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Markus Kiderlen and Daniel Meschenmoser

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Error bounds for surface area estimators based on Crofton's formula

Markus Kiderlen* and Daniel Meschenmoser†

Abstract

According to Crofton's formula, the surface area $S(A)$ of a sufficiently regular compact set A in $\mathbb{R}^d$ is proportional to the mean of all total projections $p_A(u)$ on a linear hyperplane with normal u, uniformly averaged over all unit vectors u. In applications, $p_A(u)$ is only measured in k directions and the mean is approximated by a finite weighted sum $\widehat{S}(A)$ of the total projections in these directions. The choice of the weights depends on the selected quadrature rule. We define an associated zonotope Z (depending only on the projection directions and the weights), and show that the relative error $\widehat{S}(A)/S(A)$ is bounded from below by the inradius of Z and from above by the circumradius of Z. Applying a strengthened isoperimetric inequality due to Bonnesen, we show that the rectangular quadrature rule does not give the best possible error bounds for $d = 2$. In addition, we derive asymptotic behavior of the error (with increasing k) in the planer case. The paper concludes with applications to surface area estimation in design-based digital stereology where we show that the weights due to Bonnesen's inequality are better than the usual weights based on the rectangular rule and almost optimal in the sense that the relative error of the surface area estimator is very close to the minimal error.

Key words: surface area, perimeter, Crofton formula, minimal annulus, isoperimetric inequality, associated zonotope, digitization.

1 Introduction

One common approach to approximate the surface area $S(A)$ of an unknown set $A \subset \mathbb{R}^d$ from its digitization is based on a discretization of Crofton's formula. We discuss the worst case error introduced by the discretization of the rotational integral in dependence of the quadrature rule chosen. As the methods apply generally to surface area estimators based on Crofton's formula, we describe them in a general framework and return to its application to digital images in Section 3.

For a vector v in the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, let $v^\perp$ be the linear hyperplane with normal v and let $e_{r,v}$ be the straight line with direction v through $r \in v^\perp$. Let

*Department of Mathematical Sciences, University of Aarhus, 8000 Aarhus, Denmark
Email: kiderlen@imf.au.dk

†Insitute of Stochastics, Ulm University, 89069 Ulm, Germany
Email: daniel.meschenmoser@uni-ulm.de

$A \subset \mathbb{R}^d$ be a compact set in the class $\mathcal{U}_{PR}$ (defined in the next section). A special case of Crofton's formula (Rother and Zähle, 1990) expresses the surface area $S(A)$ of A in terms of the Euler characteristic χ of linear sections

$$S(A) = \frac{2}{\gamma_d} \int_{\mathbb{S}^{d-1}} \int_{v^\perp} \chi(A \cap e_{r,v})\, dr\, \mu(dv). \tag{1}$$

Here $\gamma_d = (2\kappa_{d-1}) / (d\kappa_d)$, where κ_d is the volume of the d-dimensional unit ball, and μ is the *normalized Haar measure* on the unit sphere $\mathbb{S}^{d-1}$ (see e.g. Schneider and Weil (1992, p. 18), but note the different normalization). The inner integral of (1)

$$p_v = \int_{v^\perp} \chi(A \cap e_{r,v})\, dr \tag{2}$$

is called *total projection of A in direction v*, as it is obtained by measuring the $(d-1)$-volume of the orthogonal projection of A on $v^\perp$ with multiplicities. If total projections can be determined exactly for finitely many directions $v_1, \ldots, v_k \in \mathbb{S}^{d-1}$, say, a k-point quadrature rule can be used to discretize the outer integral in (1) and one obtains the approximation

$$\widehat{S}_k^d(A) = \frac{2}{\gamma_d} \sum_{i=1}^{k} c_i p_{v_i}, \tag{3}$$

which depends on the choice of weights $c_1, \ldots, c_k \geq 0$. To assure that the quadrature rule is exact whenever A is a ball, we assume throughout that the weights sum up to 1. If, for example, the rectangular rule is chosen in the planar case, then the weights are proportional to the arc-lengths of the corresponding spherical Voronoi cells generated by $\{v_1, \ldots, v_k\}$ on $\mathbb{S}^1$. This geometric interpretation generalizes readily to higher dimensions. If $\{P_1, \ldots, P_k\}$ is the spherical Voronoi tessellation of $\mathbb{S}^{d-1}$ generated by the set of projection directions $\{v_1, \ldots, v_k\}$ with $v_i \in P_i$ then the weights

$$c_i^V = \mu(P_i), \qquad i = 1, \ldots, k,$$

will be called *Voronoi weights* associated to $\{v_1, \ldots, v_k\}$. These weights are commonly used in applications for $d = 2, 3$.

The discretization of the spherical integral introduces a bias, which typically depends on the set A. We are interested in the worst case behavior. Already Steinhaus (1930) treated the special case where $d = 2$, k is even, and $\{v_1, \ldots, v_k\}$ forms an equidistant set of points in $\mathbb{S}^1$. For $\widehat{S}_k^2(A)$, given by (3) with Voronoi weights $c_1^V, \ldots, c_k^V$, he derived sharp bounds for the relative error:

$$\frac{\pi}{k} \frac{\cos(\pi/k)}{\sin(\pi/k)} \leq \frac{\widehat{S}_k^2(A)}{S(A)} \leq \frac{\pi}{k} \frac{1}{\sin(\pi/k)}. \tag{4}$$

The left hand side and the right hand side of (4) are the endpoints of the interval of all possible relative errors, as A varies. Such an interval can be established without the assumption of equidistant directions and in all dimensions. We refer to this interval as *error interval* in the following.

Using a translative Crofton formula in Section 2, we will define an origin-symmetric convex body $Z \subset \mathbb{R}^d$ associated to the discretization, only depending on the projection directions and the quadrature rule. We will show in Lemma 1 that the relative error $\widehat{S}_k^d(A)/S(A)$ is in a sharp way bounded from below by (a multiple of) the inradius of Z and from above by (a multiple of) the outer radius of Z. Thus, the thickness of the minimal annulus of Z equals the length of the error interval and describes the quality of the estimator. Given k projection directions, the quadrature rule (in other words, the values of the associated weights) that minimizes the minimal thickness of Z can typically only be determined numerically. In the planar case, we suggest to bound the thickness of the minimal annulus of Z from above by an isoperimetric deficit using a strengthened isoperimetric inequality due to Bonnesen. This isoperimetric deficit can be minimized with respect to all quadrature rules in closed form. The weights minimizing the isoperimetric deficit will be called *Bonnesen weights* and are proportional to the edges of a polygon circumscribing the unit disk and touching it exactly at the points $v_1, \dots, v_k$. We will show that Voronoi weights are not minimizing the length of the error interval by giving an example where the Bonnesen weights yield better error bounds. We will determine the asymptotic behavior (as $k \to \infty$) of the relative error for the Bonnesen weights in Theorem 4. At the end of Section 2 we will consider the case where the directions $v_1, \dots, v_k \in \mathbb{S}^1$ are chosen by systematic random sampling. We will show that the coefficient of error of $\widehat{S}_k^2(A)$ can be bounded from above by a geometric quantity involving Z (a multiple of the L_2-distance between Z and its Steiner ball), and use this to give an upper bound for the coefficient of error that only involves the area of Z.

In Section 3 the digitization of A on a randomly translated, rectangular grid will be considered. Asymptotic bounds for the expected value of the estimator for $S(A)$ in the grid will be established. "Asymptotic" relates here to increasing resolution of the grid. The vectors $v_1, \dots, v_k$ are chosen as grid directions, i.e. vectors which connect two grid points. We will consider the vectors of the 4-, 8- and 16-neighborhood in 2D and the 6- and 26-neighborhood in 3D. For all these settings the Voronoi and Bonnesen weights together with the corresponding in- and circumradii r and R, respectively, will be computed analytically, except for the 26 directions in 3D where numerical methods will be used. We will compare the relative errors with the minimal error achieved by numerically optimizing the weights and show that the Bonnesen weights lead, at least in 2D, to smaller errors than the widely used Voronoi weights. Restricting to a quadratic grid in the plane and a perimeter estimator based on pairs of lattice points with lattice distance at most $n \geq 2$, will show in Theorem 7 that the asymptotic mean relative error of $\widehat{S}_k^d(A)$ for Bonnesen weights decreases as n^{-2}.

The application of Bonnesen's improved isoperimetric inequality restricts many of the above arguments to the two-dimensional case. In the last section we discuss the possibility of extensions to higher dimensions.

2 Error bounds for $\widehat{S}_k^d(A)$

Throughout the following we assume that $A \in \mathcal{U}_{PR}$ is compact, where $\mathcal{U}_{PR}$ is the family of all sets in $\mathbb{R}^d$ which can be written as a finite union of compact sets $A_1, \ldots, A_m$ with positive reach such that any intersection $\bigcap_{i\in I} A_i$ with $I \subset \{1, \ldots, m\}$ is either empty or a set of positive reach, as well. A set $A \subset \mathbb{R}^d$ has positive reach if there is a $\epsilon > 0$ such that each point in the ϵ-neighborhood of A has a unique closest point in A. In particular, *convex bodies* (nonempty compact convex subsets of $\mathbb{R}^d$) and *polyconvex sets* (finite unions of convex bodies) are elements of $\mathcal{U}_{PR}$. In view of the applications in digital stereology, it is convenient to extend the total projection mapping $v \mapsto p_v$ to $\mathbb{R}^d \setminus \{o\}$ by positive homogeneity of degree 0. The translative Crofton formula

$$p_v = \frac{1}{2\|v\|} \int_{\mathbb{S}^{d-1}} |\langle u, v\rangle| \, S(A, du) \tag{5}$$

holds for almost all $v \in \mathbb{R}^d$ (see Rataj (2002, Theorem 2.1 and Theorem 2.3)), where $S(A, \cdot)$ is the surface area measure of order $d-1$ of A and $\langle u, v\rangle$ is the usual inner product of u and v. If A is polyconvex, (5) holds for *all* $v \in \mathbb{R}^d \setminus \{o\}$. If A is in addition topologically regular (A is the closure of its interior), then $S\left(A, \mathbb{S}^{d-1}\right) = S(A)$. Without this assumption the surface area of lower dimensional parts of A contribute twice to the total mass of $S(A, \cdot)$.

If $v_1, \ldots, v_k \in \mathbb{R}^d \setminus \{o\}$ are such that (5) holds with $v = v_i$ for all $i = 1, \ldots, k$, then the definition of $\widehat{S}_k^d(A)$ in combination with (5) yields

$$\widehat{S}_k^d(A) = \frac{1}{\gamma_d} \int_{\mathbb{S}^{d-1}} h(u) \, S(A, du), \tag{6}$$

with

$$h := \sum_{i=1}^{k} \frac{c_i}{\|v_i\|} |\langle v_i, \cdot\rangle|. \tag{7}$$

The key observation is that the integrand h is the support function of a convex body. We refer the reader to Schneider (1993) for relevant notions and concepts in convex geometry and only recall the most important facts here. The support function h_K of a convex body K is given by

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\}, \qquad u \in \mathbb{S}^{d-1}.$$

Here and in the following, we consider the support function as a function on the unit sphere. For convex bodies K and M and scalars $\alpha, \beta \geq 0$, we have

$$\alpha h_K + \beta h_M = h_{\alpha K \oplus \beta M}, \tag{8}$$

where the *Minkowski addition* $\oplus$ of sets and the multiplication of a set with a scalar are understood pointwise. We will repeatedly use the monotonicity property

$$K \subset M \iff h_K \leq h_M \tag{9}$$

and the fact that the support function h_{B^d} of the Euclidean unit ball B^d in $\mathbb{R}^d$ is the constant 1. Equation (9) implies in particular that any convex body is uniquely determined by its support function. Consequently, the definition

$$\delta_2(K,M)^2 := \int_{\mathbb{S}^{d-1}} (h_K(u) - h_M(u))^2 \, du$$

for convex bodies K and M, gives rise to the so-called L_2-metric $\delta_2(\cdot,\cdot)$ on the family of convex bodies. The support function of the line segment $[-x,x]$ with endpoints $-x$ and $x \in \mathbb{R}^d$ is $|\langle x,\cdot\rangle|$. Due to (8), the function h in (7) is the support function of a finite sum Z of line segments. Such sets are called *zonotopes* and play a prominent role in functional analysis, convex and stochastic geometry; see e.g. Goodey and Weil (1993) and the references therein. Explicitly, we have

$$Z = c_1[-u_1,u_1] \oplus \cdots \oplus c_k[-u_k,u_k] \tag{10}$$

with the unit vectors $u_i = v_i/\|v_i\|$ for $i = 1,\ldots,k$. In view of (6) the approximation $\widehat{S}_k^d(A)$ can be expressed in terms of the associated zonotope Z, as

$$\widehat{S}_k^d(A) = \frac{1}{\gamma_d} \int_{\mathbb{S}^{d-1}} h_Z(u) \, S(A,du). \tag{11}$$

To obtain lower and upper bounds of $\widehat{S}_k^d(A)$, we have to find maxima and minima of h_Z. Due to (9), $r \geq 0$ is the minimum of h_Z on $\mathbb{S}^{d-1}$ if and only if rB^d is the largest ball contained in Z. Similarly, $R \geq 0$ is the maximum of h_Z, if and only if RB^d is the smallest ball containing Z. With these optimal values of $0 \leq r \leq R$, the set $RB^d \setminus rB^d$ is called the *minimal annulus of* Z. The difference $R-r$ is called the *width of the minimal annulus* and denoted by $T(Z)$. For later reference we summarize this geometric interpretation for polyconvex sets (for which (5) holds for arbitrary $v \neq o$). As formulations for $\mathcal{U}_{PR}$-sets are obtained in a straightforward manner, we will restrict to polyconvex sets from now on. In addition, topological regularity is assumed throughout, as we want to use that the total mass of $S(A,\cdot)$ is the surface area of A.

Lemma 1. *Let $A \subset \mathbb{R}^d$ be a topologically regular polyconvex set with positive surface area, and fix $k \geq 2$ and $v_1,\ldots,v_k \subset \mathbb{R}^d \setminus \{o\}$. Let $\gamma_d = (2\kappa_{d-1})/(d\kappa_d)$. If $\widehat{S}_k^d(A)$ is given by* (3)*, then the sharp bounds*

$$\frac{r}{\gamma_d} \leq \left| \frac{\widehat{S}_k^d(A)}{S(A)} \right| \leq \frac{R}{\gamma_d} \tag{12}$$

for the relative estimation error hold, where r is the smaller and R is larger radius of the minimal annulus of the zonotope Z given by (10)*.*

That the bounds in the above Lemma are sharp follows from the next example.

Example 2. *Fix $k \geq 2$, $u_1,\ldots,u_k \in \mathbb{S}^{d-1}$ and weights $c_1,\ldots,c_k \geq 0$ for a quadrature rule. Define Z according to* (10)*. Due to symmetry, the ball rB^d touches the boundary of Z in at least two antipodal points rw and $-rw$, $w \in \mathbb{S}^{d-1}$. Let A be a*

ball of $(d-1)$*-volume* $1/2$ *in the hyperplane* $w^{\perp}$*. (As* A *is lower dimensional, the proper interpretation of* $S(A)$ *is* twice *its* $(d-1)$*-dimensional Hausdorff measure, so* $S(A)=1$*.) The surface area measure of* A *is concentrated on the points* w *and* $-w$*, and* h_Z *coincides in both of these directions with* r*, so* (11) *implies*

$$\widehat{S}_k^d(A)=\frac{1}{\gamma_d}\int_{\mathbb{S}^{d-1}} rS(A,du)=\frac{r}{\gamma_d}S(A),$$

and equality holds on the left hand side of (12)*.*

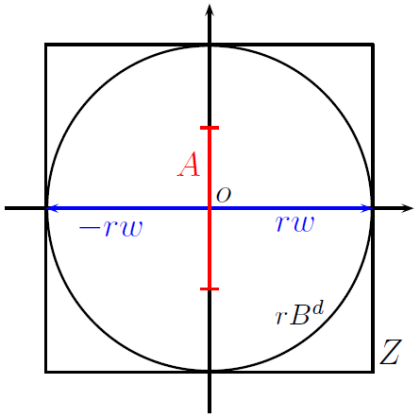

Figure 1: Visualisation for Example 2.

Figure 1 illustrates this for $d=2$*,* $k=2$*,* $u_1=(1,0)^{\top}$*,* $u_2=(0,1)^{\top}$ *and* $c_1=c_2=1/2$*. Obviously* A *is not topologically regular, but it can be approximated by a sequence of topologically regular convex bodies* (A_m) *in such a way that* $\lim_{m\to\infty}\widehat{S}_k^d(A_m)/S(A_m)=\widehat{S}_k^d(A)/S(A)$*. This implies that the left hand side of* (12) *cannot be improved, even if we restrict considerations to topologically regular sets. To show that the second inequality in* (12) *is sharp, a similar argument can be used, if* $\pm w$ *are directions for which* h_Z *becomes maximal, and thus coincides with* R*.*

In the following, we will restrict to the case $d=2$, although some of the concepts can be transferred to higher dimensions. In order that the length of the error interval of $\widehat{S}_k^d(A)$ in (6) is as small as possible, the difference $R-r$ should be as small as possible. This can be achieved by an appropriate choice of the weights $c_1,\dots,c_k$. To obtain an exact value for the integral in (6) in the case where A is a disk, we must assume that the weights sum up to one. It is not difficult to show (see e.g. Schneider (1993)), that $c_1+\cdots+c_k=1$ is equivalent to the condition that the zonotope Z given by (10) has perimeter 4. Let $\mathcal{Z}$ be the family of all zonotopes that can be written as sum of line-segments parallel to given unit vectors $u_1,\dots,u_k$. Let $\mathcal{Z}_4$ be the family of those $Z\in\mathcal{Z}$ that have perimeter 4. We are therefore faced with the problem of finding a zonotope $Z^*\in\mathcal{Z}_4$ that satisfies

$$T(Z^*)=\min\{T(Z)\,:\,Z\in\mathcal{Z}_4\}. \tag{13}$$

If

$$Z^*=c_1^*[-u_1,u_1]\oplus\cdots\oplus c_k^*[-u_k,u_k],$$

then $c_1^*, \ldots, c_k^* \geq 0$ are the best weights in (6), in the sense that among all weights summing up to one they yield the shortest interval of possible relative errors. A solution Z^* of the optimization problem (13) always exists due to a compactness argument based on the Blaschke selection theorem.

For asymptotic results, it is enough to replace the objective function in (13) by a simpler one. Bonnesen (1929) improved the isoperimetric inequality for an arbitrary planar convex body K, stating that

$$\frac{S^2(K)}{4\pi} - V(K) \geq \frac{\pi}{4} T^2(K), \tag{14}$$

where $S(K)$ and $V(K)$ are perimeter and area of K, respectively. For $K = Z \in \mathcal{Z}_4$ we have $S(Z) = 4$ and the left hand side of (14) is minimal for the zonotope $\widetilde{Z} \in \mathcal{Z}_4$ that has the greatest area. According to a classical result of Lindelöf (1869), $\widetilde{Z}$ is characterized among all zonotopes in $\mathcal{Z}_4$ by the fact that it circumscribes a circle. Due to origin-symmetry, this circle is the incircle of $\widetilde{Z}$, centered at the origin, and with radius $\widetilde{r}_k$. This allows an explicit construction of $\widetilde{Z}$. Up to scaling with the factor $1/\widetilde{r}_k$ the zonotope $\widetilde{Z}$ coincides with the polytope

$$\widetilde{P} := \bigcap_{i=1}^{k} \left\{ x \in \mathbb{R}^2 : |\langle u_i, x \rangle| \leq 1 \right\} \tag{15}$$

obtained by intersecting all supporting half-planes of the unit disk with outer normal in $\{\pm u_1, \ldots, \pm u_k\}$. We now assume without loss of generality that the vectors $u_1, \ldots, u_k$ all are located on the positive half-sphere $\{(\cos\varphi, \sin\varphi) : 0 \leq \varphi < \pi\}$ and are ordered according to increasing angles with respect to the x-axis. We write $\sphericalangle(u, v) \in [0, \pi]$ for the (smaller) angle between the unit vectors u and v. Let α_i be the outer angle of the vertex between the ith and the $(i+1)$st edge (i.e. α_i is the angle of the normal cone at this vertex, in other words $\pi - \alpha_i$ is the usual inner angle, see Figure 2). Explicitly, we have

$$\alpha_i = \begin{cases} \sphericalangle(u_i, u_{i+1}), & i = 1, \ldots, k-1 \\ \alpha_k = \pi - \sphericalangle(u_1, u_k) \end{cases} \tag{16}$$

as the normal of the $(k+1)$st edge is $-u_1$. As the length of the ith edge of $\widetilde{P}$ is $\tan(\alpha_i/2) + \tan(\alpha_{i-1}/2)$ and $S(\widetilde{Z}) = 4$, we obtain

$$\widetilde{Z} = \widetilde{c}_1 [-u_1, u_1] \oplus \cdots \oplus \widetilde{c}_k [-u_k, u_k]$$

with

$$\widetilde{c}_i = \widetilde{r}_k \frac{\tan(\alpha_i/2) + \tan(\alpha_{i-1}/2)}{2}, \tag{17}$$

and

$$\widetilde{r}_k = \left(\sum_{i=1}^{k} \tan(\alpha_i/2) \right)^{-1}, \tag{18}$$

where we have put $\alpha_0 = \alpha_k$. This and $V(\widetilde{Z}) = 2\widetilde{r}_k$ was also derived by Knebelman (1941). The weights $\widetilde{c}_i$ are based on the application of Bonnesen's inequality (14)

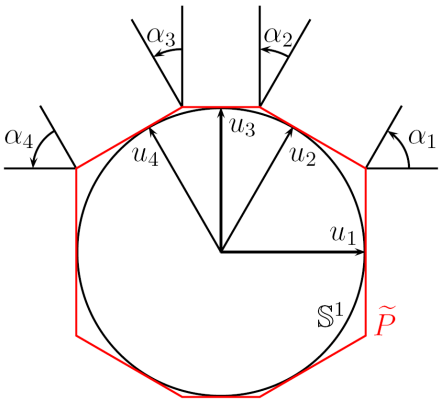

Figure 2: Construction of the angles α_i and the polytope $\widetilde{P}$ with $k = 4$.

and will be called *Bonnesen weights* in the following. The ith weight $\widetilde{c}_i$ is the relative length of the ith edge of the polygon with facet normals $\pm u_i$ which circumscribes a circle of radius $\widetilde{r}_k$. The outer radius $\widetilde{R}_k$ of $\widetilde{Z}$ is the largest distance of a vertex of $\widetilde{Z}$ from the origin, and this is

$$\widetilde{R}_k = \widetilde{r}_k \max_{i=1}^{k} \frac{1}{\cos\left(\alpha_i/2\right)}. \tag{19}$$

Summarizing, we have shown the following. If $\widehat{S}_k^d\left(A\right)$ is an estimator of $S\left(A\right) > 0$ given by (6) with the Bonnesen weights $c_i = \widetilde{c}_i$, $i = 1, \ldots, k$, from (17), then the relative errors obey

$$\frac{\pi}{2}\widetilde{r}_k \leq \frac{\widehat{S}_k^d\left(A\right)}{S\left(A\right)} \leq \frac{\pi}{2}\widetilde{R}_k. \tag{20}$$

These error bounds are sharp; see Example 2.

It should be noted that (14) is always a strict inequality unless K is a disk. As (14) is used for zonotopes here, this approach will not necessarily lead to the optimal choice of the weights. However, the choice may be better than choices for the weights motivated by usual quadrature rules.

Example 3. *We consider the integrand in* (6) *with* $k = 3$ *and the directions* $u_i = \big(\cos(\varphi_i), \sin(\varphi_i)\big)$, $i = 1, 2, 3$, *where*

$$\varphi_1 = 0, \qquad \varphi_2 = \frac{\pi}{16}, \qquad \varphi_3 = \frac{\pi}{8}.$$

As mentioned before, the rectangular quadrature rule leads to the Voronoi weights c_i^V*: The* i*th weight is the normalized length of the Voronoi arc corresponding to* u_i *(arc in* $\mathbb{S}^1$ *of all points closer to* u_i *than to any other point in* $\{\pm u_1, \pm u_2, \pm u_3\}$*). This gives*

$$c_i^V = \frac{\varphi_{i+1} - \varphi_{i-1}}{2\pi},$$

where we assumed π*-periodicity. For the present example, we obtain*

$$\mathbf{c}^V = \left(\frac{15}{32}, \frac{1}{16}, \frac{15}{32}\right)$$

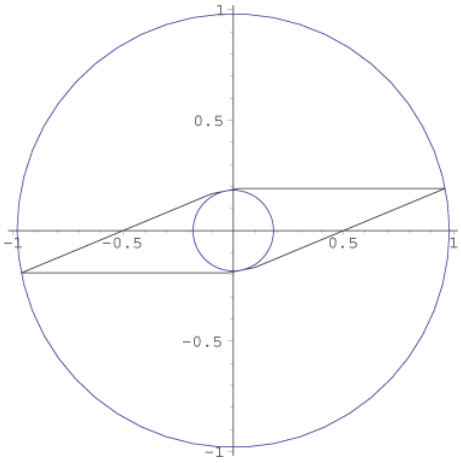

Figure 3: The zonoid associated to Voronoi weights with its minimal annulus.

and the corresponding zonotope Z_V *has inradius* $r_V \approx 0.18290$ *and circumradius* $R_V \approx 0.98199$*; see Figure 3. Thus, the width of the minimal annulus is approximately* 0.79909*. Using instead the Bonnesen weights yields approximately*

$$\widetilde{\mathbf{c}} = (0.49057, 0.01885, 0.49057)\,,$$

leading to the inradius $\widetilde{r} \approx 0.191412$ *and circumradius* $\widetilde{R} \approx 0.981147$*, respectively. The width of the minimal annulus is now approximately* 0.789735*, which is an improvement of about* 1%*.*

To formulate an asymptotic result, we have to specify how close the set of the directions $u_1, \ldots, u_k$ is to a set of equidistant directions. Following Gardner *et al.* (2006), we introduce the *symmetrized spread* Δ_k^* of $u_1, \ldots, u_k$ by

$$\Delta_k^* = \max_{u \in \mathbb{S}^1} \min_{1 \le i \le k} \min \left\{ \|u - u_i\|\,, \|u - (-u_i)\| \right\}.$$

Geometrically, Δ_k^* is the maximal distance of a unit vector from the set $\{\pm u_1, \ldots, \pm u_k\}$. In particular, $\{\pm u_1, \ldots, \pm u_k\}$ is a Δ_k^*-net in $\mathbb{S}^1$. For α_i defined by (16), we have

$$2 \sin \frac{\alpha_i}{4} \le \Delta_k^*, \qquad i = 1, \ldots, k. \tag{21}$$

The following theorem shows that the choice $c_i = \widetilde{c}_i$ leads to a relative error of $\widehat{S}_k^d(A)$ that depends quadratically on Δ_k^*. Here we only consider sampling sets $\{u_1, \ldots, u_k\}$ such that every closed sub-arc of $\mathbb{S}^1$ of length $\pi/2$ contains at least one point of $\{\pm u_1, \ldots, \pm u_k\}$. Equivalently, $\Delta_k^* \le \sqrt{2 - \sqrt{2}}$.

Theorem 4. *Let* $A \subset \mathbb{R}^2$ *be a polyconvex set with positive perimeter. Let* $k \ge 2$ *and* $\{v_1, \ldots, v_k\} \subset \mathbb{R}^2 \setminus \{0\}$ *such that the symmetrized spread of the vectors* $u_i = v_i / \|v_i\|$*,* $i = 1, \ldots, k$*, is* $\Delta_k^* \le \sqrt{2 - \sqrt{2}}$*. If* $\widehat{S}_k^d(A)$ *in* (3) *is calculated using the Bonnesen weights* $c_i = \widetilde{c}_i$*,* $i = 1, \ldots, k$*, from* (17)*, then the relative error obeys*

$$\left| \frac{\widehat{S}_k^d(A) - S(A)}{S(A)} \right| \le \frac{\pi^2}{3} \left(\Delta_k^*\right)^2. \tag{22}$$

Proof. From (20) we get

$$\pi\widetilde{r}_k - 2 \le 2\left(\frac{\widehat{S}_k^d(A) - S(A)}{S(A)}\right) \le \pi\widetilde{R}_k - 2. \tag{23}$$

We estimate the left hand side of (23). In view of (21), we have

$$\alpha_i/2 \le 2\arcsin(\Delta_k^*/2) \le \pi/4 \quad \text{for all } i = 1, \ldots, k.$$

Taylor's theorem implies

$$\tan(\alpha_i/2) \le \alpha_i/2 + c'(\alpha_i/2)^3, \qquad \text{for all } i = 1, \ldots, k,$$

where $c' = \frac{8}{3}$ is the third derivative of $\tan(x)/3!$ evaluated at $\pi/4$. Relations (18), (21) and $\arcsin(x) \le \frac{\pi}{2}x$, $0 \le x \le 1$, imply that

$$\pi\widetilde{r}_k \ge \frac{\pi}{\sum_{i=1}^{k}\left(\frac{\alpha_i}{2}\left(1 + c'\left(\frac{\alpha_i}{2}\right)^2\right)\right)} \ge \frac{2}{1 + \frac{c'\pi^2}{4}(\Delta_k^*)^2},$$

and this gives

$$\pi\widetilde{r}_k - 2 \ge -\frac{2\pi^2}{3}(\Delta_k^*)^2.$$

The right hand side of (23) can be estimated in an even easier way using the fact that the perimeter of the incircle of $\widetilde{Z}$ is bounded by $S(\widetilde{Z}) = 4$ and hence $\widetilde{r}_k \le 2/\pi$. Together with (19) this gives

$$\pi\widetilde{R}_k - 2 \le \frac{2}{1 - (\Delta_k^*)^2}(\Delta_k^*)^2 \le \frac{2}{\sqrt{2} - 1}(\Delta_k^*)^2,$$

as $\Delta_k^* \le \sqrt{2 - \sqrt{2}}$. Putting things together we arrive at

$$\left|\frac{\widehat{S}_k^d(A) - S(A)}{S(A)}\right| \le c(\Delta_k^*)^2$$

with $c = \max\{\frac{\pi^2}{3}, \frac{1}{\sqrt{2}-1}\} = \frac{\pi^2}{3}$. □

The example of equidistant sampling shows that quadratic behavior is the best possible.

Example 5. *Consider the special case where $u_1, \ldots, u_k$ are equidistant on the upper half circle, meaning that $u_i = (\cos(i\pi/k), \sin(i\pi/k))$, $i = 1, \ldots, k$. Hence*

$$\Delta_k^* = 2\sin\frac{\pi}{4k} \sim \frac{\pi}{2k}, \qquad k \to \infty.$$

By symmetry arguments, the weights leading to the minimal width of the corresponding minimal annulus must all be equal and thus $c_1 = \cdots = c_k = 1/k$ and $c_i = \widetilde{c}_i$ for

$i = 1, \ldots, k$. *According to* (18) *and* (19)*, the inner and outer radii of the associated zonotope are*

$$\widetilde{r}_k = \left(k \tan\left(\frac{\pi}{2k}\right)\right)^{-1},$$

and

$$\widetilde{R}_k = \widetilde{r}_k \left(\cos\left(\frac{\pi}{2k}\right)\right)^{-1} = \left(k \sin\left(\frac{\pi}{2k}\right)\right)^{-1};$$

cf. (4)*. Therefore, the width of the minimal annulus is*

$$\widetilde{R}_k - \widetilde{r}_k = \frac{\pi}{4} k^{-2} + O\left(k^{-4}\right), \qquad k \to \infty.$$

This shows that $\widetilde{R}_k - \widetilde{r}_k$*, and thus the relative worst case error, are of order* $(\Delta_k^*)^2$*.*

Instead of using the above geometric arguments to obtain asymptotics for the worst case error, one might also use methods from *optimum quantization*; see Gruber (2004). Among other important applications, this theory yields asymptotic minimum errors of numerical integration for classes of Hölder continuous functions. As the function $g_u : v \mapsto |\langle v, u \rangle|$, and hence the function p_v in (5) are Lipschitz continuous, optimum quantization gives an upper bound for the worst case error depending linearly on Δ_k. This suboptimal rate is due to the fact that the class of Hölder continuous functions with Hölder exponent 1 is considerably larger than its subspace spanned by $\{g_u : u \in \mathbb{S}^1\}$.

The associated zonotope for quadrature rules can also be used in the context of a semi-randomized approach, which generalizes systematic random sampling designs. The idea of this design based approach is to evaluate the total projections of the randomly rotated set ϑA in k directions. In other words, given k vectors $v_1, \ldots, v_k \in \mathbb{S}^{d-1}$ and weights $c_1, \ldots, c_k$, the estimator for $S(A)$ is defined by

$$\widehat{S}_k^d(\vartheta A) = \frac{2}{\gamma_d} \sum_{i=1}^{k} c_i p_{\vartheta^{-1} v_i}, \tag{24}$$

where ϑ is a random rotation whose distribution is the normalized Haar measure on the compact group SO_d of proper rotations. Clearly, (24) defines a *random* variable and Crofton's formula implies that this variable is an unbiased estimator for $S(A)$. In particular, if $d = 2$ and the set $\{\pm v_1, \ldots, \pm v_k\}$ is equidistant in $\mathbb{S}^1$, the estimator $\widehat{S}_k^d(\vartheta A)$ in (24) is the one obtained from systematic random sampling. Moran (1966) considered this special case and gave worst case bounds for the variance of $\widehat{S}_k^d(\vartheta A)$. His approach allows a geometric interpretation which is not restricted to the planar setting: Let Z be, again, the zonotope associated to a fixed quadrature rule. From (11) and the unbiasedness of the estimator, we get

$$\begin{aligned}
\operatorname{Var}\left(\widehat{S}_k^d(\vartheta A)\right) &= \mathbb{E}_\vartheta \left(\widehat{S}_k^d(\vartheta A) - S(A)\right)^2 \\
&= \gamma_d^{-2} \mathbb{E}_\vartheta \left(\int_{\mathbb{S}^{d-1}} \left(h_Z(\vartheta u) - \gamma_d\right) S(A, du)\right)^2 \\
&= \gamma_d^{-2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \mathbb{E}_\vartheta \left(\left(h_Z(\vartheta u) - \gamma_d\right)\left(h_Z(\vartheta v) - \gamma_d\right)\right) S(A, du) S(A, dv).
\end{aligned}$$

Hölder's inequality implies that

$$\begin{aligned}
&\mathbb{E}_\vartheta\left(\left(h_Z\left(\vartheta u\right)-\gamma_d\right)\left(h_Z\left(\vartheta v\right)-\gamma_d\right)\right)\\
&\quad\leq\left(\mathbb{E}_\vartheta\left(h_Z\left(\vartheta u\right)-\gamma_d\right)^2\right)^{1/2}\left(\mathbb{E}_\vartheta\left(h_Z\left(\vartheta v\right)-\gamma_d\right)^2\right)^{1/2}\\
&\quad=\int_{\mathbb{S}^{d-1}}\left(h_Z\left(u\right)-\gamma_d\right)^2\frac{du}{\varpi_d},
\end{aligned}$$

where $\varpi_d = d\kappa_d$ is the surface area of $\mathbb{S}^{d-1}$. Hence

$$\operatorname{Var}\left(\widehat{S}_k^d\left(\vartheta A\right)\right)\leq\frac{S\left(A\right)^2}{\gamma_d^2\varpi_d}\delta_2^2\left(Z,\gamma_d B^d\right), \tag{25}$$

where $\delta_2\left(\cdot,\cdot\right)$ denotes the L_2-metric defined earlier. This inequality is sharp, as equality holds here for example whenever A is a circular disk of dimension $d-1$. The quadrature rule is exact when $A = B^d$ and thus $\widehat{S}_k^d\left(B^d\right) = \varpi_d$, which implies that $2\gamma_d$ is the mean width $\overline{w}\left(Z\right)$ of Z and $\gamma_d B^d$ is the *Steiner ball* of Z; see Schneider (1993, p. 353). Hence, the coefficient of variation $\sqrt{\operatorname{Var}(\widehat{S}_k^d\left(\vartheta A\right))}/S(A)$ is bounded from above by a multiple of the L_2-distance of Z to its Steiner ball. Restricting to the two-dimensional case, an easily computable upper bound of the right hand side of (25) is again given by a strengthened isoperimetric inequality. Using Fourier analysis, Groemer (1990) showed that

$$\frac{S^2\left(Z\right)}{4\pi}-V\left(Z\right)\geq\frac{3}{2}\delta_2^2\left(Z,\gamma_2 B^2\right), \tag{26}$$

and discussed the close relation of this result to Bonnesen's inequality (14). This can be used to estimate the right hand side of (25). As $S\left(Z\right)=4$ and $\gamma_2 = 2/\pi$, we obtain

$$\frac{\sqrt{\operatorname{Var}\left(\widehat{S}_k^2\left(\vartheta A\right)\right)}}{S\left(A\right)}\leq\sqrt{\frac{1}{3}-\frac{\pi}{12}V\left(Z\right)}.$$

As before, the area of Z is easily calculated yielding an upper bound for the coefficient of variation. This bound is not sharp any more, but becomes better with increasing k.

3 Error bounds for digital surface area estimators

In this section we consider digitizations of a topological polyconvex set $A\subset\mathbb{R}^d$ on rectangular grids and assume that the directions v_i are given by difference vectors of grid points. For example, we consider the usual 4- and 8-neighborhoods in the plane. We compute Voronoi- and Bonnesen-weights together with the associated in- and circumradii explicitly and give asymptotic error bounds for $\widehat{S}\left(A\right)$ for increasing resolution of the digitization.

To digitize A we consider the rectangular point grid $\mathbb{G} = \delta_1\mathbb{Z}\times\delta_2\mathbb{Z}\times\cdots\times\delta_{d-1}\mathbb{Z}\times\mathbb{Z}$, where $\delta_1,\ldots,\delta_{d-1}>0$ are the lattice distances in the directions of the

axes, and we used the lattice distance in the last coordinate direction as unit. A *grid cell* is any d-dimensional cuboid $z+[0,\delta_1]\times\cdots\times[0,\delta_{d-1}]\times[0,1]\,,z\in\mathbb{G}$. Motivated by design based stereology, we consider digitizations of A by a randomly translated grid. With the random variable ξ, uniformly distributed in an arbitrarily chosen grid cell, the random grid $\xi+\mathbb{G}$ is a stationary random closed set and is called a *stationary grid* in Kiderlen and Rataj (2005).

In order to increase resolution, we scale the grid by a factor $t>0$ and denote the digitization of A in the scaled grid $t\,(\xi+\mathbb{G})$ by $\Delta_t\,(A)$. Let Q be a non-empty compact set, called the *sampling element*. We assume that each grid point $x\in t(\xi+\mathbb{G})$ is the center of a small sampling window $x+tQ$, which can be thought of as a pixel or voxel. The *pixel digitization* consists of all grid points x for which this sampling window $x+tQ$ hits the set A. Hence $\Delta_t\,(A)=\left(A\oplus t\check{Q}\right)\cap t\,(\xi+\mathbb{G})$, where $\check{Q}$ is the reflection of Q at the origin. For $Q=\{o\}$ the pixel digitization reduces to the *Gauss digitization* (sometimes called *hit-or-miss digitization*) containing all points of the scaled grid in A. All results on error bounds in this section will be stated for the pixel digitization and therefore also hold for the Gauss digitization.

We fix a set A and estimate its surface area $S\,(A)$ from the information available in its digitization $\Delta_t\,(A)$ using a discretized Crofton formula, cf. Ohser and Mücklich (2000). We will focus on the asymptotic error of this estimator when the grid spacing gets finer, i.e. when the digitized set $\Delta_t\,(A)$ becomes a better approximation of the original set A. The function p_v given by (2) can easily be estimated from the digitized set $\Delta_t\,(A)$ by comparing the values of neighboring points, if v is a grid direction. For any vector $v\in\mathbb{G}\setminus\{o\}$ such an estimator is given by

$$\widehat{p}_v\,(t)=\frac{t^{d-1}}{\|v\|}\#\left\{x\in t\,(\xi+\mathbb{G}):x\in\Delta_t\,(A)\,,x+tv\in t\,(\xi+\mathbb{G})\setminus\Delta_t\,(A)\right\}.$$

This estimator counts the number of points x in the digitized set $\Delta_t\,(A)$ such that $x+tv$ does not lie in the digitization of A. From Kiderlen and Rataj (2005, Theorem 5) it follows that this estimator is asymptotically unbiased, i.e.

$$\lim_{t\searrow 0}\mathbb{E}\widehat{p}_v\,(t)=p_v. \tag{27}$$

Having chosen k vectors $v_1,\ldots,v_k\in\mathbb{G}$ and scalars $c_1,\ldots,c_k\geq 0$, one can define

$$\widehat{S}_k^d\,(A;t)=\frac{2}{\gamma_d}\sum_{i=1}^{k}c_i\widehat{p}_{v_i}\,(t)\,, \tag{28}$$

and it follows from (27) that $\widehat{S}_k^d\,(A;t)$ is an asymptotically unbiased estimator for $\widehat{S}_k^d\,(A)$, as $t\searrow 0$. Note that the estimator given by (28) can be calculated from the knowledge of the digitization $\Delta_t\,(A)$ of A alone. $\widehat{S}_k^d\,(A;t)$ behaves approximately like a discretized Crofton integral, when t is small. Thus, the methods and results in Section 2 can be applied to obtain asymptotic error bounds.

To illustrate this approach we discretize the Crofton integral using only directions parallel to the coordinate axes: in $\mathbb{R}^d$ we choose $2d$ directions $v_i=-v_{d+i}=\delta_i e_i$, $i=1,\ldots,d-1$ and $v_d=-v_{2d}=e_d$ where e_i denotes the ith unit vector. Due to symmetry, the weights leading to a minimal error interval are all equal, $c_i=$

$1/(2d), i = 1, \ldots, 2d$, and coincide with the Voronoi weights. The zonotope Z defined in (10) is given by $Z = \left[-\frac{1}{d}, \frac{1}{d}\right]^d$ and it has inradius $r = 1/d$ and circumradius $R = 1/\sqrt{d}$. Lemma 1 and the asymptotic unbiasedness of $\widehat{S}_{2d}^d(A;t)$ imply

$$\frac{1}{d\gamma_d} \leq \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_{2d}^d(A;t)}{S(A)} \leq \frac{1}{\sqrt{d}\gamma_d}.$$

In the planar case we have $\gamma_2 = 2/\pi$ and

$$0.785 \approx \frac{\pi}{4} \leq \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_4^2(A;t)}{S(A)} \leq \frac{\pi}{4}\sqrt{2} \approx 1.111,$$

which means the asymptotic relative error is 21.5 % in the worst case. In three dimensions, the asymptotic relative error is at most 33.3 % as $\gamma_3 = 1/2$ and

$$0.667 \approx \frac{2}{3} \leq \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_6^3(A;t)}{S(A)} \leq \frac{2}{3}\sqrt{3} \approx 1.155.$$

Due to Stirling's formula we have $\sqrt{d}\gamma_d \to \sqrt{2/\pi}$ as $d \to \infty$. As $\sqrt{d}\gamma_d$ is decreasing in d, we have

$$0 \leq \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_{2d}^d(A;t)}{S(A)} \leq \sqrt{\frac{\pi}{2}} \approx 1.253.$$

for all d, where $\sqrt{\pi/2}$ is the best upper bound that holds uniformly in d. We do not obtain a non-trivial uniform lower bound, as there are $d \in \mathbb{N}$ and sets A in $\mathbb{R}^d$ with $S(A) = 1$, but such that $\widehat{S}_{2d}^d(A)$ is arbitrarily close to 0. Due to the large worst case error even in low dimensions, the above choice of directions is not recommended for practical applications. Instead, a larger number of grid directions should be used. On the other hand, the estimator of $S(A)$ in (28) is based on asymptotic considerations and becomes less reliable when the lengths of the vectors v_i are large. One would therefore restrict to vectors with bounded length, i.e. vectors contained in a small circle around the origin. The vectors $v_1, \ldots, v_k$ are elements of the grid $\mathbb{G}$, so for computational reasons it is easier to choose them as directions in the cuboid $\Gamma_n^d(\delta) = [0, \delta_1(n-1)] \times \cdots \times [0, \delta_{d-1}(n-1)] \times [0, n-1] \cap \mathbb{G}$, where n is a fixed positive integer. To be specific, we choose $\{v_1, \ldots, v_k\}$ to be the set

$$\mathcal{V}_n^{(d)} := \left\{x - y : x, y \in \Gamma_n^d(\delta)\right\} \setminus \{o\}.$$

For $n = 2$, the most common choice in applications, we have

$$\mathcal{V}_2^{(d)} = \{-\delta_1, 0, \delta_1\} \times \cdots \times \{-\delta_{d-1}, 0, \delta_{d-1}\} \times \{-1, 0, 1\} \setminus \{o\}$$

and $k = \#\mathcal{V}_2^{(d)} = 3^d - 1$. We determine asymptotic worst case errors for $n = 2$ and $n = 3$ in the planar case and for $n = 2$ in dimension $d = 3$.

In the planar case for $n = 2$, the $k = 3^2 - 1 = 8$ directions are the vertices and diagonals of the cuboid $\Gamma_2^2(\delta) = [0, \delta_1] \times [0, 1] \cap \mathbb{G}$. They connect a given central pixel with all pixels in its 8-neighborhood. Both the Voronoi weights and the Bonnesen weights can be computed analytically. It turns out that the Voronoi weights and

the Bonnesen weights do not coincide, and therefore also the corresponding in- and circumradii are different. In the following let $\beta = \arccos(\delta_1/\sqrt{\delta_1^2+1})$. For the Voronoi case, the inradius $r_{\mathrm{Vor}}(\delta_1)$ and the circumradius $R_{\mathrm{Vor}}(\delta_1)$ are given by

$$r_{\mathrm{Vor}}(\delta_1) = \begin{cases} \frac{1}{\pi}\beta + \frac{\delta_1}{2\sqrt{\delta_1^2+1}}, & \text{if } \delta_1 \le 1, \\ \frac{1}{2} - \frac{1}{\pi}\beta + \frac{1}{2\sqrt{\delta_1^2+1}}, & \text{otherwise,} \end{cases}$$

and

$$R_{\mathrm{Vor}}(\delta_1) = \begin{cases} \frac{1}{\pi}\left[\beta^2 + \left(\frac{\pi}{2} - \beta + \frac{\pi}{2\sqrt{\delta_1^2+1}}\right)^2\right]^{1/2}, & \text{if } \delta_1 \le 1, \\ \frac{1}{\pi}\left[\left(\beta + \frac{\pi}{2}\frac{\delta_1}{\sqrt{\delta_1^2+1}}\right)^2 + \left(\frac{\pi}{2} - \beta\right)^2\right]^{1/2}, & \text{otherwise,} \end{cases}$$

respectively. In the Bonnesen case the inradius $r_{\mathrm{Bon}}(\delta_1)$ is given by

$$r_{\mathrm{Bon}}(\delta_1) = \delta_1\left(2\sqrt{\delta_1^2+1}\,(\delta_1+1) - 2\left(\delta_1^2+1\right)\right)^{-1}$$

and the circumradius $R_{\mathrm{Bon}}(\delta_1)$ is given by

$$R_{\mathrm{Bon}}(\delta_1) = \begin{cases} r_{\mathrm{Bon}}(\delta_1)\sqrt{2}\left(1 + \left(\frac{1}{\delta_1^2}+1\right)^{-1/2}\right)^{-1/2}, & \text{if } \delta_1 \le 1 \\ r_{\mathrm{Bon}}(\delta_1)\sqrt{2}\left(1 + (\delta_1^2+1)^{-1/2}\right)^{-1/2}, & \text{otherwise.} \end{cases}$$

Neither the Voronoi nor the Bonnesen weights are optimal in the sense that they minimize the length of the asymptotic error interval. The optimal weights were found by numerically solving a constrained minimization problem using MATLAB. It turns out that the Bonnesen weights are very close to the optimum as can be seen from Figure 4 which shows the thickness of the minimal annulus for Voronoi, Bonnesen, and optimized weights, respectively, depending on $\delta_1 \in [0,1]$. For $\delta_1 > 1$ the thickness of the minimal annulus equals the thickness of the minimal annulus for $1/\delta_1$. This corresponds to a rotation of the grid by 90°.

It is easy to see that in the case of a square grid ($\delta_1 = 1$) the width $R - r$ of the minimal annulus is minimal if and only if all weights are equal, $c_1 = \cdots = c_8 = 1/8$, a choice which coincides with the Voronoi and the Bonnesen weights. The corresponding zonotope Z is a regular octagon with side length $1/2$ and facets parallel to the vectors $v_1, \ldots, v_8$. Z has circumradius $R = \sqrt{4+2\sqrt{2}}/4$ and inradius $r = \left(1+\sqrt{2}\right)/4$. With $\gamma_2 = 2/\pi$ we get

$$0.948 \approx \frac{\pi}{8}\left(1+\sqrt{2}\right) \le \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_8^2(A;t)}{S(A)} \le \frac{\pi}{8}\sqrt{4+2\sqrt{2}} \approx 1.026.$$

This was also obtained in Kiderlen and Jensen (2003).

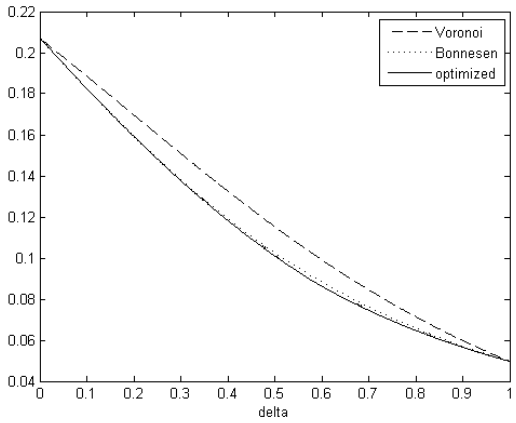

Figure 4: Thickness of the minimal annulus for eight directional vectors on a rectangular grid in 2D depending on δ_1 for Voronoi, Bonnesen, and optimized weights, respectively (see the text for details).

We consider also the case $n = 3$ in the plane, i.e. we use all 16 directions in the cuboid $\Gamma_3^2(\delta) = [0, 2\delta_1] \times [0, 2] \cap \mathbb{G}$. Now even in the special case when $\delta_1 = 1$ the 16 weights leading to a shortest asymptotic error interval are not equal. We refrain from explicitly stating the formulas for the in- and circumradii because of their complexity. The results are qualitatively similar to the case $n = 2$. The use of the Bonnesen weights yields a smaller thickness of the minimal annulus than the use of the Voronoi weights. The minimal thickness achieved with optimized weights is only slightly better than for Bonnesen weights; see Figure 5.

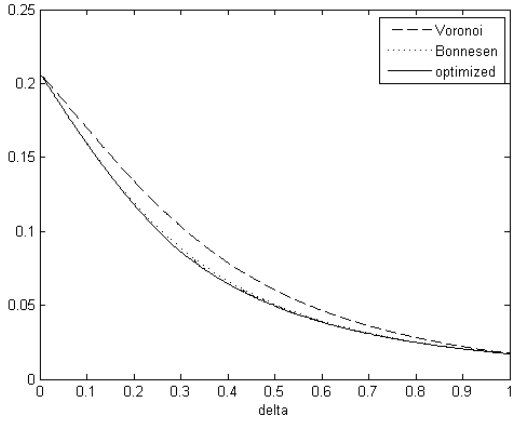

Figure 5: Thickness of the minimal annulus for 16 directional vectors on a rectangular grid in 2D depending on δ_1 for Voronoi, Bonnesen, and optimized weights, respectively (see the text for details).

In three dimensions, we restrict considerations to cubic grids ($\delta_1 = \delta_2 = 1$) and $n = 2$. The $k = 3^3 - 1 = 26$ directional vectors v_i consist of the 6 directions along the edges ($i = 1, \ldots, 6$), 12 diagonals of the faces ($i = 7, \ldots, 18$) and 8 spatial diagonals ($i = 19, \ldots, 26$) of the unit cube $[0, 1]^3$. Hence, the surface area estimator

is based on comparison of pixels with neighbors in the so-called 26-neighborhood. The Voronoi weights can be calculated as the relative sizes of the Voronoi cells on the unit sphere generated by $\{v_1/\|v_1\|, \ldots, v_k/\|v_k\|\}$. They are given by

$$c_i = \begin{cases} \frac{1}{2} - \frac{2}{\pi} \arccos\Big(\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}\sqrt{3-\sqrt{3}}} \sin\left(\frac{\pi}{8}\right) \Big) \approx 0.0457779, & \text{if } i = 1, \ldots, 6, \\ \frac{1}{2} - \frac{1}{\pi} \arccos\Big(\frac{\sqrt{6}-2}{2\sqrt{3-\sqrt{6}}} \sin\left(\frac{\pi}{8}\right) \Big) \approx 0.0369806, & \text{if } i = 7, \ldots, 18, \\ \frac{1}{2} - \frac{3}{2\pi} \arccos\Big(\frac{\left(2-\sqrt{3}\right)\left(2-\sqrt{6}\right)+2}{4\sqrt{3-\sqrt{3}}\sqrt{3-\sqrt{6}}} \Big) \approx 0.0351956, & \text{if } i = 19, \ldots, 26. \end{cases}$$

The zonotope Z is the convex hull of all points of the form $\sum_{i=1}^{26} \varepsilon_i v_i/\|v_i\|$, where $(\varepsilon_1, \ldots, \varepsilon_{26})$ runs through all vectors in $\{-1, 0, 1\}^{26}$. The *quickhull*-algorithm (Barber *et al.*, 1996) was used to find this convex hull. Z has 96 vertices, inradius $r_{\mathrm{Vor}} \approx 0.463312$, circumradius $R_{\mathrm{Vor}} \approx 0.511386$, and thickness of minimal annulus $T_{\mathrm{Vor}} \approx 0.0480748$. As $\gamma_3 = 1/2$ we obtain the rounded bounds

$$0.927 \leq \lim_{t \searrow 0} \frac{\mathbb{E}\widehat{S}_{26}^3(A;t)}{S(A)} \leq 1.023.$$

Although the definition of the Bonnesen weights was restricted to the planar case, it can naturally be generalized to higher dimensions. We will do so for comparison with the established Voronoi weights. Recall the construction for the Bonnesen weights in the plane for given directions $v_1, \ldots, v_k \in \mathbb{R}^2 \setminus \{o\}$. We defined $u_i = v_i/\|v_i\|$, $i = 1, \ldots, k$, and constructed the polygon $\widetilde{P}$ in (15) with outer unit vectors $\pm u_1, \ldots, \pm u_k$ circumscribing the unit ball. We then chose $\widetilde{c}_i$ proportional to the length of the edge of $\widetilde{P}$ with outer unit normal u_i. In higher dimensions, for given $v_1, \ldots, v_k \in \mathbb{R}^d \setminus \{o\}$, we set $u_i = v_i/\|v_i\|$, $i = 1, \ldots, k$, and

$$\widetilde{P} := \bigcap_{i=1}^{k} \left\{ x \in \mathbb{R}^d : |\langle u_i, x \rangle| \leq 1 \right\}$$

in complete analogy to (15). Hence $\widetilde{P}$ is the polytope circumscribing the unit ball with facet normals in $\{\pm u_1, \ldots, \pm u_k\}$. We then choose $\widetilde{c}_i$ proportional to the $(d-1)$-dimensional volume of the facet of $\widetilde{P}$ which has outer unit normal u_i. In the present three-dimensional example (with $\delta_1 = \delta_2 = 1$ and all directions in Γ_2^3) the Bonnesen weights were computed with *quickhull* and are given by

$$\widetilde{c}_i \approx \begin{cases} 0.0465894, & \text{if } i = 1, \ldots, 6, \\ 0.0367439, & \text{if } i = 7, \ldots, 18, \\ 0.0349421, & \text{if } i = 19, \ldots 26. \end{cases}$$

The associated inradius is $r_{\mathrm{Bon}} \approx 0.462424$, the circumradius is $R_{\mathrm{Bon}} \approx 0.511243$, and thickness of minimal annulus is $T_{\mathrm{Bon}} \approx 0.0488187$. This shows that Bonnesen weights are not better than Voronoi weights for $3^d - 1$ directions in dimension $d = 3$. But this is not surprising because they are based on Bonnesen's inequality (14)

which is valid only for two-dimensional convex bodies. In Section 4 we discuss how the approach from Section 2 could be extended to higher dimensions.

Finally, we show how the asymptotic relative error bounds in the case of a quadratic ($\delta_1 = 1$) planar grid depend on the choice of n. To do so, the symmetrized spread of the normalized vectors of $\mathcal{V}_n^{(2)}$ must be determined.

Lemma 6. *For $n \geq 2$ the symmetrized spread of $\left\{x/\left\|x\right\| : x \in \mathcal{V}_n^{(2)}\right\}$ is equal to*

$$d_n = \sqrt{2 - \sqrt{2\left(1 + \frac{n-1}{\sqrt{1+(n-1)^2}}\right)}} \leq \frac{1}{2\,(n-1)}. \tag{29}$$

Proof. Let Δ be the symmetrized spread of $D := \{x/\left\|x\right\| : x \in \mathcal{V}_n^{(2)}\}$ and set $m := n-1$. Clearly, the arc $C \subset \mathbb{S}^1$ in the first quadrant with endpoints $(1,0)^\top$, $(m^2+1)^{-1/2}\,(m,1)^\top \in D$ does not contain any other points in D and thus the spread Δ is at least the distance of the midpoint of C to one of its endpoints. Hence

$$\Delta \geq \sqrt{2\left(1-\cos\frac{\varphi}{2}\right)} = \sqrt{2\left(1-\sqrt{\frac{1+\cos\varphi}{2}}\right)} = d_n,$$

where $\varphi = \arccos\left(m/\sqrt{1+m^2}\right)$ is the length of C. This interpretation of d_n also shows the inequality in (29), as d_n cannot be larger then half the distance of $(1,0)^\top$ from $(1,1/m)^\top$. To show that $\Delta \leq d_n$, let $v, v' \in D$ two points such that the sub-arc of $\mathbb{S}^1$ connecting them does not contain any other points of D. Using reflections and translations leaving $\mathbb{Z}^2$ invariant, we may assume that $v = x/\left\|x\right\|$ and $v' = x'/\left\|x'\right\|$ where $x, x' \in \mathcal{V}_n^{(2)}$ and the angles they form with the x-axis are at most $\pi/4$. We refer to Figure 6 for a sketch of the situation. The cone between the rays spanned by x and x' cannot contain any other points of $\mathcal{V}_n^{(2)}$ in its interior. Let y (y') denote the point in $\mathcal{V}_n^{(2)} \cap \{(m,t) : t \geq 0\}$ with largest (smallest) second coordinate below (above) this cone. Then the length of the arc in $\mathbb{S}^1$ with endpoints v and v' is at most the length of the arc $C' \subset \mathbb{S}^1$ with endpoints $y/\left\|y\right\|$ and $y'/\left\|y'\right\|$. The segment $[y, y']$ does not contain any points of $\mathcal{V}_n^{(2)}$, so y and y' are distance one apart, and the length of C' is bounded from above by the length of C. Here we use the fact that among all unit intervals in $\{(m,t)^\top : t \geq 0\}$, the interval $[(m,0)^\top, (m,1)^\top]$ has the largest gnomonic projection. This gives $\Delta \leq d_n$, as v and v' where arbitrary in D. □

In view of Theorem 4, Lemma 6 yields an estimate for the asymptotic relative error using the Bonnesen-weights $c_i = \widetilde{c}_i$ whenever $n \geq 2$.

Theorem 7. *Let $A \subset \mathbb{R}^2$ be a topologically regular, polyconvex set with positive perimeter, $n \geq 2$, and $\delta_1 = 1$. Let $\widehat{S}_k^d(A;t)$ be defined by (28), where $\{v_1, \ldots, v_k\} = \mathcal{V}_n^{(2)}$ and $c_i = \widetilde{c}_i$, $i = 1, \ldots, k$, are the Bonnesen weights. Then the asymptotic relative mean error obeys*

$$\lim_{t \searrow 0} \left| \frac{\mathbb{E}\widehat{S}_k^d(A;t) - S(A)}{S(A)} \right| \leq \frac{\pi^2}{12}\,(n-1)^{-2}.$$

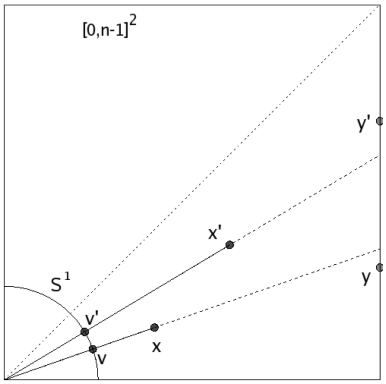

Figure 6: Construction to determine the symmetrized spread in Lemma 6: the points x and x' in $\mathcal{V}_n^{(2)}$ are contained in the cube $[0, n-1]^2$ and their normalizations v, v' in $\mathbb{S}^1$.

4 Extension to higher dimensions

Large parts of the present worst case analysis for quadrature rules, including the use of an associated zonotope Z, is not restricted to the two-dimensional setting. In order to find an easily accessible upper bound for the width of the minimal annulus, an extension of Bonnesen's and Groemer's refined isoperimetric inequalities (14) and (26) to higher dimensions (which is known) is not suitable, as it involves the surface area of Z. Instead, a strengthened version of Uhrysohn's inequality is appropriate. It reads

$$\left(\frac{\overline{w}\,(Z)}{\overline{w}\,(B^d)}\right)^d - \frac{V_d\,(Z)}{V_d\,(B^d)} \geq c_d\,(Z)\,\delta_2\left(Z, \gamma_d B^d\right)^2. \tag{30}$$

Here c_d is an explicitly known constant, depending on d, the mean width $\overline{w}\,(Z)$ of Z, and the second intrinsic volume of Z. This inequality is a special case of a whole family of geometric inequalities derived by Groemer and Schneider (1991), who also showed that it implies the Bonnesen type inequality

$$\left(\frac{\overline{w}\,(Z)}{\overline{w}\,(B^d)}\right)^d - \frac{V_d\,(Z)}{V_d\,(B^d)} \geq c'_d\,(Z)\,(R-r)^{(d+3)/2}. \tag{31}$$

The constant c'_d, again, depends on d, $\overline{w}\,(Z)$ and the second intrinsic volume of Z. For $d = 2$ inequality (30) coincides with (26), and (31) is of the same form as (14), but with a weaker exponent. However, these inequalities seem to be of limited value for applications, as it appears that (31) does not exhibit the optimal exponent, and it is not easy to determine the zonotope with given mean width which minimizes the left hand side of (30) or (31) when $d > 2$.

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