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Integrability of seminorms

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Abstract

We study integrability and equivalence of L^p-norms of polynomial chaos elements. Relying on known results for Banach space valued polynomials, a simple technique is presented to obtain integrability results for random elements that are not necessarily limits of Banach space valued polynomials. This enables us to prove integrability results for a large class of seminorms of stochastic processes and to answer, partially, a question raised by C. Borell (1979, Séminaire de Probabilités, XIII, 1–3).

Keywords: integrability; chaos processes; seminorms; regulary varying distributions

AMS Subject Classification: 60G17; 60B11; 60B12; 60E15

1 Introduction

Let T denote a countable set, $X = (X_t)_{t\in T}$ a stochastic process and N a seminorm on $\mathbb{R}^T$. This paper focuses on integrability and equivalence of L^p-norms of $N(X)$. Of particular interest is the supremum and the p-th variation norm given by

$$N(f) = \sup_{t\in T}|f(t)| \quad \text{and} \quad N(f) = \sup_{n\geq 1}\Big(\sum_{i=1}^{k_n}|f(t_i^n) - f(t_{i-1}^n)|^p\Big)^{1/p},\ p \geq 1, \tag{1.1}$$

for $f \in \mathbb{R}^T$. In the p-th variation case we assume moreover $T = [0,1] \cap \mathbb{Q}$ and $\pi_n = \{0 = t_0^n < \cdots < t_{k_n}^n = 1\}$ are nested subdivisions of T satisfying $\cup_{n=1}^{\infty}\pi_n = T$. Note that if N is given by (1.1), $B = \{x \in \mathbb{R}^T : N(x) < \infty\}$ and $\|x\| = N(x)$ for $x \in B$, then $(B, \|\cdot\|)$ is a non-separable Banach space when T is infinite.

1.1 Set-up

Let $(\Omega, \mathcal{F}, P)$ denote a probability space. For each $p > 0$ and real random variable X we let $\|X\|_p := E[|X|^p]^{1/p}$, which defines a norm when $p \geq 1$; moreover, let $\|X\|_\infty := \inf\{t \geq 0 : P(|X| \leq t) = 1\}$. When B is a Banach space, $L^p(P;B)$ denotes the space of all strongly measurable random elements, X, satisfying $\|X\|_{L^p(P;B)} =$

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$E[\|X\|^p]^{1/p} < \infty$. Throughout the paper I denotes a set, $\mathcal{H} = \{Z_t : t \in I\}$ a family of real-valued independent random variables, $d \geq 1$ a natural number and F a locally convex Hausdorff topological vector space (l.c.TVS) with dual space F^*. Following Fernique [9], a map N from F into $[0, \infty]$ is called a pseudo-seminorm if for all $x, y \in F$ and $\lambda \in \mathbb{R}$, we have

$$N(\lambda x) = |\lambda| N(x) \qquad \text{and} \qquad N(x+y) \leq N(x) + N(y).$$

An F-valued random element X is called a tetrahedral polynomial of order d if it is of the form

$$X = x_0 + \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1, \ldots, i_k} \prod_{j=1}^{k} Z_{t_{i_j}},$$

for some $n \geq 1$, $x_0, x_{i_1,\ldots,i_k} \in F$ and $t_1, \ldots, t_n$ different elements in I. $\mathcal{P}^d_{\mathcal{H}}$ denotes the set of all real-valued d-order tetrahedral polynomials and $\overline{\mathcal{P}}^d_{\mathcal{H}}$ its closure in probability. Inspired by Ledoux and Talagrand [17], and similar to Arcones and Giné [1] in the Gaussian case, we introduce the following definition:

Definition 1.1. An F-valued random element X is said to be a weak chaos element of order d if for all $n \geq 1$ and all $(x_i^*)_{i=1}^n \subseteq F^*$ there exists $(Y_i)_{i=1}^n \subseteq \overline{\mathcal{P}}^d_{\mathcal{H}}$ such that $(x_1^*(X), \ldots, x_n^*(X))$ equals $(Y_1, \ldots, Y_n)$ in distribution. The space of all F-valued weak polynomial chaos elements of order d is denoted *weak*-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$. Similarly, a real-valued stochastic process $(X_t)_{t \in T}$ is said to be a weak chaos process of order d if for all $n \geq 1$ and $(t_i)_{i=1}^n \subseteq T$ there exists $(Y_i)_{i=1}^n \subseteq \overline{\mathcal{P}}^d_{\mathcal{H}}$ such that $(X_{t_1}, \ldots, X_{t_n})$ equals $(Y_1, \ldots, Y_n)$ in distribution.

Weak chaos processes appear in the context of multiple integral processes, see e.g. Krakowiak and Szulga [15] for the α-stable case. Rademacher chaos processes are applied repeatedly when studying U-statistics, see de la Peña and Giné [8]. They are also used to study infinitely divisible chaos processes, see Marcus and Rosiński [18], Rosiński and Samorodnitsky [23], Basse and Pedersen [2] and others. Using the results of the present paper, A. Basse and S.-E. Graversen (2009, Chaos processes and semimartingales, work in progress) extend some well-known results on Gaussian semimartingales (see Jain and Monrad [12] and Stricker [26]) to a large class of chaos processes.

We shall need the following condition on $\mathcal{H}$, denoted C_q.

- For $q \in (0, \infty)$, C_q is said to be satisfied if there exists $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in (0, \infty)^4$ such that for all $s \geq \beta_1$ and $t \in I$ we have

$$P(|Z_t| > \beta_3) \geq \beta_4 \qquad \text{and} \qquad E[|Z_t|^q, |Z_t| > s] \leq \beta_2 s^q P(|Z_t| > s).$$

- C_∞ is said to be satisfied if $\mathcal{H} \subseteq L^1$ and

$$\beta := \sup_{t \in I} \Big(\frac{\|Z_t - E[Z_t]\|_\infty}{\|Z_t - E[Z_t]\|_2} \Big) < \infty.$$

Often the Z_t's will be identically distributed. Then $\mathcal{H}$ satisfies C_q for all $q \in (0, \alpha)$ for $\alpha > 0$ if $x \mapsto P(|Z_t| > x)$ is regulary varying with index $-\alpha$ or Z_t follows a Pareto-like distribution with index α (see Rosiński and Woyczyński [24]), that is

$$0 < \liminf_{x\to\infty} x^\alpha P(|Z_t| > x) \leq \limsup_{x\to\infty} x^\alpha P(|Z_t| > x) < \infty.$$

The regulary varying case follows by Karamata's Theorem (see [3, Theorem 1.5.11]). In particular, if the common distribution is symmetric α-stable for some $\alpha \in (0, 2)$ $\mathcal{H}$ satisfies C_q for all $q \in (0, \alpha)$. If the common distribution is Poisson, exponential, Gamma or Gaussian then C_q is satisfied for all $q > 0$. Finally $\mathcal{H}$ satisfies C_∞ if and only if the common distribution has compact support.

1.2 Integrability of seminorms

Let T denote a countable set, $X = (X_t)_{t\in T}$ a real-valued stochastic process and N a measurable pseudo-seminorm on $\mathbb{R}^T$ such that $N(X) < \infty$ a.s. For X Gaussian Fernique [9, Thèoréme 1.2.3] shows that $e^{\varepsilon N(X)^2}$ is integrable for some $\varepsilon > 0$. This result is extended to Gaussian chaos processes by Borell [4, Theorem 4.1]. When X is infinitely divisible Rosiński and Samorodnitsky [22, Lemma 2.2] provide conditions on the Lévy measure ensuring integrability of $e^{\varepsilon N(X)}$ for some $\varepsilon > 0$. Moreover, if X is α-stable for some $\alpha \in (0, 2)$, de Acosta [7, Theorem 3.2] shows that $N(X)^p$ is integrable for all $p < \alpha$. See also Hoffmann-Jørgensen [10] for further results.

Let X be a Rademacher process of the form $X_t = \sum_{n=1}^\infty x_n(t)Z_n$, where $x_n \in \mathbb{R}^T$ satisfies $\sum_{n=1}^\infty x_n(t)^2 < \infty$ for all $t \in T$ and $(Z_n)_{n\geq 1}$ is a Rademacher sequence, that is, a sequence of independent variables such that $P(Z_n = \pm 1) = 1/2$ for all $n \geq 1$. For some class of pseudo-seminorms N, including the supremum and the p-th variation norm, Ledoux and Talagrand [17, Theorem 4.8] shows that $e^{\varepsilon N(X)^2}$ is integrable for all $\varepsilon > 0$ if $N(x_n) < \infty$ for all $n \geq 1$. However, a symmetrization argument shows that the last assumption is always satisfied. Ledoux and Talagrand [17] also obtain a similar result for second-order Rademacher chaos processes. To the best of our knowledge, the literature does not contain general results on integrability of $N(X)$ for Rademacher chaos processes of order $d > 2$. However, this will be a special case of the results presented below.

Let $X = (X_t)_{t\in T}$ be a weak chaos process of order d as in Definition 1.1 and assume $\mathcal{H}$ satisfies C_q for some $q \in (0, \infty]$. Let N be a lower semicontinuous pseudo-seminorm on $\mathbb{R}^T$ satisfying $N(X) < \infty$ a.s. Examples are the supremum and the p-th variation norm. In this paper we show p-integrability and equivalence of L^p-norms of $N(X)$ for all finite $p \leq q$ and in the case $q = \infty$ that $e^{\varepsilon N(X)^{2/d}}$ is integrable for some $\varepsilon > 0$. In particular, we show integrability of $e^{\varepsilon N(X)^{2/d}}$ if X is a Rademacher chaos process of any order $d \geq 1$.

Borell [6] studies, under the condition

$$\sup_{t\in I} \frac{\|Z_t - E[Z_t]\|_q}{\|Z_t - E[Z_t]\|_2} < \infty, \qquad q \in (2, \infty], \tag{1.2}$$

integrability of Banach space valued random elements which are limits in probability of tetrahedral polynomials. For $q = \infty$, (1.2) is C_∞ but when $q < \infty$ (1.2) is weaker than C_q, at least when $\mathcal{H}$ consists of centered random variables. As shown in Borell [6], (1.2) implies equivalence of L^p-norms for Hilbert space valued tetrahedral polynomials for $p \leq q$, but not for Banach space valued tetrahedral polynomials except in the case $q = \infty$. Contrary to Borell [6] and others, we consider random elements which are not necessarily limits of tetrahedral polynomials. This enables us to obtain the above mentioned integrability results.

1.3 Preliminary results

The following lemma is for $q = \infty$ a consequence of Borell [6, Theorem 4.1]. For $q \in (1, \infty)$ it is taken from the proof of Kwapień and Woyczyński [16, Theorem 6.6.2]. Finally, using Kwapień and Woyczyński [16, Remark 6.9.1] the result is seen to hold also for $q \in (0, 1]$.

Lemma 1.2. *Let B denote a Banach space and assume $\mathcal{H}$ satisfies C_q for $q \in (0, \infty]$ and if $q < \infty$ that $\mathcal{H}$ consists of symmetric variables. Then for all B-valued tetrahedral polynomials X of order d and all $0 < p < r \leq q$ with $r < \infty$, we have*

$$\|X\|_{L^r(P;B)} \leq k_{p,r,d,\beta} \|X\|_{L^p(P;B)},$$

where $k_{p,r,d,\beta}$ depends only on p, q, d and β. If $q = \infty$ and $p \geq 2$ we may choose $k_{p,r,d,\beta} = A_d \beta^{2d} r^{d/2}$, where A_d only depends on d.

Remark 1.3. By applying Lemma 3.1 (see Appendix) in the proof of Borell [6, Theorem 4.1] it follows that we may choose $A_d = 2^{d^2/2+2d}$.

For $q = \infty$, Lemma 1.2 is only stated for $2 \leq p < r$ in Borell [6]. However, a standard application of Hölder's inequality shows that it is valid for all $0 < p < r$ (see e.g. Pisier [19, Lemme 1.1]). In the case $q < \infty$, Lemma 1.2 is an extension of Hoffmann-Jørgensen [11, Section 6]. The following consequence of the Paley-Zygmund inequality can be found in Krakowiak and Szulga [14, Corollary 1.4].

Lemma 1.4. *Let $(X_n)_{n\geq 1}$ be random variables for which there exist $C > 0$ and $0 < p < q$ such that $\|X_n\|_q \leq C\|X_n\|_p < \infty$ for all $n \geq 1$. If $(X_n)_{n\geq 1}$ converges in probability (respectively, is tight) then $(X_n)_{n\geq 1}$ converges in L^p (respectively, is bounded in L^q).*

2 Main results

The main results of the paper are Theorem 2.1 and Theorem 2.5. An F-valued random element X is said to be a.s. separably valued if $P(X \in A) = 1$ for some separable closed subset A of F.

Theorem 2.1. *Let F denote a metrizable l.c.TVS, $X \in$ weak-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ an a.s. separably valued random element and N a lower semicontinuous pseudo-seminorm on F such that $N(X) < \infty$ a.s. Assume $\mathcal{H}$ satisfies C_q for some $q \in (0, \infty]$ and if $q < \infty$ that $\mathcal{H}$ consists of symmetric variables. Let $k_{p,r,d,\beta}$ be given as in Lemma 1.2. Then for all finite $0 < p < r \leq q$ we have*

$$\|N(X)\|_r \leq k_{p,r,d,\beta} \|N(X)\|_p < \infty,$$

and in the case $q = \infty$ that $E[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e2^{d+5}\beta^4 \|N(X)\|_2^{2/d})$.

For $q = \infty$, Theorem 2.1 answers in the case where the pseudo-seminorm is lower semicontinuous a question raised by Borell [5] concerning integrability of pseudo-seminorms of Rademacher chaos elements. This additional assumption is satisfied in most examples, in particular in the examples in (1.1). By Lemma 1.4 we have the following corollary to Theorem 2.1.

Corollary 2.2. *Let F and $\mathcal{H}$ be as in Theorem 2.1 and N a continuous seminorm on F. Then given $(X_n)_{n\geq 1} \subseteq$ weak-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ all a.s. separably valued such that $\lim_n X_n = 0$ in probability we have $\|N(X_n)\|_p \to 0$ for all finite $p \in (0, q]$.*

Theorem 2.1 relies on the following two lemmas together with an application of Lemma 1.2 on the Banach space l^n_∞ that is $\mathbb{R}^n$ equipped with the sup norm. First, arguing as in Fernique [9, Lemme 1.2.2] we have.

Lemma 2.3. *Assume F is a strongly Lindelöf l.c.TVS. Then a pseudo-seminorm N on F is a lower semicontinuous if and only if there exists $(x^*_n)_{n\geq 1} \subseteq F^*$ such that $N(x) = \sup_{n\geq 1} |x^*_n(x)|$ for all $x \in F$.*

Proof. The *if*-implication is trivial. To show the *only if*-implication let $A := \{x \in F : N(x) \leq 1\}$. Then A is convex and balanced since N is a pseudo-seminorm and closed since N is lower semicontinuous. Thus by the Hahn-Banach theorem, see Rudin [25, Theorem 3.7], for all $x \notin A$ there exists $x^* \in F^*$ such that $|x^*(y)| \leq 1$ for all $y \in A$ and $x^*(y) > 1$, showing that

$$A^c = \bigcup_{x \in A^c} \{y \in F : |x^*(y)| > 1\}.$$

Since F is strongly Lindelöf, there exists $(x_n)_{n\geq 1} \subseteq A^c$ such that

$$A^c = \bigcup_{n=1}^{\infty} \{y \in F : |x^*_n(y)| > 1\},$$

implying that $A = \{y \in F : \sup_{n\geq 1} |x^*_n(y)| \leq 1\}$. Thus by homogeneity we have $N(y) = \sup_{n\geq 1} |x^*_n(y)|$ for all $y \in F$. $\square$

Lemma 2.4. *Let $n \geq 1$, $0 < p < q$ and $C > 0$ be given such that*

$$\|X\|_{L^q(P;l^n_\infty)} \leq C\|X\|_{L^p(P;l^n_\infty)} < \infty, \tag{2.1}$$

for all l^n_∞-valued tetrahedral polynomials X of order d. Then for all $(X_k)_{k=1}^n \subseteq \overline{\mathcal{P}}^d_{\mathcal{H}}$ we have

$$\Big\| \max_{1\leq k\leq n} |X_k| \Big\|_q \leq C \Big\| \max_{1\leq k\leq n} |X_k| \Big\|_p < \infty.$$

Proof. For each $k \in \{1, \dots, n\}$ choose $(X_{k,m})_{m\geq 1} \subseteq \mathcal{P}^d_{\mathcal{H}}$ such that $\lim_m X_{k,m} = X_k$ in probability. Set $U_m = \sup_{1\leq k\leq n} |X_{k,m}|$, $U = \sup_{1\leq k\leq n} |X_k|$ and $Y_m = (X_{1,m}, \dots, X_{n,m})$.

Then for each $m \geq 1$ Y_m is an l_∞^n-valued tetrahedral polynomial of order d and since $\|Y_m\|_\infty = \sup_{1\leq k\leq n}|X_{k,m}|$ we have by (2.1)

$$\|U_m\|_q \leq C\|U_m\|_p < \infty, \qquad \text{for all } m \geq 1. \tag{2.2}$$

Since $\lim_m U_m = U$ in probability, (2.2) shows together with Lemma 1.4 that $\lim_m U_m = U$ in L^p and hence

$$\|U\|_q \leq \liminf_{m\to\infty}\|U_m\|_q \leq C \liminf_{m\to\infty}\|U_m\|_p = C\|U\|_p < \infty.$$

□

Proof of Theorem 2.1. Since X is a.s. separably valued we may and will assume that F is separable. Hence according to Lemma 2.3 there exists $(x_n^*)_{n\geq 1} \subseteq F^*$ such that $N(x) = \sup_{n\geq 1}|x_n^*(x)|$ for all $x \in F$. For $n \geq 1$, let $X_n := x_n^*(X)$. By assumption we may choose $(Y_{k,n})_{k=1}^n \subseteq \overline{\mathcal{P}}_{\mathcal{H}}^d$ such that $(Y_{k,n})_{k=1}^n$ equals $(X_n)_{k=1}^n$ in distribution for all $n \geq 1$. In particular with

$$U_n := \sup_{1\leq k\leq n}|Y_{k,n}|, \qquad n \geq 1,$$

we have that $(U_n)_{n\geq 1}$ converges in distribution to $N(X)$. For finite $0 < p < r \leq q$ let $c = k_{p,r,d,\beta}$. Combining Lemma 1.2 and 2.4 shows $\|U_n\|_q \leq c\|U_n\|_p < \infty$ for all $n \geq 1$, and hence by Lemma 1.4, $\{U_n^p : n \geq 1\}$ is uniformly integrable, implying that

$$\|N(X)\|_r \leq \liminf_{n\to\infty}\|U_n\|_r \leq c \liminf_{n\to\infty}\|U_n\|_p = c\|N(X)\|_p < \infty.$$

Finally, the exponential integrability under C_∞ follows by Remark 1.3 since

$$E[e^{\varepsilon N(X)^{2/d}}] \leq 1 + \sum_{k=1}^d \|N(X)\|_{2k/d}^{2k/d} + \sum_{k=d+1}^\infty \big(\varepsilon 2^{d+5}\beta^4\|N(X)\|_2^{2/d}/d\big)^k \frac{k^k}{k!}.$$

This completes the proof. □

Let T denote a countable set and let $F = \mathbb{R}^T$ equipped with the product topology. F is then a separable and locally convex Fréchet space and all $x^* \in F^*$ are of the form $x \mapsto \sum_{i=1}^n \alpha_i x(t_i)$, for some $n \geq 1$, $t_1, \ldots, t_n \in T$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Thus $X. \in \textit{weak-}\overline{\mathcal{P}}_{\mathcal{H}}^d(F)$ if and only if $(X_t)_{t\in T}$ is a weak chaos process of order d. Rewriting Theorem 2.1 in the case $F = \mathbb{R}^T$ we obtain the following result.

Theorem 2.5. *Assume $\mathcal{H}$ satisfies C_q for some $q \in (0, \infty]$ and if $q < \infty$ that $\mathcal{H}$ consists of symmetric variables. Let T denote a countable set, $(X_t)_{t\in T}$ a weak chaos process and N a lower semicontinuous pseudo-seminorm on $\mathbb{R}^T$ (equipped with the product topology) such that $N(X) < \infty$ a.s. Then for all finite $0 < p < r \leq q$ we have*

$$\|N(X)\|_r \leq k_{p,r,d,\beta}\|N(X)\|_p < \infty,$$

and in the case $q = \infty$ that $E[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e2^{d+5}\beta^4\|N(X)\|_2^{2/d})$.

Theorem 2.1 and 2.5 can be improved considerably in the Gaussian case. Let $\mathcal{G}$ denote a vector space of Gaussian random variables and $\overline{\Pi}^d_{\mathcal{G}}$ the closure in probability of the random variables of the form $p(Z_1, \dots, Z_n)$, where $n \geq 1$, $Z_1, \dots, Z_n \in \mathcal{G}$ and $p \colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most d. For each Banach space B, the inequality (2.1) in Borell [5] for Rademacher variables, Lemma 3.1 (see Appendix) and a central limit theorem argument show

$$\|X\|_{L^r(P;B)} \leq 2^{d^2/2+d}\Big(\frac{r-1}{p-1}\Big)^{d/2}\|X\|_{L^p(P;B)}, \qquad 1 < p < r < \infty, \tag{2.3}$$

for all B-valued polynomials X in $\mathcal{G}$ of order at most d (not necessarily tetrahedral). For details on the central limit theorem argument see e.g. Kwapień and Woyczyński [16, page 163]. Using (2.3), the proofs of Theorem 2.1 and 2.5 show also:

Proposition 2.6. *Let T denote a countable set and $X = (X_t)_{t\in T}$ a process satisfying that for all $n \geq 1$ and $(t_i)_{i=1}^n \subseteq T$ there exists $(Y_i)_{i=1}^n \subseteq \overline{\Pi}^d_{\mathcal{G}}$ such that $(X_{t_1}, \dots, X_{t_n})$ equals $(Y_1, \dots, Y_n)$ in distribution. Then for all lower semicontinuous pseudo-seminorms N on $\mathbb{R}^T$ satisfying $N(X) < \infty$ a.s. we have $E[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e2^{d+3}\|N(X)\|_2^{2/d})$. Moreover, for all $0 < p < r < \infty$ we have*

$$\|N(X)\|_r \leq k_{p,r,d}\|N(X)\|_p < \infty,$$

where $k_{p,r,d}$ only depends on p, r and d, and for $p > 1$ we may choose $k_{p,r,d} = 2^{d^2/2+d}[(r-1)/(p-1)]^{d/2}$.

The integrability of $e^{\varepsilon N(X)^{2/d}}$ for some $\varepsilon > 0$ is a consequence of the seminal work Borell [4, Theorem 4.1]. However, the above presented proof is very simple and provides also explicit constants for integrability of $e^{\varepsilon N(X)^{2/d}}$. The next result is known from Arcones and Giné [1, Theorem 3.1] for general Gaussian polynomials.

Proposition 2.7. *Assume $\mathcal{H}$ consists of symmetric random variables satisfying C_q for some $q \in [2, \infty]$. Let F denote a Banach space and X an F-valued tight element with $x^*(X) \in \overline{\mathcal{P}}^d_{\mathcal{H}}$ for all $x^* \in F^*$. Then there exists $x_0, x_{i_1,\dots,i_k} \in F$ and $\{t_n : n \geq 1\} \subseteq I$ such that*

$$X = \lim_{n\to\infty}\Big(x_0 + \sum_{k=1}^d \sum_{1\leq i_1<\dots<i_k\leq n} x_{i_1,\dots,i_k}\prod_{j=1}^k Z_{t_{i_j}}\Big) \qquad \text{a.s. and in } L^p(P;F),$$

for all finite $p \leq q$.

Proof. We follow Arcones and Giné [1, Lemma 3.4]. Since X is tight we may and do assume F is separable, which implies that $F_1^* := \{x^* \in F^* : \|x^*\| \leq 1\}$ is metrizable and compact in the weak*-topology by the Banach-Alaoglu theorem (see Rudin [25, Theorem 3.15+3.16]). Moreover, the map $x^* \mapsto x^*(X)$ from F_1^* into L^0 is trivially weak*-continuous and thus a weak*-continuous map into L^2 by Corollary 2.2. This shows that $\{x^*(X) : x^* \in F_1^*\}$ is compact in L^2 and hence separable. By definition of $\overline{\mathcal{P}}^d_{\mathcal{H}}$, this implies that there exists a countable set $\{t_n : n \geq 1\} \subseteq I$ such that

$$x^*(X) = \sum_{A\in N_d} a(A, x^*)Z_A, \qquad \text{in } L^2,$$

for some $a(A, x^*) \in \mathbb{R}$, where $N_d = \{A \subseteq \mathbb{N} : |A| \leq d\}$ and $Z_A = \prod_{i\in A} Z_{t_i}$ for $A \in N_d$. For $A \in N_d$, the map $x^* \mapsto a(A, x^*)$ from F^* into $\mathbb{R}$ is linear and weak*-continuous and hence there exists $x_A \in F$ such that $a(A, x^*) = x^*(x_A)$, showing that

$$x^*(X) = \lim_{n\to\infty} x^*\Big(\sum_{A\in N_d^n} x_A Z_A\Big), \qquad \text{in } L^2, \tag{2.4}$$

where $N_d^n = \{A \in N_d : A \subseteq \{1, \ldots, n\}\}$. Since F is separable, (2.4) and Kwapień and Woyczyński [16, Theorem 6.6.1] show that

$$\lim_{n\to\infty} \sum_{A\in N_d^n} x_A Z_A = X \qquad \text{a.s.}$$

By Corollary 2.2 the convergence also takes place in $L^p(P; F)$ for all finite $p \leq q$, which completes the proof. □

The above proposition gives rise to the following corollary.

Corollary 2.8. *Assume $\mathcal{H}$ consists of symmetric random variables satisfying C_q for some $q \in [2, \infty]$. Let T denote a set, $V(T) \subseteq \mathbb{R}^T$ a separable Banach space where the map $f \mapsto f(t)$ from $V(T)$ into $\mathbb{R}$ is continuous for all $t \in T$ and $X = (X_t)_{t\in T}$ a stochastic process with sample paths in $V(T)$ satisfying $X_t \in \overline{\mathcal{P}}_{\mathcal{H}}^d$ for all $t \in T$. Then there exists $x_0, x_{i_1,\ldots,i_k} \in V(T)$ and $\{t_n : n \geq 1\} \subseteq I$ such that*

$$X = \lim_{n\to\infty} \Big(x_0 + \sum_{k=1}^{d} \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\ldots,i_k} \prod_{j=1}^{k} Z_{t_{i_j}}\Big) \tag{2.5}$$

a.s. in $V(T)$ and in $L^p(P; V(T))$ for all finite $p \leq q$.

Proof. For $t \in T$, let $\delta_t \colon V(T) \to \mathbb{R}$ denote the map $f \mapsto f(t)$. Since $V(T)$ is a separable Banach space and $\{\delta_t : t \in T\} \subseteq V(T)^*$ separate points in $V(T)$ we have

(i) the Borel σ-field on $V(T)$ equals the cylindrical σ-field $\sigma(\delta_t : t \in T)$,

(ii) $\{\sum_{i=1}^n \alpha_i \delta_{t_i} : \alpha_i \in \mathbb{R},\ t_i \in T,\ n \geq 1\}$ is sequentially weak*-dense in $V(T)^*$,

see e.g. Rosiński [21, page 287]. By (i) we may regard X as a random element in $V(T)$ and by (ii) it follows that $x^*(X) \in \overline{\mathcal{P}}_{\mathcal{H}}^d$ for all $x^* \in V(T)^*$. Hence the result is a consequence of Proposition 2.7. □

Borell [6, Theorem 5.1] shows Corollary 2.8 assuming (1.2), T is a compact metric space, $V(T) = C(T)$ and $X \in L^q(P; V(T))$. By assuming C_q instead of the weaker condition (1.2) we can omit the assumption $X \in L^q(P; V(T))$. Note also that by Theorem 2.5 the last assumption is satisfied under C_q. When $\mathcal{H}$ consists of symmetric α-stable random variables and $d = 1$ Corollary 2.8 is known from Rosiński [21, Corollary 5.2]. The separability assumption on $V(T)$ in Corollary 2.8 is crucial. Indeed, for all $p > 1$, Jain and Monrad [13, Proposition 4.5] construct a separable centered Gaussian process $X = (X_t)_{t\in[0,1]}$ with sample paths in the non-separable Banach space B_p of functions of finite p-th variation on $[0, 1]$ such that the support of P_X is a non-separable subset of B_p. This shows that X can not be given by (2.5). Since Gaussian measures on Banach spaces are concentrated on closed separable subsets (see Borell [4, Theorem 8.2]), P_X is not even a Gaussian measure on B_p.

3 Appendix

The next lemma is a very useful tool when studying chaos elements. The explicit constants in Remark 1.3 and (2.3) are a consequence of this result.

Lemma 3.1. *Let V denote a vector space, N a seminorm on V, $\varepsilon \in (0,1)$ and $x_0, \dots, x_d \in V$.*

$$\text{If } N\Big(\sum_{k=0}^{d} \lambda^k x_k\Big) \le 1 \text{ for all } \lambda \in [-\varepsilon, \varepsilon] \quad \text{then} \quad N\Big(\sum_{k=0}^{d} x_k\Big) \le 2^{d^2/2+d}\varepsilon^{-d}. \tag{3.1}$$

Proof. Assume first that $x_0, \dots, x_d \in \mathbb{R}$. By induction in d, let us show:

$$\text{If } \Big|\sum_{k=0}^{d} \lambda^k x_k\Big| \le 1 \text{ for all } \lambda \in [-\varepsilon, \varepsilon] \quad \text{then} \quad \Big|\sum_{k=0}^{d} x_k\Big| \le 2^{d^2/2+d}\varepsilon^{-d}. \tag{3.2}$$

For $d = 1, 2$ (3.2) follows by a straightforward argument, so assume $d \ge 3$, (3.2) holds for $d-1$ and that the left-hand side of (3.2) holds for d. We have

$$\Big|\sum_{k=0}^{d} \lambda^k (\varepsilon^k x_k)\Big| \le 1, \qquad \text{for all } \lambda \in [-1, 1],$$

which by Pólya and Szegö [20, Aufgabe 77] shows that $|x_d \varepsilon^d| \le 2^d$ and hence $|x_d| \le 2^d \varepsilon^{-d}$. For $\lambda \in [-\varepsilon, \varepsilon]$, the triangle inequality yields

$$\Big|\sum_{k=0}^{d-1} \lambda^k x_k\Big| \le 1 + 2^d, \quad \text{and hence} \quad \Big|\sum_{k=0}^{d-1} \lambda^k \frac{x_k}{1+2^d}\Big| \le 1.$$

The induction hypothesis implies

$$\Big|\sum_{k=0}^{d-1} x_k\Big| \le \varepsilon^{-(d-1)} 2^{(d-1)^2+(d-1)}(1+2^d),$$

and hence another application of the triangle inequality shows that

$$\begin{aligned}\Big|\sum_{k=0}^{d} x_k\Big| &\le \varepsilon^{-d}2^d + \varepsilon^{-(d-1)}2^{(d-1)^2/2+(d-1)}(1+2^d) \\ &\le \varepsilon^{-d}2^{d^2/2+d}\Big(2^{-d^2/2} + 2^{-1/2-d} + 2^{-1/2}\Big),\end{aligned}$$

which is less than or equal to $\varepsilon^{-d}2^{d^2/2+d}$ since $d \ge 3$. This completes the proof of (3.2).

Now let $x_0, \dots, x_d \in V$. Since N is a seminorm, Hahn-Banach theorem (see Rudin [25, Theorem 3.2]) shows that there exists a family Λ of linear functionals on V such that

$$N(x) = \sup_{F \in \Lambda} |F(x)|, \qquad \text{for all } x \in V.$$

Assuming that the left-hand side of (3.1) is satisfied we have

$$\Big|\sum_{k=0}^{d} \lambda^k F(x_k)\Big| \le 1, \qquad \text{for all } \lambda \in [-\varepsilon, \varepsilon] \text{ and all } F \in \Lambda,$$

which by (3.2) shows

$$\Big|F\Big(\sum_{k=0}^{d} x_k\Big)\Big| = \Big|\sum_{k=0}^{d} F(x_k)\Big| \le 2^{d(d-1)}\varepsilon^{-d}, \qquad \text{for all } F \in \Lambda.$$

This completes the proof. □

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