THIELE CENTRE

FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

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Research Report

No. 07 | June 2009

MULTIVARIATE SUPOU PROCESSES

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Abstract

Univariate superpositions of Ornstein-Uhlenbeck (OU) type processes, called supOU processes, provide a class of continuous time processes capable of exhibiting long memory behaviour. This paper introduces multivariate supOU processes and gives conditions for their existence and finiteness of moments. Moreover, the second order moment structure is explicitly calculated, and examples exhibit the possibility of long range dependence.

Our supOU processes are defined via homogeneous and factorisable Lévy bases. We show that the behaviour of supOU processes is particularly nice when the mean reversion parameter is restricted to normal matrices and especially to strictly negative definite ones.

For finite variation Lévy bases we are able to give conditions for supOU processes to have locally bounded càdlàg paths of finite variation and to show an analogue of the stochastic differential equation of OU type processes, which has been suggested in [2] in the univariate case. Finally, as an important special case, we introduce positive semi-definite supOU processes.

AMS 2000 subject classifications: Primary 60G10,60H20; secondary 60E07, 60G51, 60G57

Keywords and phrases: Lévy bases, long memory, Ornstein-Uhlenbeck type processes, normal matrices, positive semi-definite stochastic processes, second order moment structure, stochastic differential equation

1 Introduction

Lévy-driven Ornstein-Uhlenbeck (OU) type processes are extensively used in applications as elements in continuous time models for observed time series. One area where they are often applied is mathematical finance (see e.g. [10]), especially in the OU type stochastic volatility model of [6]. An OU type process is given as the solution of a stochastic differential equation of the form

$$dX_t = -aX_t dt + dL_t \tag{1.1}$$

with L being a Lévy process (see e.g. [30] for a comprehensive introduction) and $a \in \mathbb{R}$. Typically, one is interested mainly in stationary solutions of (1.1). Provided $a > 0$ and $E(\ln(|L| \vee 1)) < \infty$, the SDE (1.1) has a unique stationary solution given by

$$X_t = \int_{-\infty}^{t} e^{-a(t-s)} dL_s.$$

However, in many applications the dependence structure exhibited by empirical data is found to be not in good accordance with that of OU type processes which have autocorrelation functions of the form e^{-ah} for positive lags h. In many data sets a more complex and often a (quasi)long memory behaviour of the autocorrelation function is encountered. OU type processes could be replaced by fractional OU type processes (see [21] or [22], for instance) to have long memory effects included in the model. However, in this case many desirable properties are lost, in particular fractional OU type processes do no longer have jumps. An alternative to obtain long memory from OU type processes and still to have jumps is to add up countably many independent OU type processes, i.e.

$$X_t = \sum_{k=1}^{\infty} w_i \int_{-\infty}^{t} e^{-a_i(t-s)} dL_{i,s}$$

with independent identically distributed Lévy processes $(L_i)_{i\in\mathbb{N}}$ and appropriate $a_i > 0$, $w_i > 0$ with $\sum_{i=1}^{\infty} w_i = 1$. Intuitively we can likewise "add" (i.e. integrate) up independent OU type processes with all parameter values $a > 0$ possible. The resulting processes are called supOU processes and have been introduced in [2] where it has also been established that they may exhibit long range dependence. For a comprehensive treatment regarding the theory and use of univariate supOU processes in finance we refer to [7].

So far supOU processes have only been considered in the univariate case. However, in many applications it is crucial to model several time series with a joint model and so flexible multivariate models are important. Therefore, in this paper we introduce and study multivariate supOU processes. Due to the appearance of matrices and the related peculiarities our theory is not a straightforward extension of the univariate results. Multivariate (d-dimensional) OU type processes (see e.g. [31] or [17]) are defined as the solutions of SDEs of the form

$$dX_t = AX_t dt + dL_t \tag{1.2}$$

with L a d-dimensional Lévy process and A a $d \times d$ matrix. Provided all eigenvalues of A have strictly negative real part and $E(\ln(\|L\| \vee 1)) < \infty$, we have again a unique stationary solution given by

$$X_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s.$$

Intuitively our multivariate supOU processes are obtained by "adding up" independent OU type processes with all possible parameters A, i.e. we consider all $d \times d$ matrices A with eigenvalues of strictly negative real parts. It turns out later on that the behaviour of supOU becomes easier when we restrict A to come only from a nice subset, like the negative definite matrices.

The remainder of this paper is structured as follows. The next section starts with a brief overview of important notation and conventions used in this paper and is followed in Section 2.2 by a comprehensive introduction into Lévy bases and the related integration theory, which will be needed to define supOU processes. In Section 3 we first define multivariate supOU processes and provide existence conditions in Section 3.1. Thereafter, we discuss the existence of moments and derive the second order structure. For the finite variation case we show important path properties in Section 3.3. Besides establishing that we have càdlàg paths of bounded variation, we give an analogue of the stochastic differential equation (1.2) for supOU processes and its proof. In particular, this proves a conjecture in [2], which has not yet been shown in any non-degenerate set-up. We conclude that section with several examples illustrating the behaviour and properties of supOU processes and showing that they may exhibit long memory. Finally, in Section 4 we use our results to define positive semi-definite supOU processes and analyse their properties. These processes are important for applications like stochastic volatility modelling, since they may be used to describe the stochastic dynamics of a latent covariance matrix.

2 Background and preliminaries

2.1 Notation

We denote the set of real $m \times n$ matrices by $M_{m,n}(\mathbb{R})$. If $m = n$, we simply write $M_n(\mathbb{R})$ and denote the group of invertible $n \times n$ matrices by $GL_n(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_n$, the (closed) positive semi-definite cone by $\mathbb{S}_n^+$ and the open positive definite cone by $\mathbb{S}_n^{++}$ (likewise $\mathbb{S}_n^{--}$ are the strictly negative definite matrices etc.). I_n stands for the $n \times n$ identity matrix. The tensor (Kronecker) product of two matrices A, B is written as $A \otimes B$. vec denotes the well-known vectorisation operator that maps the $n \times n$ matrices to $\mathbb{R}^{n^2}$ by stacking the columns of the matrices below one another. For more information regarding the tensor product and vec operator we refer to [15, Chapter 4]. The spectrum of a matrix is denoted by $\sigma(\cdot)$. Finally, A^* is the transpose (adjoint) of a matrix $A \in M_{m,n}(\mathbb{R})$ and A_{ij} stands for the entry of A in the ith row and jth column.

Norms of vectors or matrices are denoted by $\|\cdot\|$. If the norm is not further specified, then it is understood that we take the Euclidean norm or its induced operator norm, respectively. However, due to the equivalence of all norms none of our results really depends on the choice of norms.

For a complex number z we denote by $\Re(z)$ its real part and by $\Im(z)$ its imaginary part. Moreover, the indicator function of a set A is written 1_A.

A mapping $f : V \to W$ is said to be $\mathscr{V}$-$\mathscr{W}$-measurable, if it is measurable when the σ-algebra $\mathscr{V}$ is used on the domain V and the σ-algebra $\mathscr{W}$ is used on the range W. The Borel σ-algebras are denoted by $\mathscr{B}(\cdot)$ and λ typically stands for the Lebesgue measure which in vector or matrix spaces is understood to be defined as the product of the coordinatewise Lebesgue measures.

Throughout we assume that all random variables and processes are defined on a given complete probability space $(\Omega, \mathscr{F}, P)$.

Furthermore, we employ an intuitive notation with respect to the (stochastic) integration with matrix-valued integrators referring to any of the standard texts (e.g. [28]) for a comprehensive treatment of the theory of stochastic integration. Let $(A_t)_{t\in\mathbb{R}^+}$ in $M_{m,n}(\mathbb{R})$, $(B_t)_{t\in\mathbb{R}^+}$ in $M_{r,s}(\mathbb{R})$ be càdlàg and adapted processes and $(L_t)_{t\in\mathbb{R}^+}$ in $M_{n,r}(\mathbb{R})$ be a semi-martingale. Then we denote by $\int_0^t A_{s-} dL_s B_{s-}$ the matrix C_t in $M_{m,s}(\mathbb{R})$ which has ij-th element $C_{ij,t} = \sum_{k=1}^n \sum_{l=1}^r \int_0^t A_{ik,s-} B_{lj,s-} dL_{kl,s}$.

Equivalently such an integral can be understood in the sense of [24, 23] by identifying it with the integral $\int_0^t \mathbf{A}_{s-}dL_s$ with $\mathbf{A}_t$ being for each fixed t the linear operator $M_{n,r}(\mathbb{R}) \to M_{m,s}(\mathbb{R})$, $X \mapsto A_t X B_t$. Analogous notation is used in the context of integrals with respect to random measures.

Finally, integrals of the form $\int_A \int_B f(x,y)m(dx,dy)$ are understood to be over the set A in x and over B in y.

2.2 Lévy bases

To lay the foundations for the definition of vector-valued supOU processes, we give now a summary of Lévy bases and the related integration theory. In this context recall that a d-dimensional Lévy process can be understood as an $\mathbb{R}^d$-valued random measure on the real numbers. If $L = (L_t)_{t\in\mathbb{R}}$ is a d-dimensional Lévy process, this measure is simply determined by $L((a,b]) = L(b) - L(a)$ for all $a, b \in \mathbb{R}, a < b$.

Define now $M_d^- := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$ and $\mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ to be the bounded Borel sets of $M_d^- \times \mathbb{R}$. Note that M_d^- is obviously a cone, but not a convex one (cf. [14], for instance). Moreover, we obviously have $\overline{M_d^-} = \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0] + i\mathbb{R}\}$.

Definition 2.1. *A family $\Lambda = \{\Lambda(B) : B \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)\}$ of $\mathbb{R}^d$-valued random variables is called an* $\mathbb{R}^d$-valued Lévy basis *on $M_d^- \times \mathbb{R}$ if:*

(*a*) *the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$,*

(*b*) *for any $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, \ldots, B_n \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ the random variables $\Lambda(B_1), \ldots, \Lambda(B_n)$ are independent and*

(*c*) *for any pairwise disjoint sets $B_i \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ with $i \in \mathbb{N}$ which satisfies $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$ the series $\sum_{n=1}^\infty \Lambda(B_n)$ converges almost surely and $\Lambda\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n=1}^\infty \Lambda(B_n)$.*

In the literature Lévy bases are also often called infinitely divisible independently scattered random measures (abbreviated i.d.i.s.r.m.) instead.

In the following we will only consider Lévy bases, which are homogeneous (in time) and factorisable (into the effects of one underlying infinitely divisible distribution and a probability distribution on M_d^-), i.e. their characteristic function is of the form

$$E\left(\exp(iu^*\Lambda(B))\right) = \exp\left(\varphi(u)\Pi(B)\right) \tag{2.1}$$

for all $u \in \mathbb{R}^d$ and $B \in \mathscr{B}_b\left(M_d^-(\mathbb{R}) \times \mathbb{R}\right)$. Here $\Pi = \pi \times \lambda$ is the product of a probability measure π on $M_d^-(\mathbb{R})$ and the Lebesgue measure λ on $\mathbb{R}$ and

$$\varphi(u) = iu^*\gamma - \frac{1}{2}u^*\Sigma u + \int_{\mathbb{R}^d} \left(e^{iu^*x} - 1 - iu^*x 1_{[-1,1]}(\|x\|)\right)\nu(dx) \tag{2.2}$$

is the cumulant transform of an infinitely divisible distribution on $\mathbb{R}^d$ with Lévy-Khintchine triplet (γ, Σ, ν), i.e. $\gamma \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}_d^+$ and ν is a Lévy measure – a Borel measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty$. The quadruple $(\gamma, \Sigma, \nu, \pi)$ determines the distribution of the Lévy basis completely and is henceforth referred to as the "generating quadruple" (cf. [11]).

The Lévy process L defined by

$$L_t = \Lambda(M_d^- \times (0,t]) \text{ and } L_{-t} = \Lambda(M_d^- \times (-t,0)) \text{ for } t \in \mathbb{R}^+$$

has characteristic triplet (γ, Σ, ν) and is called "the underlying Lévy process".

For more information on $\mathbb{R}^d$-valued Lévy bases see [29] and [26].

A Lévy basis has a Lévy-Itô decomposition.

Theorem 2.2 (Lévy-Itô decomposition)**.** *Let Λ be a homogeneous and factorisable $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$. Then there exists a modification $\tilde{\Lambda}$ of Λ which is also a Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ such that there exists an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}^G$ on $M_d^- \times \mathbb{R}$ with generating quadruple $(0, \Sigma, 0, \pi)$ and an independent Poisson random measure μ on $(\mathbb{R}^d \times M_d^- \times \mathbb{R}, \mathscr{B}(\mathbb{R}^d \times M_d^- \times \mathbb{R}))$ with intensity measure $\nu \times \pi \times \lambda$ which satisfy*

$$\begin{aligned}\tilde{\Lambda}(B) = \gamma(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\|x\|\le 1}\int_B x(\mu(dx, dA, ds) - ds\pi(dA)\nu(dx)) \\ + \int_{\|x\|>1}\int_B x\mu(dx, dA, ds) \qquad (2.3)\end{aligned}$$

for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ and all $\omega \in \Omega$.

Provided $\int_{\|x\|\le 1} \|x\|\nu(dx) < \infty$, it holds that

$$\tilde{\Lambda}(B) = \gamma_0(\pi \times \lambda)(B) + \tilde{\Lambda}^G(B) + \int_{\mathbb{R}^d}\int_B x\mu(dx, dA, ds)$$

for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$ with

$$\gamma_0 := \gamma - \int_{\|x\|\le 1} x\nu(dx). \qquad (2.4)$$

Furthermore, the integral with respect to μ exists as a Lebesgue integral for all $\omega \in \Omega$.

Here an $\mathbb{R}^d$-valued Lévy basis $\tilde{\Lambda}$ on $M_d^- \times \mathbb{R}$ is called a *modification* of a Lévy basis Λ if $\tilde{\Lambda}(B) = \Lambda(B)$ a.s. for all $B \in \mathscr{B}_b(M_d^- \times \mathbb{R})$. For the necessary background on the integration with respect to Poisson random measures we refer to [16, Section 2.1] and [18, Lemma 12.13].

Proof. This follows immediately from [26, Theorem 4.5], because the control measure m is given by $m(B) = \left(\|\gamma\| + \mathrm{tr}(\Sigma) + \int_{\mathbb{R}^d}(1 \wedge \|x\|^2)\nu(dx)\right)(\pi \times \lambda)(B)$ which is trivially continuous due to the presence of the Lebesgue measure. The second part is an immediate consequence, as no compensation for the small jumps is needed if $\int_{\|x\|\le 1} \|x\|\nu(dx) < \infty$. □

From now on we assume without loss of generality that all Lévy bases are such that they have the Lévy-Itô decomposition (2.3).

In the following we need to define integrals of deterministic functions with respect to a Lévy basis Λ. Following [29], for simple functions $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$,

$$f(x) = \sum_{i=1}^{m} a_i 1_{B_i}(x)$$

with $m \in \mathbb{N}$, $a_i \in M_d(\mathbb{R})$ and $B_i \in \mathscr{B}_b\left(M_d^- \times \mathbb{R}\right)$, and for every $B \in \mathscr{B}\left(M_d^- \times \mathbb{R}\right)$ we define the integral

$$\int_B f(x)\Lambda(dx) = \sum_{i=1}^m a_i \Lambda(B \cap B_i).$$

A $\mathscr{B}\left(M_d^- \times \mathbb{R}\right)$-$\mathscr{B}\left(M_d(\mathbb{R})\right)$-measurable function $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ is said to be Λ-*integrable* if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ Lebesgue almost everywhere and for all $B \in \mathscr{B}\left(M_d^- \times \mathbb{R}\right)$ the sequence $\int_B f_n(x)\Lambda(dx)$ converges in probability. For Λ-integrable f we set $\int_B f(x)\Lambda(dx) = \operatorname{plim}_{n\to\infty} \int_B f_n(x)\Lambda(dx)$. As in [29] well-definedness of the integral is ensured by [32].

The following result is a straightforward generalisation of [29, Propositions 2.6, 2.7] to $\mathbb{R}^d$-valued Lévy bases and follows along the same lines.

Proposition 2.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form (2.1) and $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ a $\mathscr{B}\left(M_d^- \times \mathbb{R}\right)$-$\mathscr{B}\left(M_d(\mathbb{R})\right)$-measurable function. Then f is Λ-integrable if and only if*

$$\int_{M_d^-}\int_{\mathbb{R}}\Big\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x\left(1_{[-1,1]}(\|f(A,s)x\|) - 1_{[-1,1]}(\|x\|)\right)\nu(dx)\Big\| ds\pi(dA) < \infty, \quad (2.5)$$

$$\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\Sigma f(A,s)^*\| ds\pi(dA) < \infty, \quad (2.6)$$

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} (1 \wedge \|f(A,s)x\|^2)\nu(dx)ds\pi(dA) < \infty. \quad (2.7)$$

If f is Λ-integrable the distribution of $\int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic triplet $(\gamma_{\text{int}}, \Sigma_{\text{int}}, \nu_{\text{int}})$ given by

$$\gamma_{\text{int}} = \int_{M_d^-}\int_{\mathbb{R}}\Big(f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x\left(1_{[-1,1]}(\|f(A,s)x\|) - 1_{[-1,1]}(\|x\|)\right)\nu(dx)\Big) ds\pi(dA), \quad (2.8)$$

$$\Sigma_{\text{int}} = \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Sigma f(A,s)^* ds\pi(dA), \quad (2.9)$$

$$\nu_{\text{int}}(B) = \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_B(f(A,s)x)\nu(dx)ds\pi(dA) \textit{ for all Borel sets } B \subseteq \mathbb{R}^d. \quad (2.10)$$

When the underlying Lévy process has finite variation we can do ω-wise Lebesgue integration.

Proposition 2.4. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic quadruple $(\gamma, 0, \nu, \pi)$ satisfying $\int_{\|x\|\le 1}\|x\|\nu(dx) < \infty$ and define γ_0 as in (2.4) , i.e. $\varphi(u) = iu^*\gamma_0 + \int_{\mathbb{R}^d}\left(e^{iu^*x} - 1\right)\nu(dx)$. Additionally, let $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ be a*

$\mathscr{B}\left(M_d^- \times \mathbb{R}\right)$-$\mathscr{B}\left(M_d(\mathbb{R})\right)$-measurable function satisfying

$$\int_{M_d^-}\int_{\mathbb{R}} \|f(A,s)\| ds\pi(dA) < \infty, \tag{2.11}$$

$$\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} (1 \wedge \|f(A,s)x\|)\nu(dx)ds\pi(dA) < \infty. \tag{2.12}$$

Then

$$\begin{aligned}\int_{M_d^-}\int_{\mathbb{R}} & f(A,s)\Lambda(dA,ds) \\ &= \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\gamma_0 ds\pi(dA) + \int_{\mathbb{R}^d}\int_{M_d^-}\int_{\mathbb{R}} f(A,s)x\mu(dx,dA,ds)\end{aligned} \tag{2.13}$$

and the right hand side is a Lebesgue integral for every $\omega \in \Omega$ (Conditions (2.11) *and* (2.12) *are also necessary for this). Moreover, the distribution of*

$$\int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Lambda(dA,ds)$$

is infinitely divisible with characteristic function

$$\begin{aligned} E\Big(\exp\Big(iu^*\int_{M_d^-}\int_{\mathbb{R}} & f(A,s)\Lambda(dA,ds)\Big)\Big) \\ &= \exp\Big(iu^*\gamma_{\mathrm{int},0} + \int_{\mathbb{R}^d}\left(e^{iu^*x}-1\right)\nu_{\mathrm{int}}(dx)\Big), \quad u \in \mathbb{R}^d,\end{aligned}$$

where

$$\gamma_{\mathrm{int},0} = \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\gamma_0 ds\pi(dA), \tag{2.14}$$

$$\nu_{\mathrm{int}}(B) = \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} 1_B(f(A,s)x)\nu(dx)ds\pi(dA) \text{ for all Borel sets } B \subseteq \mathbb{R}^d. \tag{2.15}$$

Proof. Follows from the Lévy-Itô decomposition and the usual integration theory with respect to Poisson random measures (see [18, Lemma 12.13]). □

Remark 2.5. *All results of this section remain valid when replacing M_d^- with $M_k(\mathbb{R})$, $k \in \mathbb{N}$, or any measurable subset of a finite dimensional real vector space and when considering integration of functions $f : M_k(\mathbb{R}) \times \mathbb{R} \to M_{m,d}(\mathbb{R})$. We decided to state all our results with M_d^- as this set will be used mainly in the following and it reduces the notational burden.*

3 Multidimensional supOU processes

In this section we will introduce supOU processes taking values in $\mathbb{R}^d$ with $d \in \mathbb{N}$ and analyse their properties. This extends to a multivariate setting the theory of univariate supOU processes as introduced in [2] and studied further e.g. in [12].

3.1 Definition and existence

We define a d-dimensional supOU process as a process of the form (3.4) below.

Theorem 3.1. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ satisfying*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty \tag{3.1}$$

and assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+ \backslash \{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that:

$$\|e^{As}\| \leq \kappa(A) e^{-\rho(A)s} \,\forall\, s \in \mathbb{R}^+, \quad \pi - \textit{almost surely}, \tag{3.2}$$

and

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty. \tag{3.3}$$

Then the process $(X_t)_{t\in\mathbb{R}}$ given by

$$X_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds) \tag{3.4}$$

is well-defined for all $t \in \mathbb{R}$ and stationary. The distribution of X_t is infinitely divisible with characteristic triplet $(\gamma_X, \Sigma_X, \nu_X)$ given by

$$\gamma_X = \int_{M_d^-} \int_{\mathbb{R}^+} \Big(e^{As}\gamma + \int_{\mathbb{R}^d} e^{As} x \left(1_{[-1,1]}(\|e^{As}x\|) - 1_{[-1,1]}(\|x\|) \right) \nu(dx) \Big) ds \pi(dA), \tag{3.5}$$

$$\Sigma_X = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As} \Sigma e^{A^* s} ds \pi(dA), \tag{3.6}$$

$$\nu_X(B) = \int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} 1_B(e^{As}x) \nu(dx) ds \pi(dA) \textit{ for all Borel sets } B \subseteq \mathbb{R}^d. \tag{3.7}$$

Proof. The stationarity is obvious once the well-definedness is shown. Using Proposition 2.3 it follows that necessary and sufficient conditions for the integral to exist are given by

$$\int_{M_d^-} \int_{\mathbb{R}^+} \left\| e^{As}\gamma + \int_{\mathbb{R}^d} e^{As} x \left(1_{[-1,1]}(\|e^{As}x\|) - 1_{[-1,1]}(\|x\|) \right) \nu(dx) \right\| ds \pi(dA) < \infty, \tag{3.8}$$

$$\int_{M_d^-} \int_{\mathbb{R}^+} \|e^{As} \Sigma e^{A^* s}\| ds \pi(dA) < \infty, \tag{3.9}$$

$$\int_{M_d^-} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} (1 \wedge \|e^{As}x\|^2) \nu(dx) ds \pi(dA) < \infty. \tag{3.10}$$

First we show (3.10):

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}(1\wedge\|e^{As}x\|^2)\nu(dx)ds\pi(dA)\\
&\quad\le\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}(1\wedge\kappa(A)^2e^{-2\rho(A)s}\|x\|^2)\nu(dx)ds\pi(dA)\\
&\quad=\int_{M_d^-}\int_{\|x\|>1/\kappa(A)}\frac{\ln(\kappa(A)\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad+\int_{M_d^-}\int_{\|x\|\le 1/\kappa(A)}\frac{\kappa(A)^2\|x\|^2}{2\rho(A)}\nu(dx)\pi(dA).
\end{aligned}$$

The finiteness of the first integral follows from (3.1), (3.3), $\kappa(A)\ge 1$ and ν being a Lévy measure, which imply

$$\begin{aligned}
&\int_{M_d^-}\int_{\|x\|>1/\kappa(A)}\frac{\ln(\kappa(A)\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\quad\le\int_{M_d^-}\int_{\|x\|>1}\frac{\ln(\kappa(A))+\ln(\|x\|)+1/2}{\rho(A)}\nu(dx)\pi(dA)\\
&\qquad+\int_{M_d^-}\int_{\|x\|\le 1}\frac{3\kappa(A)^2\|x\|^2}{2\rho(A)}\nu(dx)\pi(dA)\\
&\quad=\int_{M_d^-}\frac{\ln(\kappa(A))}{\rho(A)}\pi(dA)\int_{\|x\|>1}\nu(dx)\\
&\qquad+\int_{M_d^-}\frac{1}{\rho(A)}\pi(dA)\int_{\|x\|>1}(\ln(\|x\|)+1/2)\nu(dx)\\
&\qquad+\int_{M_d^-}\frac{3\kappa(A)^2}{2\rho(A)}\pi(dA)\int_{\|x\|\le 1}\|x\|^2\nu(dx)<\infty.
\end{aligned}$$

Likewise the finiteness of the second integral is implied by (3.3), $\kappa(A)\ge 1$ and $\int_{\|x\|\le 1}\|x\|^2\nu(dx)<\infty$, as ν is a Lévy measure.

Next (3.9) follows from

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\|e^{As}\Sigma e^{A^*s}\|ds\pi(dA)\\
&\quad\le\|\Sigma\|\int_{M_d^-}\int_{\mathbb{R}^+}\kappa(A)^2e^{-2\rho(A)s}ds\pi(dA)=\|\Sigma\|\int_{M_d^-}\frac{\kappa(A)^2}{2\rho(A)}\pi(dA)
\end{aligned}$$

and (3.3).

Turning to (3.8) we have from (3.3) that:

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\left\|e^{As}\gamma\right\|ds\pi(dA)\\
&\quad\le\|\gamma\|\int_{M_d^-}\int_{\mathbb{R}^+}\kappa(A)e^{-\rho(A)s}ds\pi(dA)=\|\gamma\|\int_{M_d^-}\frac{\kappa(A)}{\rho(A)}\pi(dA)<\infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}^+}\left\|\int_{\mathbb{R}^d} e^{As}x\left(1_{[-1,1]}(\|e^{As}x\|)-1_{[-1,1]}(\|x\|)\right)\nu(dx)\right\|ds\pi(dA)\\
&\quad\leq \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\leq 1,\|e^{As}x\|\geq 1}\left\|e^{As}x\right\|\nu(dx)ds\pi(dA)\\
&\qquad+\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\geq 1,\|e^{As}x\|\leq 1}\left\|e^{As}x\right\|\nu(dx)ds\pi(dA)\\
&\quad\leq \int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\leq 1,\|e^{As}x\|\geq 1}\left\|e^{As}x\right\|^2\nu(dx)ds\pi(dA)\\
&\qquad+\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\in\left(1,e^{\rho(A)s/2}\right)}\|x\|\kappa(A)e^{-\rho(A)s}\nu(dx)ds\pi(dA)\\
&\qquad+\int_{M_d^-}\int_{\mathbb{R}^+}\int_{\|x\|\geq e^{\rho(A)s/2}}\nu(dx)ds\pi(dA)\\
&\quad\leq \int_{\|x\|\leq 1}\|x\|^2\nu(dx)\int_{M_d^-}\frac{\kappa(A)^2}{2\rho(A)}\pi(dA)+\int_{M_d^-}\frac{2\kappa(A)}{\rho(A)}\pi(dA)\int_{\|x\|>1}\nu(dx)\\
&\qquad+\int_{M_d^-}\frac{2}{\rho(A)}\pi(dA)\int_{\|x\|>1}\ln(\|x\|)\nu(dx)<\infty
\end{aligned}$$

with the finiteness following from (3.1), (3.3) and ν being a Lévy measure.

That the distribution of X_t is infinitely divisible and has the stated characteristic triplet follows now immediately from Proposition 2.3. □

Remark 3.2. *(i) The necessary and sufficient conditions for the existence of the multivariate supOU process X are* (3.8), (3.9), (3.10). *However, as they are obviously very intricate to check in concrete situations, it seems to be appropriate to replace them by the sufficient conditions* (3.1), (3.2), (3.3). *One particular advantage of these conditions is that they involve only integrals with respect to either ν or π, but not with respect to both.*

(ii) Note also that for $d=1$ the conditions above become the necessary and sufficient conditions of [12], as we can then take $\kappa(A)=1$ and $\rho(A)=-A$.

(iii) By looking at the Jordan decomposition one can see that pointwise there is for any $A\in M_d^-$ a constant $\kappa\in[1,\infty)$ and a $\rho\in(0,-\max(\Re(\sigma(A)))]$ such that $\|e^{As}\|\leq\kappa e^{-\rho s}$ for all $s\in\mathbb{R}^+$. If A is diagonalisable, it is possible to choose $\rho(A)=-\max(\Re(\sigma(A)))$ and $\kappa(A)=\|U\|\|U^{-1}\|$ with $U\in GL_d(\mathbb{C})$ being such that UAU^{-1} is diagonal. So (3.2) *essentially demands that this choice has to be done measurably in A (see especially Example 3.5 for a concrete example).*

In some applications like stochastic volatility modelling, for instance, one is particularly interested in the case where the underlying Lévy process is of finite variation and the supOU process is defined via ω-wise integration. The following result is proved using Proposition 2.4 together with variations of the arguments of the proof of Theorem 3.1.

Proposition 3.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ satisfying*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty \quad \text{and} \quad \int_{\|x\|\leq 1} \|x\|\nu(dx) < \infty \tag{3.11}$$

and assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+ \backslash \{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that:

$$\|e^{As}\| \leq \kappa(A)e^{-\rho(A)s} \,\forall\, s \in \mathbb{R}^+, \quad \pi - \text{almost surely}, \tag{3.12}$$

and

$$\int_{M_d^-} \frac{\kappa(A)}{\rho(A)} \pi(dA) < \infty. \tag{3.13}$$

Then the process $(X_t)_{t\in\mathbb{R}}$ given by

$$\begin{aligned} X_t &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds) \\ &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \gamma_0 ds \pi(dA) + \int_{\mathbb{R}^d} \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} x \mu(dx, dA, ds) \end{aligned}$$

is well-defined as a Lebesgue integral for all $t \in \mathbb{R}$ and $\omega \in \Omega$ and X is stationary.

Remark 3.4. *If* (3.3) *is satisfied for a Lévy basis, then* (3.13) *is also satisfied.*

We shall not develop the general case further, but consider two special cases which appear to be sufficient for most purposes. We define $M_d^{N-} := \{A \in M_d(\mathbb{R}) :$ A is normal and $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}\}$.

Proposition 3.5. *(i) Assume that $\pi(M_d^{N-}) = 1$, then* (3.2) *or* (3.12) *are satisfied with $\kappa(A) = 1$ and $\rho(A) = -\max(\Re(\sigma(A)))$. Moreover,* (3.3) *or* (3.13) *are implied by*

$$-\int_{M_d^{N-}} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) < \infty. \tag{3.14}$$

(ii) Assume that there are a $K \in \mathbb{N}$ and diagonalisable $A_1, \ldots, A_K \in M_d^-(\mathbb{R})$ such that $\pi(\{\lambda A_i : i = 1, \ldots, K; \lambda \in \mathbb{R}^+ \backslash \{0\}\}) = 1$. Then (3.2) *or* (3.12) *are satisfied with $\kappa(A) = C$ for some $C \in [1, \infty)$ and $\rho(A) = -\max(\Re(\sigma(A)))$. Moreover,* (3.3) *or* (3.13) *are implied by*

$$-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) < \infty. \tag{3.15}$$

In dimension one these are again the well-known necessary and sufficient conditions. Observe also that the eigenvalues are continuous (and hence measurable) in A, because they are the zeros of the characteristic polynomial.

Proof. Part (i) follows immediately from the fact that all normal matrices are unitarily diagonalisable.

Likewise, (ii) is a consequence of the above mentioned pointwise bound and the fact that this can be turned into a global one, because for fixed $i = 1, \ldots, N$ the matrices $\{\lambda A_i\}_{\lambda \in \mathbb{R}^+ \setminus \{0\}}$ are all diagonalised by the same invertible matrices. □

In (i) the mean reversion parameter A of the superimposed OU type processes is restricted to normal matrices and in (ii) to finitely many rays $\{\lambda A_i\}_{\lambda \in \mathbb{R}^+ \setminus \{0\}}$.

Remark 3.6. *(i) Typically one will, in general, not consider normal matrices for A as in (i), but only negative definite ones, since this allows one to use well-known distributions on the positive definite matrices (see e.g. [13]) for π. In the case (ii) possible π can be obtained by using arbitrary distributions on $\mathbb{R}^+$ along the rays and positive weights summing to one for the different rays.*

(ii) Intuitively (3.14) *and* (3.15) *mean that π may not put too much mass on elements with very slow exponential decay rates.*

3.2 Finiteness of moments and second order structure

Before we look at the second order structure, we give sufficient conditions ensuring the finiteness of moments.

Theorem 3.7. *Let X be a stationary d-dimensional supOU process driven by a Lévy basis Λ satisfying the conditions of Theorem 3.1.*

(i) If

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{3.16}$$

for $r \in (0, 2]$, then X has a finite r-th moment, i.e. $E(\|X_t\|^r) < \infty$.

(ii) If $r \in (2, \infty)$ and

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \quad \int_{M_d^-} \frac{\kappa(A)^r}{\rho(A)} \pi(dA) < \infty, \tag{3.17}$$

then X has a finite r-th moment, i.e. $E(\|X_t\|^r) < \infty$.

In connection with the above results observe that the underlying Lévy process L has an r-th moment, i.e. $E(\|L_1\|^r) < \infty$, for $r \in \mathbb{R}^+$ if and only if $\int_{\|x\|>1} \|x\|^r \nu_L(dx) < \infty$.

Proof. Using [30, Corollary 25.8] we have to show $\int_{\|x\|>1} \|x\|^r \nu_X(dx) < \infty$.

Now,

$$\begin{aligned}
\int_{\|x\|>1} \|x\|^r \nu_X(dx) &= \int_{M_d^-} \int_0^\infty \int_{\mathbb{R}^d} \|e^{As}x\|^r 1_{(1,\infty)}(\|e^{As}x\|)\nu(dx)ds\pi(dA) \\
&\le \int_{M_d^-} \int_0^\infty \int_{\mathbb{R}^d} \kappa(A)^r e^{-r\rho(A)s} \|x\|^r 1_{(1,\infty)}(\kappa(A)e^{-\rho(A)s}\|x\|)\nu(dx)ds\pi(dA) \\
&= \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \int_0^{\frac{\ln(\kappa(A)\|x\|)}{\rho(A)}} \kappa(A)^r e^{-r\rho(A)s}\|x\|^r ds\nu(dx)\pi(dA) \\
&= \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r\|x\|^r}{r\rho(A)} \left(1 - \frac{1}{\kappa(A)^r\|x\|^r}\right)\nu(dx)\pi(dA) \\
&= \int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r\|x\|^r - 1}{r\rho(A)} \nu(dx)\pi(dA).
\end{aligned}$$

That $\int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{1}{r\rho(A)}\nu(dx)\pi(dA) < \infty$ has already been shown in the proof of Theorem 3.1.

Moreover, we obtain

$$\begin{aligned}
&\int_{M_d^-} \int_{\|x\|>1/\kappa(A)} \frac{\kappa(A)^r\|x\|^r}{\rho(A)}\nu(dx)\pi(dA) \\
&\le \int_{M_d^-} \int_{\|x\|>1} \frac{\kappa(A)^r\|x\|^r}{\rho(A)}\nu(dx)\pi(dA) + \int_{M_d^-} \int_{\|x\|\le 1} \frac{\kappa(A)^{r\vee 2}\|x\|^{r\vee 2}}{\rho(A)}\nu(dx)\pi(dA).
\end{aligned}$$

Hence, (i) and (ii) follow, since ν is a Lévy measure, using also (3.3) for (i). □

Remark 3.8. *In the set-up of Proposition 3.5 (i) (and analogously in (ii))*

$$-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))}\pi(dA) < \infty, \qquad \int_{\|x\|>1} \|x\|^r \nu(dx) < \infty. \tag{3.18}$$

imply (3.14) *and* (3.17) *(respectively* (3.16)*).*

Theorem 3.9. *Let X be a stationary d-dimensional supOU process driven by a Lévy basis Λ satisfying the conditions of Theorem 3.1 and assume $\int_{\mathbb{R}^d} \|x\|^2\nu(dx) < \infty$. Then $E(\|X_0\|^2) < \infty$ and we have*

$$E(X_0) = -\int_{M_d^-} A^{-1}\Big(\gamma + \int_{|x|>1} x\nu(dx)\Big)\pi(dA) \tag{3.19}$$

$$\operatorname{var}(X_0) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\Big(\Sigma + \Big(\int_{\mathbb{R}^d} xx^*\nu(dx)\Big)\Big)\pi(dA) \tag{3.20}$$

$$\operatorname{cov}(X_h, X_0) = -\int_{M_d^-} e^{Ah}(\mathscr{A}(A))^{-1}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big)\pi(dA) \text{ for } h \in \mathbb{N}. \tag{3.21}$$

with $\mathscr{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto AX + XA^*$.

Moreover, it holds that

$$\lim_{h\to\infty} \operatorname{cov}(X_h, X_0) = 0. \tag{3.22}$$

Proof. The finiteness of the second moments follows from Theorem 3.7. Using the formulae of Theorem 3.1 and [30, Example 25.12] we obtain

$$E(X_0) = \gamma_X + \int_{\|x\|>1} x\nu_X(dx) = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As}\Big(\gamma + \int_{\|x\|>1} x\nu(dx)\Big) ds\pi(dA).$$

Noting that $\frac{d}{ds} A^{-1} e^{As} = e^{As}$, integrating over s gives (3.19).

Likewise we get

$$\mathrm{var}(X_0) = \Sigma_X + \int_{\mathbb{R}^d} xx^*\nu_X(dx) = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big) e^{A^*s} ds\pi(dA)$$

which implies (3.20) by integrating over s.

Finally,

$$\begin{aligned}
\mathrm{cov}(X_h, X_0) =& \mathrm{cov}\Big(\int_{M_d^-} \int_{-\infty}^{h} e^{A(h-s)}\Lambda(dA, ds), \int_{M_d^-} \int_{-\infty}^{0} e^{-As}\Lambda(dA, ds)\Big)\\
=& \mathrm{cov}\Big(\int_{M_d^-} \int_{-\infty}^{0} e^{A(h-s)}\Lambda(dA, ds), \int_{M_d^-} \int_{-\infty}^{0} e^{-As}\Lambda(dA, ds)\Big)\\
=& \int_{M_d^-} e^{Ah}\Big(\int_{-\infty}^{0} e^{-As}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big) e^{-A^*s} ds\Big)\pi(dA) \qquad (3.23)\\
=& -\int_{M_d^-} e^{Ah}(\mathscr{A}(A))^{-1}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big)\pi(dA)
\end{aligned}$$

since Λ is a Lévy basis and hence the random measures Λ on $M_d^- \times (0, h]$ and on $M_d^- \times (-\infty, 0]$ are independent.

From (3.23) one obtains

$$\begin{aligned}
&\Big\|\int_{M_d^-} e^{Ah}\Big(\int_{-\infty}^{0} e^{-As}\Big(\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big) e^{-A^*s} ds\Big)\pi(dA)\Big\|\\
&\leq \int_{M_d^-} \int_{-\infty}^{0} \kappa(A)^2 e^{\rho(A)(2s-h)} ds\pi(dA) \Big\|\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big\|\\
&\leq \int_{M_d^-} \frac{\kappa(A)^2}{2\rho(A)}\pi(dA)\Big\|\Sigma + \int_{\mathbb{R}^d} xx^*\nu(dx)\Big\| < \infty
\end{aligned}$$

and therefore $\lim_{h\to\infty} e^{Ah} = 0$ for all $A \in M_d^-$, and dominated convergence establish (3.22). □

3.3 Some important path properties

In this section we show for a supOU process X a representation, which generalises the SDE that governs OU type processes, and we derive important path properties of X. That "SDE representation" – identity (3.28) below – has been conjectured in the univariate case in [2], where neither a proof nor conditions for its validity have been given. Below we are able to show these results for finite variation Lévy bases, which are naturally appearing in applications like stochastic volatility modelling. The properties which we establish are especially important in the context of integration, since they imply that, if X is the integrator, then pathwise Lebesgue integration can be carried out, and, when X is the integrand, the theory of stochastic integrals of càdlàg processes with respect to semimartingales (see [28], for instance) respectively the L^2-theory of e.g. [25] applies. Likewise, the integrated process is of importance in certain applications.

Below the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$ generated by Λ is defined by $\mathscr{F}_t$ being the σ-algebra generated by the set of random variables $\{\Lambda(B) : B \in \mathscr{B}(M_d^- \times (-\infty, t])\}$ for $t \in \mathbb{R}$.

Theorem 3.10. *Let X be a supOU process as in Proposition 3.3. Then:*

(i) $X_t(\omega)$ is $\mathscr{B}(\mathbb{R})\times\mathscr{F}$ measurable as a function of $t \in \mathbb{R}$ and $\omega \in \Omega$ and adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$ generated by Λ.

(ii) If

$$\int_{M_d^-} \kappa(A)\pi(dA) < \infty, \tag{3.24}$$

the paths of X are locally uniformly bounded in t for every $\omega \in \Omega$.

Furthermore, $X_t^+ = \int_0^t X_s ds$ exists for all $t \in \mathbb{R}^+$ and

$$\begin{aligned} X_t^+ = &\int_{M_d^-}\int_{-\infty}^{t} A^{-1}e^{A(t-s)}\Lambda(dA, ds) \\ &- \int_{M_d^-}\int_{-\infty}^{0} A^{-1}e^{-As}\Lambda(dA, ds) - \int_{M_d^-}\int_{0}^{t} A^{-1}\Lambda(dA, ds). \end{aligned} \tag{3.25}$$

(iii) Provided that

$$\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)}{\rho(A)}\pi(dA) < \infty \tag{3.26}$$

and

$$\int_{M_d^-} \|A\|\kappa(A)\pi(dA) < \infty \tag{3.27}$$

it holds that

$$X_t = X_0 + \int_0^t Z_u du + L_t \tag{3.28}$$

where L is the underlying Lévy process and

$$Z_u = \int_{M_d^-}\int_{-\infty}^{u} Ae^{A(u-s)}\Lambda(dA, ds) \tag{3.29}$$

for all $u \in \mathbb{R}$ with the integral existing ω-wise.
Moreover, the paths of X are càdlàg and of finite variation on compacts.

Proof. (i) is immediate from the definition of X_t as a Lebesgue integral and the measurability properties of the integrand $e^{A(t-s)}x1_{\mathbb{R}^+}(t-s)$ which as a function of t, s, A, x is $\mathscr{B}(\mathbb{R}\times\mathbb{R}\times M_d^- \times \mathbb{R}^d)$-$\mathscr{B}(\mathbb{R}^d)$-measurable.

(ii) We first show local uniform boundedness of X. Choose arbitrary $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$. Then

$$\begin{aligned} f_{T_1,T_2}(A,s,x) &:= \sup_{t\in[T_1,T_2]} \left\| e^{A(t-s)} x 1_{\mathbb{R}^+}(t-s)\right\| \\ &\leq \left(\kappa(A)e^{-\rho(A)(T_1-s)}1_{(-\infty,T_1]}(s) + \kappa(A)1_{(T_1,T_2]}(s)\right)\|x\| \end{aligned}$$

for all $A \in M_d^-(\mathbb{R}), s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ and

$$\begin{aligned} \sup_{t\in[T_1,T_2]} \|X_t\| \leq & \int_{M_d^-}\int_{\mathbb{R}} f_{T_1,T_2}(A,s,\gamma_0)ds\pi(dA) \\ & + \int_{\mathbb{R}^d}\int_{M_d^-}\int_{\mathbb{R}} f_{T_1,T_2}(A,s,x)\mu(dx,dA,ds). \end{aligned}$$

Therefore we only have to show the ω-wise existence and finiteness of the integral on the right hand side. This is, however, an immediate consequence of the above upper bound, (3.24), Proposition 2.4 and arguments as in the proof of Theorem 3.1 noting that

$$\begin{aligned} &\int_{M_d^-}\int_{T_1}^{T_2}\int_{\mathbb{R}^d}(1\wedge\kappa(A)\|x\|)\nu(dx)ds\pi(dA) \\ &\quad \leq (T_2-T_1)\Big(\int_{M_d^-}\int_{\|x\|\leq 1}\kappa(A)\|x\|\nu(dx)\pi(dA) + \int_{M_d^-}\int_{\|x\|>1} 1\nu(dx)\pi(dA)\Big). \end{aligned}$$

Turning to X_t^+ the existence follows immediately from the local boundedness. Noting that we have actually proved the local boundedness of

$$\int_{M_d^-}\int_{-\infty}^{t}\|e^{A(t-s)}\gamma_0\|ds\pi(dA) + \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^{t}\|e^{A(t-s)}x\|\mu(dx,dA,ds)$$

above, we can use Fubini to obtain

$$\begin{aligned} X_t^+ =& \int_{M_d^-}\int_{-\infty}^{t}\int_{0\vee u}^{t} e^{A(s-u)}\gamma_0 dsdu\pi(dA) \\ &+ \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^{t}\int_{0\vee u}^{t} e^{A(s-u)}xds\mu(dx,dA,du) \\ =& \int_{M_d^-}\int_{-\infty}^{t} A^{-1}e^{A(s-u)}\gamma_0\Big|_{s=(0\vee u)}^{t} du\pi(dA) \\ &+ \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^{t} A^{-1}e^{A(s-u)}x\Big|_{s=(0\vee u)}^{t} ds\mu(dx,dA,du), \end{aligned}$$

which establishes (3.25) by straightforward calculations.

(iii) Using similar calculations as before the existence of Z_u as an ω-wise integral follows from Proposition 2.4 and (3.26). Similarly to (ii) one sees that under (3.27)

Z is locally uniformly bounded in u. Hence, one can use Fubini to obtain

$$\begin{aligned}\int_0^t Z_u du &= \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^t\int_{0\vee s}^t Ae^{A(u-s)}xdu\mu(dx,dA,ds)\\ &\quad+\int_{M_d^-}\int_{-\infty}^t\int_{0\vee s}^t Ae^{A(u-s)}\gamma_0 duds\pi(dA)\\ &= \int_{\mathbb{R}^d}\int_{M_d^-}\int_{-\infty}^t e^{A(u-s)}x\Big|_{u=(0\vee s)}^t \mu(dx,dA,ds)\\ &\quad+\int_{M_d^-}\int_{-\infty}^t e^{A(u-s)}\gamma_0\Big|_{u=(0\vee s)}^t ds\pi(dA)\\ &= X_t - X_0 - L_t\end{aligned}$$

which establishes (3.28). That X has cádlág paths of finite variation is now an immediate consequence of this integral representation. □

Remark 3.11. *(i) Condition* (3.24) *is always true if* π *is concentrated on the normal matrices or on finitely many rays and hence especially in dimension* $d = 1$. *Moreover, it could be replaced by the weaker but rather impracticable condition that* $f_{[T_1,T_2]}(A, s, x) \wedge 1$ *is integrable with respect to* $\pi \times \lambda \times \nu$ *and that, for any fixed* x, $f_{[T_1,T_2]}(A, s, x)$ *is integrable with respect to* $\pi \times \lambda$ *(cf. [20, Proposition 2.1] for a very related result whose proof is similar in spirit to ours, but uses a series representation instead of the Lévy-Itô decomposition).*

(ii) Intuitively (3.27) *means that* π *does not place too much mass on the elements of* M_d^- *with high norm and thus very fast exponential decay rates.*

If π *is concentrated on the normal matrices or finitely many diagonalisable rays, then* (3.26) *and* (3.27) *become*

$$-\int_{M_d^-}\frac{(\|A\|\vee 1)}{\max\Re(\sigma(A))}\pi(dA)<\infty \ \text{ and } \int_{M_d^-}\|A\|\pi(dA)<\infty. \tag{3.30}$$

In particular, the second condition simply means that π *has a finite first moment.*

If π *is concentrated on* $\mathbb{S}_d^{--}$, *then we have* $\|A\| = -\min(\sigma(A))$ *and* (3.26) *becomes*

$$\int_{\mathbb{S}_d^{--}}\frac{(\min(\sigma(A))\wedge -1)}{\max(\sigma(A))}\pi(dA)<\infty, \tag{3.31}$$

so it can be seen as a condition on the spread between the different exponential decay rates measured by the eigenvalues. It is easy to see that in dimension $d = 1$, *it is equivalent to* $\int_{\mathbb{R}^-}(-1/A)\,\pi(dA) < \infty$, *which is part of the necessary and sufficient conditions for the existence of the supOU process.*

3.4 Examples and long range dependence

Like in the univariate case, the expression (3.21) does not imply that we necessarily have an exponential decay of the autocovariance function and thus a short memory process. On the contrary we can easily obtain a long memory process, as the following examples exhibit. Note that this illustrates that (3.22) is not obvious and indeed requires a detailed proof as above.

Apart from showing that multivariate supOU processes may exhibit long range dependence, the purpose of this section is to analyse some concrete examples and their properties.

Regarding long range dependence, there is unfortunately basically no general theory developed in the multivariate case so far. So below we mean by long range dependence simply that at least one element of the autocovariance function decays asymptotically like $h^{-\alpha}$ for the lag h going to infinity and for some $\alpha \in (0;1)$. Intuitively this should clearly be a case when one may appropriately speak of long range dependence. Establishing a general theory for multivariate long range dependence seems to be very important, but is beyond the scope of this paper.

Example 3.1. Let Λ be a d-dimensional Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ with ν satisfying $\int_{\mathbb{R}^d} \|x\|^2 \nu(dx) < \infty$ and π being given as the distribution of RB with a diagonalisable $B \in M_d^-$ and R being a real $\Gamma(\alpha, \beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \backslash \{0\}$. Hence, R has probability density $f(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} 1_{\mathbb{R}^+}(r)$, and from

$$\begin{aligned} -\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) &= \frac{-\beta^\alpha}{\max(\Re(\sigma(B)))\Gamma(\alpha)} \int_{\mathbb{R}^+} r^{\alpha-2} e^{-\beta r} dr \\ &= \frac{-\beta^\alpha}{\max(\Re(\sigma(B)))\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} = \frac{-\beta}{\alpha \max(\Re(\sigma(B)))} \end{aligned}$$

we conclude that (3.18) holds. Consequently the process

$$X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds)$$

exists, is stationary and has finite second moments.

For the autocovariance function for positive lags h we find

$$\begin{aligned} \operatorname{cov}(X_h, X_0) &= -\int_{M_d^-} e^{Ah} (\mathscr{A}(A))^{-1} \operatorname{vec}\Big(\Sigma + \int_{\mathbb{R}^d} xx^* \nu(dx)\Big) \pi(dA) \\ &= \int_{\mathbb{R}^+} e^{Bhr - \beta I_d r} r^{\alpha-2} dr \Big(-\frac{\beta^\alpha}{\Gamma(\alpha)} \mathscr{B}^{-1} \Big(\Sigma + \int_{\mathbb{R}^d} xx^* \nu(dx)\Big)\Big) \end{aligned}$$

with $\mathscr{B} : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto BX + XB^*$. Let now $U \in GL_d(\mathbb{C})$ and $\lambda_1, \lambda_2, \ldots, \lambda_d \in (-\infty, 0) + i\mathbb{R}$ be such that

$$UBU^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d \end{pmatrix}.$$

Then, from $\int_0^\infty t^{z-1} e^{-kt} dt = \Gamma(z) k^{-z}$ for all $z, k \in (0, \infty) + i\mathbb{R}$, where the power is defined via the principal branch of the complex logarithm (see [1, p. 255]), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^+} e^{Bhr-\beta I_d r} r^{\alpha-2} dr \\
&= U \int_{\mathbb{R}^+} \exp\left(-r\left(\beta I_d - \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d \end{pmatrix} h\right)\right) r^{\alpha-2} dr U^{-1} \\
&= \Gamma(\alpha-1) U \left(\beta I_d - \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d \end{pmatrix} h\right)^{1-\alpha} U^{-1} \\
&= \Gamma(\alpha-1) \left(\beta I_d - Bh\right)^{1-\alpha}.
\end{aligned}$$

Above the $(1-\alpha)$-th power of a matrix is understood to be defined via spectral calculus as usual.

Hence,

$$\operatorname{cov}(X_h, X_0) = -\frac{\beta^\alpha}{\alpha-1} \left(\beta I_d - Bh\right)^{1-\alpha} \mathscr{B}^{-1}\Big(\Sigma + \int_{\mathbb{R}^d} xx^* \nu(dx)\Big)$$

and thus we have a polynomially decaying autocovariance function. For $\alpha \in (1,2)$ we obviously get long memory.

Another question is whether we have the nice path properties of Theorem 3.10. Hence, assume additionally that $\int_{\|x\|\le 1} \|x\| \nu(dx) < \infty$. In our example Condition (3.24) is trivially satisfied and so the paths of X are locally uniformly bounded in t. Regarding Condition (3.30) the second part is equivalent to

$$\|B\| \int_0^\infty r^\alpha e^{-\beta r} dr < \infty,$$

which is always true, as any Gamma distribution has a finite mean. Denoting the density of the $\Gamma(\alpha,\beta)$-distribution by $f_{\alpha,\beta}(r)$, one obtains for the first part

$$\begin{aligned}
-\int_0^\infty \frac{(r\|B\| \vee 1)}{r \max(\Re(\sigma(B)))} f_{\alpha,\beta}(r) dr = &-\int_0^{\|B\|^{-1}} \frac{1}{r \max(\Re(\sigma(B)))} f_{\alpha,\beta}(r) dr \\
&- \int_{\|B\|^{-1}}^\infty \frac{\|B\|}{\max(\Re(\sigma(B)))} f_{\alpha,\beta}(r) dr,
\end{aligned}$$

which is obviously finite. Hence, the conditions of Theorem 3.10 (iii) are satisfied and thus the paths are càdlàg and of finite variation, and (3.28) is valid.

Example 3.2. The previous example has an immediate extension to the case when π is concentrated on several rays instead of a single one as above. Assume we have $w_1, \ldots, w_m \in [0,1]$ with $\sum_{i=1}^m w_i = 1$ and diagonalisable $B_1, \ldots, B_m \in M_d^-$ and define π_i to be the probability measure of the random variable $R_i B_i$ with R_i being

$\Gamma(\alpha_i, \beta_i)$ distributed with $\alpha_i > 1$, $\beta_i \in \mathbb{R}^+\backslash\{0\}$. If ν is as above and $\pi = \sum_{i=1}^m w_i \pi_i$, we get for the multivariate supOU process X

$$\operatorname{cov}(X_h, X_0) = -\sum_{i=1}^m \left(\frac{w_i \beta_i^{\alpha_i}}{\alpha_i - 1} \left(\beta_i I_d - B_i h\right)^{1-\alpha_i} \mathscr{B}_i^{-1} \right) \Big(\Sigma + \int_{\mathbb{R}^d} x x^* \nu(dx) \Big)$$

with $\mathscr{B}_i : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto B_i X + X B_i^*$.

Assuming now $\int_{\|x\|\leq 1} \|x\| \nu(dx) < \infty$, it is likewise straightforward to see that Conditions (3.24) and (3.30) are satisfied. Hence, the paths of X are locally uniformly bounded in t, càdlàg and of finite variation, and (3.28) is valid.

Example 3.3. A similar result can be obtained if we restrict the mean reversion parameter A to the strictly negative definite matrices $\mathbb{S}_d^{--}$ and define π as a probability distribution on the proper convex cone $\mathbb{S}_d^{--}$ as follows. Let $\mathbf{S}_d^{--}$ denote the intersection of the unit sphere in $\mathbb{S}_d$ with $\mathbb{S}_d^{--}$, let $\alpha : \mathbf{S}_d^{--} \to (1, \infty)$, $\beta : \mathbf{S}_d^{--} \to (0, \infty)$ be measurable mappings and w a probability distribution on $\mathbf{S}_d^{--}$ such that

$$-\int_{\mathbf{S}_d^{--}} \frac{\beta(v)}{\alpha(v) \max(\sigma(v))} w(dv) < \infty. \tag{3.32}$$

Now define π via

$$\pi(B) = \int_{\mathbf{S}_d^{--}} \int_0^\infty 1_B(rv) \frac{\beta(v)^{\alpha(v)}}{\Gamma(\alpha(v))} r^{\alpha(v)-1} e^{-\beta(v) r} dr w(dv)$$

for any Borel set $B \in M_d^-(\mathbb{R})$. Then π is a probability distribution concentrated on $\mathbb{S}_d^{--}$.

Using this π in the above set-up means that the mean reversion parameter is no longer necessarily restricted to finitely many rays. Moreover, similar calculations to the ones in Example 3.1 give

$$-\int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) = \int_{\mathbf{S}_d^{--}} \frac{-\beta(v)}{\alpha(v) \max(\sigma(v))} w(dv) < \infty.$$

Hence, (3.18) holds and the process $X_t = \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds)$ exists, is stationary and has finite second moments. Likewise we get for the autocovariance function

$$\begin{aligned} &\operatorname{cov}(X_h, X_0) \\ &\quad = -\Big(\int_{\mathbf{S}_d^{--}} \frac{\beta(v)^{\alpha(v)}}{\alpha(v) - 1} \left(\beta(v) I_d - v h\right)^{1-\alpha(v)} (\mathscr{V}(v))^{-1} w(dv) \Big) \Big(\Sigma + \int_{\mathbb{R}^d} x x^* \nu(dx) \Big) \end{aligned}$$

with $\mathscr{V}(v) : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto vX + Xv^*$.

Turning to the path properties, assume now that $\int_{\|x\|\leq 1} \|x\| \nu(dx) < \infty$. Again Condition (3.24) is trivially satisfied and so the paths of X are locally uniformly bounded in t. Regarding condition (3.30) the second part becomes

$$\int_{M_d^-} \|A\| \pi(dA) = \int_{\mathbf{S}_d^{--}} \int_0^\infty r f_{\alpha(v),\beta(v)}(r) dr w(dv) = \int_{\mathbf{S}_d^{--}} \frac{\alpha(v)}{\beta(v)} w(dv)$$

and for the first part one obtains

$$-\int_{M_d^-} \frac{(\|A\| \vee 1)}{\max \Re(\sigma(A))}\pi(dA) = -\int_{\mathbf{S}_d^{--}} \int_0^1 \frac{1}{r \max(\sigma(v))} f_{\alpha(v),\beta(v)}(r) dr w(dv) - \int_{\mathbf{S}_d^{--}} \int_1^\infty \frac{1}{\max(\sigma(v))} f_{\alpha(v),\beta(v)}(r) dr w(dv).$$

The first summand is finite due to (3.32) and the second one is finite if

$$-\int_{\mathbf{S}_d^{--}} \left(\frac{1}{\max(\sigma(v))}\right) w(dv)$$

is finite. Hence, provided

$$-\int_{\mathbf{S}_d^{--}} \frac{1}{\max(\sigma(v))} w(dv) < \infty \qquad \text{and} \qquad \int_{\mathbf{S}_d^{--}} \frac{\alpha(v)}{\beta(v)} w(dv) < \infty,$$

the conditions of Theorem 3.10 (iii) are satisfied and thus the paths are càdlàg and of finite variation and (3.28) is valid.

Based on this we can easily give an example where we know that the supOU process exists due to Proposition 3.3, but the conditions of Theorem 3.10 (iii) are not satisfied. Assume w is a discrete distribution concentrated on the points

$$v_n = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 + (3n)^{-1} & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, n \in \mathbb{N},$$

and that $w(v_n) = \frac{6}{\pi^2 n}$, $\alpha(v_n) = 2$ and $\beta(v_n) = n^{-1}$. Then we have that

$$-\int_{\mathbf{S}_d^{--}} \frac{\beta(v)}{\alpha(v) \max(\sigma(v))} w(dv) = \frac{6}{\pi^2} \sum_{n=1}^\infty n^{-3} < \infty,$$

but

$$\int_{M_d^-} \|A\| \pi(dA) = \frac{12}{\pi^2} \sum_{n=1}^\infty n^{-1} = \infty$$

and hence Condition (3.26) is not satisfied. Observe that this means that the probability measure π we have constructed does not have a first moment, although it is defined via a polar representation where the radial parts are all univariate Gamma distributions.

Example 3.4. Let Λ be now a two-dimensional Lévy basis with generating quadruple $(\gamma, \Sigma, \nu, \pi)$ with ν satisfying $\int_{\mathbb{R}^d} \|x\|^2 \nu(dx) < \infty$. We restrict the mean reversion parameter A to $\mathbb{D}_2^{--}$, the 2 x 2 diagonal matrices with strictly negative entries on the diagonal. Hence, π is a measure on $\mathbb{D}_2^{--}$ which can be identified with $(\mathbb{R}^{--})^2$ and we assume that π has Lebesgue density

$$\pi(da_1, da_2) = \frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (-a_1)^{\alpha_1 - 1} (-a_2)^{\alpha_2 - 1} e^{\beta_1 a_1 + \beta_2 a_2} 1_{(\mathbb{R}^{--})^2}(a_1, a_2) da_1 da_2$$

with $\alpha_1, \alpha_2 > 1$ and $\beta_1, \beta_2 > 0$. So the diagonal elements are independent and their absolute values follow Gamma distributions. We obtain

$$\begin{aligned}
&-\int_{\mathbb{D}_2^{--}} \frac{1}{\max(\Re(\sigma(A)))}\pi(dA) \\
&= \int_0^\infty \int_0^\infty \frac{1}{\min(a_1,a_2)} \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(a_1)^{\alpha_1-1}(a_2)^{\alpha_2-1}e^{-\beta_1 a_1-\beta_2 a_2}da_1 da_2 \\
&\leq \int_0^\infty \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(a_1)^{\alpha_1-2}e^{-\beta_1 a_1}da_1 \int_0^\infty \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}(a_2)^{\alpha_2-1}e^{-\beta_2 a_2}da_2 \\
&\quad + \int_0^\infty \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(a_1)^{\alpha_1-1}e^{-\beta_1 a_1}da_1 \int_0^\infty \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}(a_2)^{\alpha_2-2}e^{-\beta_2 a_2}da_2 < \infty.
\end{aligned}$$

Hence, (3.18) holds and the process $X_t = \int_{M_d^-}\int_{-\infty}^t e^{A(t-s)}\Lambda(dA,ds)$ exists, is stationary and has finite second moments.

Let us now consider the individual components $X_{1,t}$, $X_{2,t}$ of X_t. Denote by $P_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x_1,x_2)^* \mapsto x_1$ the projection onto the first coordinate and define an $\mathbb{R}$-valued Lévy basis Λ_1 on $\mathbb{R}^{--}\times\mathbb{R}$ via $\Lambda_1(da_1,ds) = P(\Lambda(P_1^{-1}(da_1),ds)$ and a Lévy measure ν_1 on $\mathbb{R}$ via $\nu_1(dx_1) = \nu(P_1^{-1}(dx_1))$. Then Λ_1 has characteristic quadruple $(\gamma_1, \Sigma_{11}, \nu_1, \pi_1)$ with π_1 having Lebesgue density

$$\pi_1(da_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1))}(-a_1)^{\alpha_1-1}e^{\beta_1 a_1}1_{(\mathbb{R}^{--})}(a_1)da_1$$

and

$$X_{1,t} = \int_{\mathbb{R}^{--}}\int_{-\infty}^t e^{a_1(t-s)}\Lambda_1(da_1,ds).$$

For the autocovariance function of the first component we get

$$\mathrm{cov}(X_{1,h},X_{1,0}) = \frac{\beta_1^{\alpha_1}}{2(\alpha_1-1)}\left(\beta_1+h\right)^{1-\alpha_1}\left(\Sigma_{11}+\int_{\mathbb{R}} x_1^2\nu_1(dx_1)\right),\ h\in\mathbb{R}^+.$$

An analogous result holds for the second component $X_{2,t}$ and we have long memory in both components provided $\alpha_1, \alpha_2 \in (1,2)$.

The importance of this example is, however, that we can model the stationary distributions of X_1 and X_2, i.e. the margins of the stationary distribution of X_t, very explicitely by specifying the margins of ν, i.e. ν_1 and ν_2. From [2, Theorem 3.1, Corollary 3.1] and [12, Remark 2.2] we know that it is exactly all non-degenerate self-decomposable distributions on $\mathbb{R}$ which arise as the stationary distributions of the components. Moreover, these authors provide formulae to calculate ν_1 (or ν_2) if one wants to obtain a given stationary distribution for the component (alternatively [3, Lemma 5.1] or the refinement [27, Theorem 4.9] can be used). Hence, one can specify a two-dimensional supOU process with prescribed stationary distributions of the components by calculating the required ν_1 and ν_2 and choosing ν accordingly. The easiest way to get a possible ν is by specifying $\nu(dx_1,dx_2) = \nu_1(dx_1)\times\delta_0(x_2) + \delta_0(x_1)\times\nu_2(dx_2)$ with δ_0 denoting the Dirac distribution with unit mass at zero. In this case the components of X are independent. An easy way to get an appropriate ν and allowing for dependence is to combine ν_1 and ν_2 using a Lévy copula (see [19] and [4]).

Likewise it is again interesting to look at the path properties of Theorem 3.10. Assuming again $\int_{\|x\|\leq 1}\|x\|\nu(dx)<\infty$, Condition (3.24) is trivially satisfied and so the paths of X are locally uniformly bounded in t. Regarding the second part of condition (3.30) we have that

$$\int_{M_d^-}\|A\|\pi(dA)<\infty$$

is equivalent to

$$\int_{(\mathbb{R}^+)^2}\max(a_1,a_2)f_{\alpha_1,\beta_1}(a_1)f_{\alpha_2,\beta_2}(a_1)da_1da_2 \\ \leq \int_{\mathbb{R}^+}a_1f_{\alpha_1,\beta_1}(a_1)da_1+\int_{\mathbb{R}^+}a_2f_{\alpha_2,\beta_2}(a_1)da_2<\infty$$

which is always true. Turning to (3.31), it is implied by

$$\int_{(\mathbb{R}^+)^2}\frac{\max(a_1,a_2)}{\min(a_1,a_2)}f_{\alpha_1,\beta_1}(a_1)f_{\alpha_2,\beta_2}(a_1)da_1da_2 \\ \leq \int_{(\mathbb{R}^+)^2}\frac{a_1+a_2}{a_1}f_{\alpha_1,\beta_1}(a_1)f_{\alpha_2,\beta_2}(a_1)da_1da_2 \\ +\int_{(\mathbb{R}^+)^2}\frac{a_1+a_2}{a_2}f_{\alpha_1,\beta_1}(a_1)f_{\alpha_2,\beta_2}(a_1)da_1da_2<\infty,$$

which is easily seen to be always true. Hence, the conditions of Theorem 3.10 (iii) are satisfied and thus the paths are càdlàg and of finite variation and (3.28) is valid.

Obviously this example has a straightforward extension to general dimension d.

Example 3.5. So far we have only studied cases where we could use Proposition 3.5 and did especially never have to bother with $\kappa(A)$ in the conditions of Theorem 3.1.

In this example we will present a case where the behaviour of $\kappa(A)$ is crucial and where we show how κ and ρ can be specified in a measurable way. We define the following sets:

$$\begin{aligned}\mathscr{D}_d^- &= \{X\in M_d(\mathbb{R}): X \text{ is diagonal; all diagonal elements are strictly} \\ &\qquad \text{negative, pairwise distinct and ordered such that } \Re(x_{ii})\leq\Re(x_{jj}) \\ &\qquad \text{and } \Im(x_{ii})1_{\{x_{ii}=x_{jj}\}}\leq\Im(x_{jj})1_{\{x_{ii}=x_{jj}\}}\ \forall 1\leq i\leq j\leq n\},\\ \mathscr{S}_d &= \{X\in GL_d(\mathbb{R}): \text{ the first non-zero element in each column is } 1\},\\ \mathscr{M}_d^- &= \{SDS^{-1}: S\in\mathscr{S}_d,\, D\in\mathscr{D}_d^-\}.\end{aligned}$$

If $A=SDS^{-1}$ is in $\mathscr{M}_d^-$, the matrix D consists of the eigenvalues of A and the columns of S are the eigenvectors of A. In principle there are many possible S and D if we only demand $A=SDS^{-1}$. However, if we restrict ourselves to $S\in\mathscr{S}_d$, $D\in\mathscr{D}_d^-$, then S,D are unique, as elementary linear algebra shows. This means that the map

$$\mathfrak{M}:\mathscr{S}_d\times\mathscr{D}_d^-\to\mathscr{M}_d^-,(S,D)\mapsto SDS^{-1}$$

is bijective (and obviously continuous). We denote by $\mathfrak{M}^{-1}=(\mathfrak{S},\mathfrak{D})$ the inverse mapping. Since computing eigenvectors and eigenvalues are measurable procedures as are the orderings and normalisations involved in obtaining the diagonal matrix in $\mathscr{D}_d^-$ and the eigenvector matrix in $\mathscr{S}_d$, all these mappings are measurable. Note also that $\mathscr{D}_d^-$, $\mathscr{S}_d$, $\mathscr{M}_d^-$ are Borel sets.

Defining $\kappa : \mathscr{M}_d^- \to [1,\infty), A \mapsto \|\mathfrak{S}(A)\|\|(\mathfrak{S}(A))^{-1}\|$ and $\rho(A) = -\max(\Re(\sigma(A)))$ gives therefore measurable mappings on $\mathscr{M}_d^-$ satisfying $\|e^{As}\| \leq \kappa(A)e^{-\rho(A)s}$. Using these definitions for κ and ρ one could now specify probability distributions π on $\mathscr{M}_d^-$ and check whether Condition (3.3) is satisfied and the associated supOU process therefore exists.

However, in concrete situations it seems easier to specify a Borel probability measure $\pi_{\mathscr{S}_d\times\mathscr{D}_d^-}$ on $\mathscr{S}_d \times \mathscr{D}_d^-$ and define π as its image under $\mathfrak{M}$, i.e. $\pi(B) = \pi_{\mathscr{S}_d\times\mathscr{D}_d^-}(\mathfrak{M}^{-1}(B))$ for all Borel sets B. Assume $\pi_{\mathscr{S}_d\times\mathscr{D}_d^-} = \pi_{\mathscr{S}_d} \times \pi_{\mathscr{D}_d^-}$ is the product of two probability measures $\pi_{\mathscr{S}_d}$ on $\mathscr{S}_d$ and $\pi_{\mathscr{D}_d^-}$ on $\mathscr{D}_d^-$. Then we have

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)}\pi(dA) < \infty$$
$$\Leftrightarrow \int_{\mathscr{S}_d} \|S\|^2\|S^{-1}\|^2\pi_{\mathscr{S}_d}(dS) < \infty \text{ and } -\int_{\mathscr{D}_d^-} \frac{1}{\max(\Re(\sigma(D)))}\pi_{\mathscr{D}_d^-}(dD) < \infty.$$

That $\int_{\mathscr{S}_d} \|S\|^2\|S^{-1}\|^2\pi_{\mathscr{S}_d}(dS)$ can be finite or infinite depending on the choice of $\pi_{\mathscr{S}_d}$ is exhibited by the following example. Let $\pi_{\mathscr{S}_2}$ be a discrete measure concentrated on the points

$$S_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \text{ and } p_n := \pi_{\mathscr{S}_2}(S_n) = C_\alpha n^{-\alpha}\ \forall n \in \mathbb{N}$$

with $\alpha > 1$ and $C_\alpha = 1/\sum_{n=1}^\infty n^{-\alpha}$. Then

$$S_n^{-1} = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}.$$

Using the equivalence of all norms we get that

$$\int_{\mathscr{S}_2} \|S\|^2\|S^{-1}\|^2\pi_{\mathscr{S}_2}(dS) < \infty \Leftrightarrow C_\alpha \sum_{n=1}^\infty n^4 p_n < \infty \Leftrightarrow \alpha > 5.$$

Returning to the general example with π given via $\pi_{\mathscr{S}_d} \times \pi_{\mathscr{D}_d^-}$ and turning to path properties, we assume again $\int_{\|x\|\leq 1} \|x\|\nu(dx) < \infty$. In this finite variation case the existence conditions (3.13) become

$$\int_{\mathscr{S}_d} \|S\|\|S^{-1}\|\pi_{\mathscr{S}_d}(dS) < \infty \text{ and } -\int_{\mathscr{D}_d^-} \frac{1}{\max(\Re(\sigma(D)))}\pi_{\mathscr{D}_d^-}(dD) < \infty.$$

Furthermore, Condition (3.24) is always satisfied when the existence conditions are satisfied and so the paths of X are locally uniformly bounded in t. Straightforward arguments show that the conditions of Theorem 3.10 (iii) are satisfied and thus the paths are càdlàg and of finite variation and (3.28) is valid if

$$\int_{\mathscr{S}_d} \|S\|^2\|S^{-1}\|^2\pi_{\mathscr{S}_d}(dS) < \infty,$$
$$-\int_{\mathscr{D}_d^-} \frac{\|D\|}{\max(\Re(\sigma(D)))}\pi_{\mathscr{D}_d^-}(dD) < \infty \text{ and } \int_{\mathscr{D}_d^-} \|D\|\pi_{\mathscr{D}_d^-}(dD) < \infty.$$

By polarly decomposing $\pi_{\mathscr{D}_d^-}$ into a measure on the unit sphere in the diagonal matrices and a radial part, the long memory examples of the foregoing examples have straightforward extensions to this set-up.

4 Positive semi-definite supOU processes

Based on the previous section we now consider supOU processes which are positive semi-definite at all times. The importance of such processes is that they can be used to describe the random evolution of a latent covariance matrix over time and hence they can be used in multivariate models for heteroskedastic data, e.g. the stochastic volatility model of [9].

Let us briefly recall that a $d \times d$ positive semi-definite OU type process (see [8]) is defined as the unique càdlàg solution of the SDE

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^*)dt + dL_t, \Sigma_0 \in \mathbb{S}_d^+$$

with $A \in M_d(\mathbb{R})$ and L being a $d \times d$ matrix subordinator (see [5]), i.e. a Lévy process in $\mathbb{S}_d$ with $L_t - L_s \in \mathbb{S}_d^+ \,\forall\, s,t \in \mathbb{R}^+$, $s < t$. If $\max(\Re(\sigma(A))) < 0$ and $E(\ln(\max(\|L_1\|, 1))) < \infty$, the above SDE has the unique stationary solution

$$\Sigma_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^*(t-s)}.$$

That the linear operators $\mathbb{S}_d \to \mathbb{S}_d$ of the form $Z \mapsto AZ + ZA^*$ with some $A \in M_d(\mathbb{R})$ are *the* ones to be used for positive semi-definite OU type processes has been established in [27].

Like one has to restrict the driving Lévy process to matrix subordinators in the OU type processes, if one wants to get a positive semi-definite OU type process, one needs to impose a comparable condition on the Lévy basis below. Note that for a $d \times d$ matrix-valued Lévy-basis Λ we denote by $\mathrm{vec}(\Lambda)$ the $\mathbb{R}^{d^2}$-valued Lévy basis given by $\mathrm{vec}(\Lambda)(B) = \mathrm{vec}(\Lambda(B))$ for all Borel sets B. Moreover, observe that $\mathrm{tr}(XY^*)$ (with $X, Y \in M_d(\mathbb{R})$ and tr denoting the usual trace functional) defines a scalar product on $M_d(\mathbb{R})$ and that the vec operator is a Hilbert space isometry between $M_d(\mathbb{R})$ equipped with this scalar product and $\mathbb{R}^{d^2}$ with the usual Euclidean scalar product.

Positive semi-definite supOU processes are defined as processes of the form (4.4) below which is the analogue of (3.4).

Theorem 4.1. *Let Λ be an $\mathbb{S}_d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\gamma, 0, \nu, \pi)$ with $\gamma_0 := \gamma - \int_{\|x\|\le 1} x\nu(dx) \in \mathbb{S}_d^+$ and ν being a Lévy measure on $\mathbb{S}_d$ satisfying $\nu(\mathbb{S}_d \backslash \mathbb{S}_d^+) = 0$,*

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty \text{ and } \int_{\|x\|\le 1} \|x\|\nu(dx) < \infty. \tag{4.1}$$

Moreover, assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}^+\backslash\{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that:

$$\|e^{As}\| \le \kappa(A)e^{-\rho(A)s} \,\forall\, s \in \mathbb{R}^+, \quad \pi - \text{almost surely}, \tag{4.2}$$

and

$$\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty. \tag{4.3}$$

Then the process $(\Sigma_t)_{t\in\mathbb{R}}$ given by

$$\begin{aligned}\Sigma_t &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \Lambda(dA, ds) e^{A^*(t-s)} \qquad (4.4)\\ &= \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} \gamma_0 e^{A^*(t-s)} ds \pi(dA) + \int_{\mathbb{S}_d} \int_{M_d^-} \int_{-\infty}^t e^{A(t-s)} x e^{A^*(t-s)} \mu(dx, dA, ds)\end{aligned}$$

is well-defined as a Lebesgue integral for all $t \in \mathbb{R}$ and $\omega \in \Omega$ and Σ is stationary.

Moreover,

$$\mathrm{vec}(\Sigma_t) = \int_{M_d^-} \int_{-\infty}^t e^{(A\otimes I_d + I_d \otimes A)(t-s)} \mathrm{vec}(\Lambda)(dA, ds), \tag{4.5}$$

$\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}$ and the distribution of Σ_t is infinitely divisible with characteristic function

$$E\left(\exp\left(i\mathrm{tr}(u\Sigma_t)\right)\right) = \exp\Big(i\mathrm{tr}(u\gamma_{\Sigma,0}) + \int_{\mathbb{S}_d} \left(e^{i\mathrm{tr}(ux)} - 1\right) \nu_\Sigma(dx)\Big),\ u \in \mathbb{S}_d,$$

where

$$\gamma_{\Sigma,0} = \int_{M_d^-} \int_0^\infty e^{As} \gamma_0 e^{A^*s} ds \pi(dA), \tag{4.6}$$

$$\nu_\Sigma(B) = \int_{M_d^-} \int_0^\infty \int_{\mathbb{S}_d^+} 1_B(e^{As} x e^{A^*s}) \nu(dx) ds \pi(dA) \text{ for all Borel sets } B \subseteq \mathbb{S}_d. \tag{4.7}$$

Proof. The equivalence of (4.5) and (4.4) follows from standard results on the vectorisation operator and the tensor product (see [15]).

Next we note that $e^{(A\otimes I_d + I_d\otimes A)(t-s)} = e^A \otimes e^A$ and that $\|e^A \otimes e^A\| = \|e^A\|^2$ (using the operator norm associated with the Euclidean norm). Hence, all assertions except $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}^+$ follow immediately from Propositions 2.4 and 3.3.

However, $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}^+$ is now immediate, since the integral exists ω-wise, $e^{As} X e^{A^*s} \in \mathbb{S}_d^+ \, \forall A \in M_d(\mathbb{R}), X \in \mathbb{S}_d^+, s \in \mathbb{R}$ and $\mathbb{S}_d^+$ is a closed convex cone. □

Remark 4.2. *Like in Proposition 3.5, $\kappa(A)$ can be replaced by 1 and $\rho(A)$ by $-\max(\Re(\sigma(A)))$ in (4.3) (and also in (4.14) and (4.15) below) provided π is concentrated on the normal matrices or finitely many diagonalisable rays.*

Most importantly in the context of stochastic volatility models, which involve stochastic integrals with Σ as integrand, Theorem 3.10 also has an analogue for positive semi-definite supOU processes.

Theorem 4.3. *Let Σ be the positive semi-definite supOU process of Theorem 4.1. Then:*

(i) $\Sigma_t(\omega)$ is $\mathscr{B}(\mathbb{R})\times\mathscr{F}$ measurable as a function of $t \in \mathbb{R}$ and $\omega \in \Omega$ and adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$ generated by Λ.

(ii) If

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty, \tag{4.8}$$

the paths of Σ are locally uniformly bounded in t for every $\omega \in \Omega$.

Furthermore, $\Sigma_t^+ = \int_0^t \Sigma_s ds$ exists for all $t \in \mathbb{R}^+$ and

$$\begin{aligned}\Sigma_t^+ = & \int_{M_d^-}\int_{-\infty}^t (\mathbf{A}(A))^{-1}\left(e^{A(t-s)}\Lambda(dA,ds)e^{A^*(t-s)}\right)\\ & - \int_{M_d^-}\int_{-\infty}^0 (\mathbf{A}(A))^{-1}\left(e^{-As}\Lambda(dA,ds)e^{-A^*s}\right)\\ & - \int_{M_d^-}\int_0^t (\mathbf{A}(A))^{-1}\Lambda(dA,ds)\end{aligned} \tag{4.9}$$

with $\mathbf{A}(A) : \mathbb{S}_d \to \mathbb{S}_d, X \mapsto AX + XA^*$.

(iii) Provided that

$$-\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)^2}{\rho(A)}\pi(dA) < \infty \tag{4.10}$$

and

$$\int_{M_d^-} \|A\|\kappa(A)^2\pi(dA) < \infty \tag{4.11}$$

it holds that

$$\Sigma_t = \Sigma_0 + \int_0^t Z_u du + L_t \tag{4.12}$$

where L is the underlying matrix subordinator and

$$Z_u = \int_{M_d^-}\int_{-\infty}^u \left(Ae^{A(u-s)}\Lambda(dA,ds)e^{A^*(u-s)} + e^{A(u-s)}\Lambda(dA,ds)e^{A^*(u-s)}A^*\right) \tag{4.13}$$

for all $u \in \mathbb{R}$ with the integral existing ω-wise.
Moreover, the paths of Σ are càdlàg and of finite variation on compacts.

Formula (4.9) is of particular interest in connection with stochastic volatility modelling, as in this case the integrated volatility Σ_t^+ is a quantity of fundamental importance.

Finally, we consider the existence of moments and the second order structure which follow immediately from Theorems 3.7 and 3.9.

Proposition 4.4. *Let Σ be a stationary $\mathbb{S}_d^+$-valued supOU process driven by a Lévy basis Λ satisfying the conditions of Theorem 4.1.*

(i) If

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{4.14}$$

for $r \in (0,1]$, then Σ has a finite r-th moment, i.e. $E(\|\Sigma_t\|^r) < \infty$.

(ii) If $r \in (1,\infty)$ and

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \qquad \int_{M_d^-} \frac{\kappa(A)^{2r}}{\rho(A)} \pi(dA) < \infty, \tag{4.15}$$

then Σ has a finite r-th moment, i.e. $E(\|\Sigma_t\|^r) < \infty$.

(iii) If the conditions given in (ii) are satisfied for $r = 2$, then the second order structure of Σ is given by:

$$E(\Sigma_0) = -\int_{M_d^-} \mathbf{A}(A)^{-1}\Big(\gamma_0 + \int_{\mathbb{S}_d} x\nu(dx)\Big)\pi(dA)$$

$$\mathrm{var}(\mathrm{vec}(\Sigma_0)) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\Big(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^*\nu(dx)\Big)\pi(dA)$$

$$\begin{aligned}&\mathrm{cov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))\\ &\quad= -\int_{M_d^-} e^{(A\otimes I_d + I_d \otimes A)h}(\mathscr{A}(A))^{-1}\Big(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^*\nu(dx)\Big)\pi(dA) \text{ for } h \in \mathbb{N},\end{aligned}$$

with $\mathbf{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto AX + XA^*$ and $\mathscr{A}(A) : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R})$, $X \mapsto (A \otimes I_d + I_d \otimes A)X + X(A^* \otimes I_d + I_d \otimes A^*)$.

The Examples 3.1 to 3.5 can all be immediately adapted to the positive semi-definite set-up. More examples in connection with stochastic volatility modelling can be found in [9].

5 Conclusion

In this paper we introduced multivariate supOU processes and obtained various important properties of them. Currently we are considering their use in stochastic volatility modelling in [9]. However, there are still many important issues to be addressed which we hope to do in future work. Of particular interest is, for example, the development of good estimators for supOU models and to show properties like consistency and asymptotic normality for them. This is related to understanding better the dependence structure of supOU processes which are clearly not Markovian.

Likewise, we have shown that supOU processes allow to model long memory effects (in a specific sense). This illustrates that a detailed theory of multivariate long range dependence should be developed.

Acknowledgements

This work was initiated during a visit of the authors to the Oxford-Man Institute at the University of Oxford in December 2007. The authors are very grateful for the hospitality and support.

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