Suppose Σ is a matrix subordinator, that is Σ is càdlàg and S_+^d-valued with $\Sigma_0 = 0$, and has stationary and independent increments; cf. Barndorff-Nielsen and Pérez-Abreu (2007). The characteristic function of Σ_t is given by

$$\exp\left[\mathrm{i}\,\mathrm{tr}(\Sigma_t Z)\right] = \exp\Big[t\Big(\mathrm{i}\,\mathrm{tr}(\Gamma_0 Z) + \int_{S_+^d} (e^{\mathrm{i}\,\mathrm{tr}(UZ)} - 1)\,\rho(\mathrm{d}U)\Big)\Big], \quad Z \in S_+^d,$$

where tr denotes the trace of a matrix, $\Gamma_0 \in S_+^d$ is a constant and ρ is the Lévy measure on S_+^d, which satisfies

$$\int_{S_+^d} \min(1, \mathrm{tr}(U))\,\rho(\mathrm{d}U) < \infty.$$

Let (A, Φ) denote a representation of Σ. (Notice that if we choose

$$A = \Sigma^{11} + \cdots + \Sigma^{dd}, \tag{5.3}$$

then A is a one-dimensional subordinator).

Let X be a CGPII_d-Σ. By Proposition 3.1, X is then a Lévy process on $\mathbb{R}^d$, and by Proposition 3.4 we have the representation

$$X \stackrel{\mathcal{D}}{=} \Big(\int_0^t \Phi_s^{1/2}\,\mathrm{d}(B \circ A)_s\Big)_{t \geq 0} \tag{5.4}$$

where B is a d-dimensional standard Brownian motion independent of Σ. By Proposition 4.1, X is not an IGPII unless Σ is deterministic.

We emphasize that the representation (5.4) involves both a time change and an integral in contrast to the CPII's in Subsection 1.1 where, in the case of a generative convolution semigroup, we have the representation (1.2) defined solely in terms of a time change.

Example 5.3. A one-dimensional random variable Y is said to be of type G if it is of the form $Y \stackrel{\mathcal{D}}{=} \Psi U$ where $\Psi > 0$ and $U \sim N(0,1)$ are independent and Ψ^2 is infinitely divisible. The interest in this concept comes in particular from the fact that such a Y determines a Lévy process $X = (X_t)_{t\geq 0}$ for which $X_1 \stackrel{\mathcal{D}}{=} Y$ and $X_t = B \circ \Sigma_t$ where Σ denotes a one-dimensional subordinator having $\Sigma_1 \stackrel{\mathcal{D}}{=} \Psi^2$ and B is a one-dimensional Brownian motion independent of Σ. The definition of type G extends to d-dimensional random vectors Y with U being standard normal and $\Psi \in S_+^d$ and such that Ψ^2 is an infinitely divisible $d \times d$ matrix. This extension was discussed under the name mult$G$ in Barndorff-Nielsen and Pérez-Abreu (2002) (cf. also Barndorff-Nielsen et al. (2006)). Example 5.2 shows that also in this multivariate setting the Lévy process X determined by Y, which is given by (5.4) for a matrix subordinator Σ satisfying $\Sigma_1 \stackrel{\mathcal{D}}{=} \Psi^2$, has a convenient representation which, however, now involves integration in addition to a time change.

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