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Abstract

The present paper characterizes various properties of chaos processes which in particular includes processes where all time variables admit a Wiener chaos expansion of a fixed finite order. The main focus is on the semimartingale property, p-variation and continuity. The general results obtained are finally used to characterize when a moving average is a semimartingale.

Keywords: semimartingales; p-variation; moving averages; chaos processes; absolutely continuity

AMS Subject Classification: 60G48; 60G51; 60G17; 60G15; 60G10

1 Introduction

The present paper is concerned with various properties of chaos processes. Chaos processes includes processes for which all coordinates belongs to a Wiener chaos of a fixed finite order, infinitely divisible processes, Rademacher processes, linear processes and more general processes which are limits of tetrahedral polynomials; see Section 2 for more details. In Rosiński et al. (1993) continuity and zero-one laws are derived for some classes of chaos processes. Houdré and Pérez-Abreu (1994) and Janson (1997) provides good surveys on various aspects of chaos processes.

In the first part we extend important results for Gaussian to chaos processes. In particular that of Jain and Monrad (1982) saying that if a separable Gaussian process is of bounded variation then the L^2-expansion converge in total variation norm to the process. Together with the observation by Jeulin (1993) that the process in this case is absolutely continuous with respect to a deterministic measure. Likewise the characterization of a stationary Gaussian processes of bounded variation, Ibragimov (1973), and the canonical decomposition of a Gaussian quasimartingale, Jain and Monrad (1982), together with the extension to Gaussian semimartingales,

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Stricker (1983), are generalized. Extensions of the result on Gaussian Dirichlet processes obtained by Stricker (1988) are also given. Furthermore we prove that chaos processes admitting a p-variation for some $p \geq 1$ are almost surely continuous except on an at most countable set, generalizing a result of Itô and Nisio (1968).

In the second part we study moving averages $X = \varphi * Y$ also known as stochastic convolutions. When Y is a Brownian motion, Knight (1992) has characterized those kernels φ for which X is an $\mathcal{F}^Y$-semimartingale, and Jeulin and Yor (1993) and Basse (2009b) those φ for which X is an $\mathcal{F}^X$-semimartingale. Basse and Pedersen (2009) have characterized those φ for which X is an $\mathcal{F}^Y$-semimartingale in the case where Y is Lévy process. Moreover, Basse (2008) extends Knight's result to the spectral representation of general Gaussian processes. Using the obtained decomposition results we provide necessary and sufficient conditions on φ for X to be an $\mathcal{F}^Y$-semimartingale. This result covers in particular the case where $dY_t = \sigma_t \, dW_t$ and σ is Gaussian chaos process associated with the Brownian motion W.

2 Preliminaries

Let $(\Omega, \mathcal{B}, P)$ denote a complete probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. $T > 0$ is here a fixed positive number. A càdlàg $\mathcal{F}$-adapted process $X = (X_t)_{t\in[0,T]}$ is called an $\mathcal{F}$-semimartingale if it admits a representation

$$X_t = X_0 + A_t + M_t, \qquad t \in [0,T], \tag{2.1}$$

where M is a càdlàg $\mathcal{F}$-local martingale starting at 0 and A is a càdlàg process of bounded variation starting at 0. Furthermore, X is called a special $\mathcal{F}$-semimartingale if A in (2.1) can be chosen predictable and in this case the decomposition is unique. A special $\mathcal{F}$-semimartingale X with canonical decomposition $X = X_0 + M + A$, is said to belong to H^p for $p \geq 1$ if $E[[M]_T^{p/2} + V_A(T)^p] < \infty$. $V_A(t)$ denotes the total variation of $s \mapsto A_s$ on $[0,t]$ and $[M]_t$ the quadratic variation of M on $[0,t]$. For each càdlàg process X set $D_X = \{t \in [0,T] : P(X_t = X_{t-}) < 1\}$. Then as it is well-known D_X is at most countable and D_X is empty if and only if X is continuous in probability.

Variation of processes will be important. To simplify the notation we set for each $p \geq 1$, $X = (X_t)_{t\in[0,T]}$ and $\tau = \{0 \leq t_0 < \cdots < t_n \leq T\}$

$$|\tau| = \max_{1\leq i\leq n} |t_i - t_{i-1}| \qquad \text{and} \qquad V_X^{p,\tau} = \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|^p.$$

We say that X admits a p-variation if there exists a right-continuous process $[X]^{(p)}$ such that for all $t \in [0,T]$ $V_X^{p,\tau} \to [X]_t^{(p)}$ in probability as $|\tau| \to 0$, where τ runs through all subdivisions of $[0,t]$. Furthermore, X is said to be of bounded p-variation if $\{V_X^{p,\tau} : \tau \text{ subdivision of } [0,T]\}$ is bounded in L^0. If $p = 2$ we use the short-hand notation $[X]$ for the quadratic variation of X, that is $[X] = [X]^{(2)}$. Observe that $V_X(t) = [X]_t^{(1)}$, if $V_X(T) < \infty$ a.s.

If X admits a p-variation then it is also of bounded p-variation. Likewise if X is of bounded p-variation it is also of bounded q-th variation for all $q \geq p$ since

$p \mapsto (\sum_{i=1}^n |a_i|^p)^{1/p}$ is decreasing. If X is càdlàg and τ_n are subdivisions of $[0,T]$ such that $|\tau_n| \to 0$ then

$$\liminf_{n\to\infty} V_X^{p,\tau_n} \geq \sum_{0<s\leq T} |\Delta X_s|^p, \qquad \text{a.s.}$$

Thus using

$$P(\liminf_{n\to\infty} V_X^{p,\tau_n} > x) \leq \sup_{n\geq 1} P(V_X^{p,\tau_n} > x), \qquad \text{for all } x > 0,$$

we have that $\sum_{0<s\leq T} |\Delta X_s|^p < \infty$ a.s. if X is of bounded p-variation.

Throughout the following I denotes a set and for all $i \in I$, $\mathcal{H}_i$ is a family of independent random variables. Set $\mathcal{H} = \{\mathcal{H}_i\}_{i\in I}$. For each Banach space F and $i \in I$ let $\mathcal{P}^d_{\mathcal{H}_i}(F)$ denote the set of variables $p(Z_1, \ldots, Z_n)$ where $n \geq 1$, $Z_1, \ldots, Z_n$ different elements in $\mathcal{H}_i$ and p is an F-valued tetrahedral polynomial of order d. Recall that $p\colon \mathbb{R}^n \to F$ is called an F-valued tetrahedral polynomial of order d if there exist $x_0, x_{i_1,\ldots,i_k} \in F$ and $l \geq 1$ such that

$$p(z_1, \ldots, z_n) = x_0 + \sum_{k=1}^d \sum_{1\leq i_1<\cdots<i_k\leq l} x_{i_1,\ldots,i_k} \prod_{j=1}^k z_{i_j}.$$

Let $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ denote the closure in distribution of $\cup_{i\in I} \mathcal{P}^d_{\mathcal{H}_i}(F)$, that is, $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ is the set of all F-valued random elements X for which there exists a sequence $(X_k)_{k\geq 1} \subseteq \cup_{i\in I} \mathcal{P}^d_{\mathcal{H}_i}(F)$ converging weakly to X.

The following two conditions on $\mathcal{H}$ will be important:

(a) For $q \in (0,\infty)$ there exists $\beta_1, \beta_2 > 0$ such that for all $Z \in \cup_{i\in I} \mathcal{H}_i$ there exists $c_Z > 0$ satisfying

$$P(|Z| \geq c_Z) \geq \beta_1 \quad \text{and} \quad E[|Z|^q, |Z| > s] \leq \beta_2 s^q P(|Z| > s) \quad s \geq c_Z.$$

(b)
$$\bigcup_{i\in I} \mathcal{H}_i \subseteq L^1 \qquad \text{and} \qquad \sup_{i\in I} \sup_{Z\in\mathcal{H}_i} \Big(\frac{\|Z - E[Z]\|_\infty}{\|Z - E[Z]\|_2} \Big) = \beta_3 < \infty.$$

Notation, chaos processes. *A real-valued stochastic process $X = (X_t)_{t\in U}$ is said to be a chaos process of order d if $(X_{t_1}, \ldots, X_{t_n}) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$ for all $n \geq 1$ and $t_1, \ldots, t_n \in U$. Furthermore X is said to be a chaos process if it is a chaos process of order d for some $d \geq 1$. A chaos process X is said to satisfy C_q for $0 < q < \infty$, if the associated $\mathcal{H}$ satisfies* (a) *for the given q and if $d \geq 2$ all $Z \in \cup_{i\in I} \mathcal{H}_i$ are symmetric. Moreover, X is said to satisfy C_∞ if $\mathcal{H}$ satisfies* (b).

Following Fernique (1997) a mapping N, from a vector space V into $[0,\infty]$, is called a pseudo-seminorm if for all $\theta \in \mathbb{R}$ and $x, y \in V$ we have

$$N(\theta x) = |\theta| N(x) \qquad \text{and} \qquad N(x+y) \leq N(x) + N(y).$$

The following result, which is taken from Basse (2009a, Theorem 2.7), is crucial for this paper. Here $d \geq 1$ and $q > 0$ are given numbers.

Theorem 2.1. *Let U denote a countable set, $X = (X_t)_{t\in U}$ a chaos process of order d satisfying C_q and N a lower semi-continuous pseudo-seminorm on $\mathbb{R}^U$ equipped with the product topology such that $N(X) < \infty$ a.s. Then for all finite $p \le q$ there exists a real constant $k_{p,q,d,\beta}$, only depending on p, q, d and the β's from* (a) *and* (b)*, such that*

$$\|N(X)\|_q \le k_{p,q,d,\beta} \|N(X)\|_p < \infty.$$

Three important examples of chaos processes satisfying C_q are given as follows:

(1): Let $\mathcal{G}$ denote a vector space of Gaussian random variables, and for $d \ge 1$ $\overline{\mathcal{P}}_{\mathcal{G}}^d$ be the closure in probability of all random variables of the form $p(Z_1, \ldots, Z_n)$, where $n \ge 1$, $Z_1, \ldots, Z_n \in \mathcal{G}$ and $p\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most d (not necessarily tetrahedral). $X = (X_t)_{t\in U}$ satisfying $\{X_t : t \in U\} \subseteq \overline{\mathcal{P}}_{\mathcal{G}}^d$ is then called a Gaussian chaos process of order d, and it is in particular a chaos process satisfying C_∞ (see Basse (2009a)); in fact we may chose $I = \{0\}$ and $\mathcal{H}_0$ to be a Rademacher sequence. Recall that a Rademacher sequence is an independent, identically distributed sequence $(Z_n)_{n\ge1}$ such that $P(Z_1 = \pm 1) = \frac{1}{2}$. The key example of a Gaussian vector space $\mathcal{G}$ is

$$\mathcal{G} = \left\{ \int_0^\infty h(s)\, dW_s : h \in L^2(\mathbb{R}_+, \lambda) \right\}, \tag{2.2}$$

where W is a Brownian motion and λ is the Lebesgue measure. In this case X is a Gaussian chaos process of order d if and only if it has the following representation in terms of multiple Wiener-Itô integrals

$$X_t = \sum_{k=0}^{d} \int_{\mathbb{R}_+^k} f_{k,t}(s_1, \ldots, s_k)\, dW_{s_1} \cdots dW_{s_k}, \qquad t \in U, \tag{2.3}$$

where $f_{k,t} \in L^2(\mathbb{R}_+^k)$. Processes of the form (2.3) appear as weak limits of U-statistics, see Janson (1997, Chapter 11) and de la Peña and Giné (1999). For a detailed survey on Gaussian chaos processes and expansions, see Janson (1997), Nualart (2006) and Houdré and Pérez-Abreu (1994).

(2): Let $X = (X_t)_{t\in U}$ be given by

$$X_t = \int_S f(t, s)\, \Lambda(ds), \qquad t \in U, \tag{2.4}$$

where Λ is an independently scattered infinitely divisible random measure (or random measure for short) on some non-empty space S equipped with a δ-ring $\mathcal{S}$, and $s \mapsto f(t, s)$ are Λ-integrable deterministic functions in the sense of Rajput and Rosiński (1989). The associated $\mathcal{H} = \{\mathcal{H}_i\}_{i\in I}$ is here described by

$$\mathcal{H}_i = \{\Lambda(A_1), \ldots, \Lambda(A_n)\}, \qquad i \in I,$$

for I denoting the set of all finite collections $\{A_1, \ldots, A_n\}$ where $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{S}$. In this case X is a chaos process of order 1. For example if X

is a symmetric α-stable process separable in L^0, then X has a representation of the form (2.4) and hence it follows that it is a chaos process of order 1 satisfying C_q for all $q < \alpha$. For further examples of random measures Λ for which X given by (2.4) satisfies C_q see Basse (2009a).

(3): Assume that $(Z_n)_{n\geq 1}$ is a sequence of independent, identically distributed random variables and $x(t), x_{i_1,\dots,i_k}(t) \in \mathbb{R}$ are real numbers such that

$$X_t = x(t) + \sum_{k=1}^{d} \sum_{1\leq i_1<\dots<i_k<\infty} x_{i_1,\dots,i_k}(t) \prod_{j=1}^{k} Z_{i_j},$$

exists in probability for all $t \in U$, then $X = (X_t)_{t\in U}$ is a chaos process of order d associated to $I = \{0\}$ and $\mathcal{H}_0 = \{Z_n : n \geq 1\}$. If for some $\alpha > 0$, $x \mapsto P(|Z_1| > x)$ is regulary varying with index $-\alpha$ then $\mathcal{H}$ satisfies (a) for all $q \in (0, \alpha)$; see Bingham et al. (1989, Theorem 1.5.11). In particular, if Z_1 follows a symmetric α-stable distribution for some $\alpha \in (0, 2)$ then $\mathcal{H}$ satisfies (a) for all $q \in (0, \alpha)$. If the common distribution is Poisson, exponential, gamma or Gaussian then $\mathcal{H}$ satisfies (a) for all $q > 0$. Finally, $\mathcal{H}$ satisfies (b) if and only if Z_1 is a.s. bounded.

3 Path properties

For all $p \geq 0$ and all subset A of L^p denote by $\overline{\text{span}}_{L^p} A$ the L^p-closure of the linear span of A. Let $X = (X_t)_{t\in[0,T]}$ be a square-integrable process for which $\overline{\text{span}}_{L^2}\{X_t : t \in [0, T]\}$ is a separable Hilbert space with orthonormal basis $(U_i)_{i\geq 1}$. Let $X_t^{(n)}$ denote the n-th order L^2-expansion of X_t given by

$$X_t^{(n)} = \sum_{j=1}^{n} f_j(t) U_j, \tag{3.1}$$

where $f_j(t) = E[U_j X_t]$ for $j \geq 1$. Note that for $t \in [0, T]$, $\lim_n X_t^{(n)} = X_t$ in L^2. The above separability assumption is always satisfied if X is a càdlàg process satisfying C_q for some $q \in [2, \infty]$.

If X is càdlàg and of integrable variation μ_X denotes the Lebesgue–Stieltjes measure on $[0, T]$ induced by $t \mapsto E[V_X(t)]$. In this context we have the following extension of Jain and Monrad (1982, Theorem 1.2) and Jeulin (1993) in the Gaussian case. Here $BV([0, T])$ denotes the Banach space $\{f \in \mathbb{R}^{[0,T]} : f \text{ càdlàg and } V_f(T) < \infty\}$ equipped with the norm $\|f\|_{BV} = V_f(T) + |f(0)|$.

Theorem 3.1. *Let $X = (X_t)_{t\in[0,T]}$ denote a càdlàg process of bounded variation satisfying C_q for some $q \in [2, \infty]$. Then there exists a subsequence $(n_k)_{k\geq 1}$ such that $X^{(n_k)}$ converges a.s. to X in $BV([0, T])$ and X is a.s. absolutely continuous with respect to μ_X.*

For an α-stable process X of the form (2.4) with $1 < \alpha < 2$, it is shown in Pérez-Abreu and Rocha-Arteaga (1997, Theorem 4(b)) that if X is of bounded variation and satisfies some additional assumption then it is absolutely continuous with respect to μ_X. This situation is not covered by Theorem 3.1 since for such processes only C_q for $q \in (0, \alpha)$ is satisfied. If the sample paths of X are contained in a separable subspace of $BV([0,T])$ Theorem 3.1 follows by Basse (2009a, Corollary 2.11). On the other hand, Theorem 3.1 insures that almost all sample paths of X do belong to a separable subspace of $BV([0,T])$, more precisely to the space of functions which are absolutely continuous with respect to μ_X.

Theorem 3.1 is a direct consequence of Theorem 2.1 and the following lemma, in which X, $X^{(n)}$ and f_j are as above.

Lemma 3.2. *Assume that $X = (X_t)_{t\in[0,T]}$ is a càdlàg process of integrable variation such that $\|X_s - X_u\|_2 \leq c\|X_s - X_u\|_1$ for all $0 \leq s < u \leq T$ and some $c > 0$. Then each f_j is absolutely continuous with respect to μ_X and $\lim_n E[V_{X-X^{(n)}}(T)] = 0$.*

Proof. For $j \geq 1$ and $0 \leq s < u \leq T$ we have

$$|f_j(s) - f_j(u)| \leq \|U_j\|_2 \|X_s - X_u\|_2 \leq c\|X_s - X_u\|_1,$$

which shows that each f_j is absolutely continuous with respect to μ_X. Let ψ_j denote the density of f_j with respect to μ_X. We have

$$E[V_{X-X^{(n)}}(T)] \leq \sup_{k\geq 1} \sum_{i=1}^{a_k} \Big(\sum_{j=n+1}^{\infty} (f_j(t_i^k) - f_j(t_{i-1}^k))^2 \Big)^{1/2}, \tag{3.2}$$

where $\tau_k = \{0 = t_0^k < \cdots < t_{a_k}^k = T\}$ are nested subdivisions of $[0,T]$ satisfying $|\tau_k| \to 0$. By Jeulin (1993, Lemme 3) the right-hand side of (3.2) equals

$$\int_0^T \Big(\sum_{j=n+1}^{\infty} \psi_j(s)^2 \Big)^{1/2} \mu_X(ds).$$

Another application of Jeulin (1993, Lemme 3) yields

$$\begin{aligned}\int_0^T &\Big(\sum_{j=1}^{\infty} \psi_j(s)^2 \Big)^{1/2} \mu_X(ds) \\ &= \sup_{k\geq 1} \sum_{i=1}^{a_k} \Big(\sum_{j=1}^{\infty} (f_j(t_i^k) - f_j(t_{i-1}^k))^2 \Big)^{1/2} \leq cE[V_X(T)] < \infty.\end{aligned}$$

Thus by Lebesgue's dominated convergence theorem, $\lim_n E[V_{X-X^{(n)}}(T)] = 0$. This completes the proof. □

The equivalence of the L^1- and L^2-norms of the increments of X is crucial for Lemma 3.2 to be true. For example if X is a Poisson process with parameter $\lambda > 0$ then μ_X is proportional to the Lebesgue measure but all paths are step functions.

Corollary 3.3. *Let $X = (X_t)_{t\in[0,T]}$ be as in Theorem 3.1. Then for every Radon measure μ on $[0,T]$ there exists a unique decomposition $X_t = Y_t + A_t$ of X, where Y and A are càdlàg processes of bounded variation such that Y is absolutely continuous with respect to μ and A is singular to μ and $\{Y_t, A_t : t \in [0,T]\} \subseteq \overline{\mathrm{span}}_{L^0}\{X_t : t \in [0,T]\}$.*

Proof. Let $\mathcal{S}_0 = \overline{\operatorname{span}}_{L^0}\{X_t : t \in [0,T]\}$. Since $\mathcal{S}_0$ is L^2-closed the U_n's in (3.1) belong to $\mathcal{S}_0$. For each $j \geq 1$, decompose f_j in (3.1) as $f_j = g_j + h_j$, where g_j, h_j are càdlàg functions of bounded variation, g_j being absolutely continuous with respect to μ and h_j singular to μ. Set

$$Y_t^{(n)} = \sum_{j=1}^{n} g_j(t)U_j \quad \text{and} \quad A_t^{(n)} = \sum_{j=1}^{n} h_j(t)U_j, \qquad t \in [0,T].$$

For all $n, k \geq 1$,

$$V_{X^{(n)}-X^{(k)}}(T) = V_{Y^{(n)}-Y^{(k)}}(T) + V_{A^{(n)}-A^{(k)}}(T). \tag{3.3}$$

By Theorem 3.1 there exists a subsequence $(n_k)_{k\geq 1}$ such that $\lim_k X^{(n_k)} = X$ in the total variation norm on $[0,T]$ and so by completeness (3.3) implies that $\lim_k Y^{(n_k)}$ and $\lim_k A^{(n_k)}$ exist in total variation norm a.s. Calling these limit processes Y and A we have for all $t \in [0,T]$

$$\lim_{k\to\infty} Y_t^{(n_k)} = Y_t \qquad \text{and} \qquad \lim_{k\to\infty} A_t^{(n_k)} = A_t, \qquad \text{a.s.},$$

showing that $Y_t, A_t \in \mathcal{S}_0$. Moreover since the sets of functions which are absolutely continuous with respect to μ respectively singular to μ are closed in $BV([0,T])$ the proof of the corollary is complete. $\square$

Lemma 3.4. *Let X denote a càdlàg process process of bounded p-th variation. Then X admits an q-variation for all $q > p$ and*

$$[X]_t^{(q)} = \sum_{0<s\leq t} |\Delta X_s|^q < \infty, \qquad 0 \leq t \leq T.$$

Proof. Fixed $q > p$ and set for $0 \leq t \leq T$ and $n \geq 1$

$$X_t^n = \sum_{0<s\leq t} \Delta X_s 1_{\{|\Delta X_s|>1/n\}}, \qquad S_t = \sum_{0<s\leq t} |\Delta X_s|^q.$$

Recall that $S_t < \infty$ a.s. since X is of bounded q-variation. For all $n \geq 1$ X^n has piecewise constant sample paths and so X^n admits a q-variation and

$$[X^n]_t^{(q)} = \sum_{0<s\leq t} |\Delta X_s|^q 1_{\{|\Delta X_s|>1/n\}} \xrightarrow[n\to\infty]{} S_t \qquad \text{a.s.},\ t \in [0,T].$$

Therefore it reduces to show

$$\lim_{n\to\infty} \limsup_{|\tau|\to 0} P(|V_X^{q,\tau} - V_{X^n}^{q,\tau}| > \varepsilon) = 0 \qquad \text{for all } \varepsilon > 0. \tag{3.4}$$

Writing $\tilde{X}_t^n$ for $X_t - X_t^n$ we have for all $n \geq 1, t \in [0,T]$ and subdivisions $\tau = \{0 = t_0 < \cdots < t_k = t\}$

$$|V_X^{q,\tau} - V_{X^n}^{q,\tau}| \leq \sum_{i=1}^{k} \big||X_{t_i} - X_{t_{i-1}}|^q - |X_{t_i}^n - X_{t_{i-1}}^n|^q\big| \leq q \sum_{i=1}^{k} C_i^{q-1} |\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n|,$$

for some C_i's between $|X^n_{t_i} - X_{t_{i-1}}|$ and $|\tilde{X}^n_{t_i} - \tilde{X}_{t_{i-1}}|$, and hence by Hölder's inequality

$$\begin{aligned}
|V^{q,\tau}_X - V^{q,\tau}_{X^n}| &\leq q\Big(\sum_{i=1}^k C_i^q\Big)^{(q-1)/q}\Big(\sum_{i=1}^k |\tilde{X}^n_{t_i} - \tilde{X}^n_{t_{i-1}}|^q\Big)^{1/q} \\
&\leq q\Big(V^{q,\tau}_X + V^{q,\tau}_{X^n}\Big)^{(q-1)/q}\Big(\max_{1\leq i\leq k}|\tilde{X}^n_{t_i} - \tilde{X}^n_{t_{i-1}}|^{q-p} V^{p,\tau}_{\tilde{X}^n}\Big)^{1/q} \\
&\leq q2^{p/q}\Big(V^{q,\tau}_X + V^{q,\tau}_{X^n}\Big)^{(q-1)/q}\Big(V^{p,\tau}_X + V^{p,\tau}_{X^n}\Big)^{1/q}\max_{1\leq i\leq k}|\tilde{X}^n_{t_i} - \tilde{X}^n_{t_{i-1}}|^{(q-p)/q}.
\end{aligned}$$

Using that $\max_{1\leq i\leq k}|\tilde{X}^n_{t_i} - \tilde{X}^n_{t_{i-1}}| < 2n^{-1}$ for $|\tau|$ sufficiently small we have

$$\begin{aligned}
&\limsup_{|\tau|\to 0} P(|V^{q,\tau}_X - V^{q,\tau}_{X^n}| > \varepsilon) \\
&\quad\leq \limsup_{|\tau|\to 0} P\left(q2^{p/q}(V^{q,\tau}_X + S_t)^{(q-1)/q}(V^{p,\tau}_X + S^{p/q}_t)^{1/q}2n^{-1} > \frac{\varepsilon}{2}\right),
\end{aligned}$$

which implies (3.4) since $\{V^{p,\tau}_X : \tau\}$ is bounded in L^0. □

Proposition 3.5. *Let X denote a càdlàg process. Assume that it admits a p-variation and satisfies C_q for some $q \in [2p, \infty]$ or that it is of bounded p-variation and satisfies C_q for some $q \in (2p, \infty]$. Then a.s. X is discontinuous only on D_X, and hence X is a.s. continuous if and only if it is continuous in probability.*

In the proof we need the following two remarks concerning any càdlàg process X:

(i) If X is of integrable variation then $\mu_X(\{t\}) > 0$ if and only if $t \in D_X$.

(ii) If X admits a p-variation then $\Delta[X]^{(p)} = |\Delta X|^p$.

To prove (i) let $t > 0$ and choose $(t_n)_{n\geq 1} \subseteq [0, t)$ such that $t_n \uparrow t$. By Lebesgue's dominated convergence theorem we have

$$\mu_X(\{t\}) = \lim_{n\to\infty} E[V_X(t) - V_X(t_n)] = E\big[\lim_{n\to\infty}(V_X(t) - V_X(t_n))\big] = E[|\Delta X_t|],$$

which shows (i). For $p = 2$ (ii) follows by Jacod (1981, Lemme 3.11). The general case can be proved by imitating Jacod's proof.

Proof of Proposition 3.5. We may assume that X admits a p-variation. Indeed, if X is of bounded p-variation and satisfies C_q for some $q \in (2p, \infty]$ then according to Lemma 3.4 it admits a $\frac{q}{2}$-variation.

Assume therefore that X admits a p-variation and satisfies C_q for a $q \in [2p, \infty]$. Let $0 \leq u < t \leq T$ and choose subdivisions τ_n of $[u, t]$ such that

$$\lim_{n\to\infty} V^{p,\tau_n}_X = [X]^{(p)}_t - [X]^{(p)}_u, \qquad \text{almost surely.}$$

For $f \in \mathbb{R}^{[0,T]}$ let

$$N(f) = \limsup_{n\to\infty}(V^{p,\tau_n}_f)^{1/p}.$$

Then N is a lower semicontinuous pseudo-seminorm, and since $([X]^{(p)}_t - [X]^{(p)}_u)^{1/p} = N(X)$ a.s. it follows by Theorem 2.1 that

$$\|[X]^{(p)}_t - [X]^{(p)}_u\|_2 = \|N(X)\|^p_{2p} \le k^p_{p,2p}\|N(X)\|^p_p = k^p_{p,2p}\|[X]^{(p)}_t - [X]^{(p)}_u\|_1 < \infty.$$

For $u = 0$ this gives that $[X]^{(p)}$ is integrable and since it is increasing it is also of integrable variation. Hence by Lemma 3.2 $[X]^{(p)}$ is a.s. absolutely continuous with respect to $\mu_{[X]^{(p)}}$ and so by (i) $[X]^{(p)}$ is continuous on $D^c_{[X]^{(p)}}$. Finally, by applying (ii) it follows that X is continuous on D^c_X. Therefore, X has continuous sample paths if and only if D_X is empty, that is if X is continuous in probability. □

For $f\colon \mathbb{R} \to \mathbb{R}$, let $W_f\colon \mathbb{R} \to [0,\infty]$ denote its oscillation function given by

$$W_f(t) = \lim_{n\to\infty} \sup_{u,s\in[t-1/n,t+1/n]} |f(s) - f(u)|, \qquad t \in \mathbb{R}.$$

Itô and Nisio (1968, Theorem 1) show that each separable Gaussian process which is continuous in probability has a deterministic oscillation function. By Marcus and Rosen (2006, Theorem 5.3.7) this is also true for Rademacher processes. Furthermore, Cambanis et al. (1990) show that a very large class of infinitely divisible processes also have this property. Thus for such processes Proposition 3.5 holds even without the assumption of being of bounded p-variation. On the other hand the following example shows that Gaussian chaos processes do not in general have deterministic oscillation functions. Let $(Y_t)_{t\ge 0}$ denote a Gaussian process which is continuous in probability and has oscillation function $t \mapsto \alpha(t) \in (0,\infty)$ and such that Y_0 is non-deterministic. Then X, given by $X_t = Y_0 Y_t$, is a separable second-order Gaussian chaos process continuous in probability with oscillation function $t \mapsto |Y_0|\alpha(t)$.

3.1 The stationary increment case

According to e.g. Doob (1990), a centered and L^2-continuous process $X = (X_t)_{t\in\mathbb{R}}$ with stationary increments has a spectral measure m_X, which is the unique symmetric measure integrating $s \mapsto (1+s^2)^{-1}$ and satisfying

$$\Gamma_X(t,u) := E[(X_t - X_0)(X_u - X_0)] = \int_{\mathbb{R}} \frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\, m_X(ds).$$

Furthermore set $v_X(t) = \Gamma_X(t,t)$, and if X is stationary denote by R_X its auto covariance function, and by n_X the unique finite and symmetric measure satisfying

$$R_X(t) = E[X_t X_0] = \int_{\mathbb{R}} e^{its}\, n_X(ds), \qquad t \in \mathbb{R}.$$

Proposition 3.6. *Assume that X is an L^2-continuous process with stationary increments satisfying condition C_q for some $q \in [2,\infty]$. Then the following five conditions are equivalent:*

(i) X has a.s. càdlàg paths of bounded variation,

(ii) X *has a.s. absolutely continuous paths,*

(iii) $m_X(\mathbb{R}) < \infty$, *(iv)* $\Gamma_X \in C^2(\mathbb{R}^2; \mathbb{R})$, *(v)* $v_X \in C^2(\mathbb{R}; \mathbb{R})$.

If X *is stationary then (i)-(v) are also equivalent to* $\int_{\mathbb{R}} t^2 \, n_X(dt) < \infty$ *or* $R_X \in C^2(\mathbb{R}; \mathbb{R})$.

The Gaussian case is covered by Ibragimov (1973, Theorem 12). See also Doob (1990, page 536) for general results about mean-square differentiability. A Hermite process X with parameter $(d, H) \in \mathbb{N} \times (\frac{1}{2}, 1)$ is a Gaussian chaos process of order d with stationary increments and the same covariance function as the fractional Brownian motion with Hurst parameter H; see Maejima and Tudor (2007) for a precise definition. The corresponding spectral measure is $m_X(ds) = c_H |s|^{1-2H} \, ds$, that is a non-finite measure, and so by Proposition 3.6 X is not of bounded variation.

Proof. Assume (i), that is X has càdlàg paths of bounded variation. The stationary increments implies that μ_X equals the Lebesgue measure up to a scaling constant. Thus (i)⇒(ii) since by Theorem 3.1 X is absolutely continuous with respect to μ_X. (ii)⇒(i) is obvious. Furthermore if X is càdlàg and of bounded variation then by Proposition 3.7 below we have

$$\infty > \sup_{n \geq 1} \Big(n^2 v_X(1/n) \Big) \geq \sup_{n \geq 1} \int_{\mathbb{R}} \Big(\frac{\sin(s/n)}{s/n} \Big)^2 m_X(ds).$$

Hence by Fatou's lemma $m_X(\mathbb{R}) < \infty$ and so (i)⇒(iii). (iii)⇒(iv)⇒(v) are easy. To see that (v) implies (i) assume $v_X \in C^2(\mathbb{R}; \mathbb{R})$. Since v_X is symmetric and $v_X(0) = 0$ we have $v_X'(0) = 0$. Thus $v_X(t) = O(t^2)$ as $t \to 0$ and hence by Proposition 3.7 X is of bounded 1-variation. To show that a.a. sample paths of X are càdlàg and of bounded variation let τ_n be nested subdivisions of $[a, b]$ such that $|\tau_n| \to 0$. Using that an increasing sequence which is bounded in L^0 is a.s. bounded, $\sup_{n \geq 1} V_X^{1,\tau_n} < \infty$ a.s. Since X has sample paths of bounded variation through $\cup_{n \geq 1} \tau_n$ and is L^2-continuous we may choose a right-continuous modification of X. This modification will then have càdlàg paths of bounded variation, showing (i). The stationary case follows by similarly arguments. □

Proposition 3.7. *Let* $p \geq 1$ *and assume that* X *is an* L^2*-continuous process with stationary increments and satisfies* C_q *for some* $q \in [p, \infty]$. *Then* X *is of bounded* p*-variation if and only if* $v_X(t) = O(t^{2/p})$ *as* $t \to 0$. *Furthermore,* X *admits a* p*-variation zero, i.e.* $[X]_t^{(p)} \equiv 0$, *if and only if* $v_X(t) = o(t^{2/p})$ *as* $t \to 0$.

Proof. Assume that X is of bounded p-variation. For all $r \leq v \leq q$ there exists, according to Theorem 2.1, a constant $k_{r,v}$ such that for all subdivisions τ

$$\| (V_X^{p,\tau})^{1/p} \|_v \leq k_{r,q} \| (V_X^{p,\tau})^{1/p} \|_r < \infty. \tag{3.5}$$

Since $\{ (V_X^{p,\tau})^{1/p} : \tau \}$ is bounded in L^0, (3.5) and Krakowiak and Szulga (1986, Corollary 1.4) shows that $\sup_\tau \| (V_X^{p,\tau})^{1/p} \|_v < \infty$. In particular for $v = p$

$$\infty > \sup_\tau E[V_X^{p,\tau}] = \sup_\tau \sum_{i=1}^k E\Big[|X_{t_i} - X_{t_{i-1}}|^p \Big],$$

where $\tau = \{0 = t_0 < \cdots < t_k = T\}$. Using the equivalence of moments of X, see Theorem 2.1, it now follows that X is of bounded p-variation if and only if

$$\sup_{\tau} \sum_{i=1}^{k} v_X(t_i - t_{i-1})^{p/2} < \infty. \tag{3.6}$$

This proves the first part of the statement since (3.6) is equivalent to $v_X(t) = O(t^{2/p})$. Similar arguments show that X admits a p-variation zero if and only if

$$\lim_{|\tau|\to 0} \sum_{i=1}^{k} v_X(t_i - t_{i-1})^{p/2} = 0. \tag{3.7}$$

Thus by observing that (3.7) is satisfied if and only if $v_X(t) = o(t^{2/p})$ the proof is complete. □

By definition $v_X(t) = t^{2H}$ for a Hermite process X with parameters (d, H). Thus by Proposition 3.7 X is of bounded p-variation if and only if $p \geq \frac{1}{H}$. Moreover, X has p-variation zero if and only if $p > \frac{1}{H}$. If X is Gaussian such that v_X is concave and $\alpha := \lim_{t\to 0} v_X(t)/t^{2/p}$ exists in $\mathbb{R}$ for some $p \geq 2$ it is possible to show that X admits a p-variation; see Marcus and Rosen (2006, Theorem 10.2.3). The special case $\alpha = 0$ is included in the above Proposition 3.7, however a generalization to $\alpha > 0$ is not straightforward since the proof here relies on Borell's isoperimetric inequality in which the Gaussian assumption is crucial.

4 Semimartingales

In this section we characterize the canonical decomposition of chaos semimartingales, and in the next section this characterization is used to study when a moving average is a semimartingale.

The canonical decomposition of Gaussian quasimartingales are characterized in Jain and Monrad (1982) and their result is extended to Gaussian semimartingales in Stricker (1983). Theorem 2.1 allows us to generalize this to a much larger setting. The proof by Stricker (1983) relies on the fact that a càdlàg Gaussian process X, and in particular Gaussian semimartingales, only has jumps on D_X. If X is a chaos process satisfying C_q for some $q \in [4, \infty]$ admitting a quadratic variation we know by Proposition 3.5 that X has only jumps on D_X, allowing us to proceed as in Stricker (1983). However, in the case $q \in [1, 4)$ we need a result by Meyer (1984).

We shall need the following notation: Given a filtration $\mathcal{F}$, a process X is said to be $(\mathcal{F}, q)$-stable if $(E[X_t|\mathcal{F}_s])_{s,t\in[0,T]}$ is a chaos process satisfying C_q. In this case set $\mathcal{PC} = \overline{\text{span}}_{L^0}\{E[X_t|\mathcal{F}_s] : s, t \in [0, T]\}$.

Theorem 4.1. *Let $X = (X_t)_{t\in[0,T]}$ denote an $(\mathcal{F}, q)$-stable chaos process for some $q \in [1, \infty]$. If X is an $\mathcal{F}$-semimartingale then $X \in H^p$ for all finite $p \in [1, q]$ and $\{A_t, M_t : t \in [0, T]\} \subseteq \mathcal{PC}$, where $X = X_0 + M + A$ is the $\mathcal{F}$-canonical decomposition of X. In particular A and M are chaos processes satisfying C_q.*

Let M^d and M^c denote, respectively, the purely discontinuous and continuous martingale component of M and A^c, A^{sc} and A^d the absolutely continuous, singular *continuous respectively discrete component of A. If $q \in [4,\infty]$ then X has a.s. only jumps on D_X and has therefore a.s. continuous paths if and only if it is continuous in probability. Moreover, $\{M_t^c, M^d, A_t^c, A^{sc}, A_t^d : t \in [0,T]\} \subseteq \mathcal{PC}$, and for each $t \in [0,T]$ we have*

$$M_t^d = \sum_{s\in(0,t]\cap D_X} \Delta M_s \quad \textit{and} \quad A_t^d = \sum_{s\in(0,t]\cap D_X} \Delta A_s, \tag{4.1}$$

where both sums converge in L^p for all finite $p \le q$ and the second converges also absolutely a.s.

Proof. Consider subdivisions $\tau_n = \{0 = t_0^n < \cdots < t_{2^n}^n = T\}$ where $t_i^n = Ti2^{-n}$ for $i = 0, \dots, 2^n$. By passing to a subsequence we may assume that $\lim_{n\to\infty} V_X^{2,\tau_n}$ exists a.s. For $f\colon [0,T]\cap\mathbb{Q} \to \mathbb{R}$ define

$$\Phi(f) := \sup_{n\ge 1} \sqrt{V_f^{2,\tau_n}}. \tag{4.2}$$

Then Φ is a lower semicontinuous pseudo-seminorm on $\mathbb{R}^{[0,T]\cap\mathbb{Q}}$ and $\Phi(X) < \infty$ a.s. Since X is a chaos process satisfying C_q Theorem 2.1 shows that $E[\Phi(X)^p] < \infty$ for all finite $p \le q$. In particular $\Phi(X)$ is integrable and hence by Meyer (1984) X is a special $\mathcal{F}$-semimartingale. Denoting by A its bounded variation component Meyer (1984) shows moreover that

$$S_n^X := \sum_{i=1}^{2^n} E[X_{t_i} - X_{t_{i-1}} | \mathcal{F}_{t_{i-1}}] \xrightarrow[n\to\infty]{} A_T \qquad \text{in the weak } L^1\text{-topology.} \tag{4.3}$$

Since $\mathcal{PC}$ is L^1-closed, (4.3) shows that $A_T \in \mathcal{PC}$. Similar arguments show that $\{A_s : s \in [0,T]\} \subseteq \mathcal{PC}$ and hence also $\{M_s : s \in [0,T]\} \subseteq \mathcal{PC}$. Since X is $(\mathcal{F}, q)$-stable this shows that A and M are chaos processes satisfying C_q. Thus by arguing as above we have $E[[M]_T^{p/2}] < \infty$ for all finite $p \le q$. Moreover define for $f\colon [0,T]\cap\mathbb{Q} \to \mathbb{R}$

$$\Psi(f) := \sup_{n\ge 1} V_f^{1,\tau_n}.$$

Then Ψ is a lower semicontinuous pseudo-seminorm on $\mathbb{R}^{[0,T]\cap\mathbb{Q}}$ and $\Psi(A) < \infty$ a.s.. Hence by Theorem 2.1, $E[V_A(T)^p] < \infty$ for all finite $p \le q$ implying that $X \in H^p$ for all finite $p \le q$.

To prove the second part assume $q \ge 4$. By Corollary 3.3, $A^c, A^{sc}, A^d \subseteq \mathcal{PC}$, since $A \subseteq \mathcal{PC}$. We claim that $D_A \subseteq D_X$. Assume on the contrary there exists a number $t \in D_A \setminus D_X$. Then

$$\Delta A_t = E[\Delta A_t | \mathcal{F}_{t-}] = -E[\Delta M_t | \mathcal{F}_{t-}] = 0, \qquad \text{a.s.}$$

contradicting the assumption that $t \in D_A$. Hence D_A and therefore also D_M are contained in D_X. By Proposition 3.5, A and M are continuous on D_A^c respectively D_M^c, implying that they are continuous on D_X^c. This shows that A^d is of the form (4.1). Set

$$(Y_t)_{t\in[0,T]} = \left(\int_0^t 1_{D_X^c}(s)\, dM_s\right)_{t\in[0,T]} \quad \text{and} \quad (U_t)_{t\in[0,T]} = \left(\int_0^t 1_{D_X}(s)\, dM_s\right)_{t\in[0,T]}.$$

Since $(\Delta Y_t)_{t\in[0,T]} = (1_{D_X^c}(t)\Delta M_t)_{t\in[0,T]}$ and M is continuous on D_X^c, Y is a continuous martingale. On the other hand for every continuous bounded martingale N we have

$$\langle U, N\rangle_t = \int_0^t 1_{D_X}(s)\, d\langle M, N\rangle_s = 0,$$

since $\langle M, N\rangle$ is continuous and D_X is countable. Thus U is a purely discontinuous martingale, and so U and Y are the purely discontinuous respectively the continuous martingale component of M. Finally, since D_X is countable,

$$U_t = \sum_{s\in(0,t]\cap D_X} \Delta M_s,$$

where the sum converges in probability and therefore also in L^p for all finite $p \le q$ according to Theorem 2.1. □

Essentially due to Föllmer (1981) a process X is called an $\mathcal{F}$-Dirichlet processes if it can be decomposed as

$$X = Y + A,$$

where Y is an $\mathcal{F}$-semimartingale and A is $\mathcal{F}$-adapted, continuous and has quadratic variation zero. A Dirichlet process X is said to be special if it has a decomposition $X = Y + A$ where Y is a special semimartingale. In this case X has a unique decomposition

$$X = X_0 + M + A^c + A^d,$$

where M is a local martingale, A^d is a predictable pure jump process of bounded variation and A^c is a continuous process of quadratic variation zero. We have the following extension of Stricker (1988, Theorem 1):

Proposition 4.2. *Let X denote an $(\mathcal{F}, q)$-stable chaos process for some $q \in [4, \infty]$. If X is an $\mathcal{F}$-Dirichlet process then it is special, has almost surely only jumps on D_X and $M_t, A_t^d, A_t^c \in \mathcal{PC}$ for all $t \in [0, T]$. Furthermore, M is a true martingale belonging to H^p for all finite $p \le q$ and A^d is a pure jump process of integrable variation having almost surely only jumps on D_X. Finally, A^c is of zero energy, that is $\lim_{|\tau|\to 0} E\big[V_{A^c}^{2,\tau}\big] = 0$.*

Proof. Let Φ be given as in (4.2). Arguing as in Theorem 4.1 it follows that $E[\Phi(X)^p] < \infty$ for all finite $p \le q$. Hence by Stricker (1988, Theorem 1) X is special and $S_n^X \to A_T$ in the weak L^1-topology, where $A_t = A_t^d + A_t^c$. Since $\mathcal{PC}$ is L^1-closed we have $A_T \in \mathcal{PC}$ and similar $M_t, A_t \in \mathcal{PC}$ for all $t \in [0, T]$. Assume there exists $t \in D_A \setminus D_X$. Due to the fact that A is $\mathcal{F}$-predictable we have

$$\Delta A_t = E[\Delta A_t|\mathcal{F}_{t-}] = -E[\Delta M_t|\mathcal{F}_{t-}] = 0, \qquad \text{a.s.}$$

which contradicts $t \in D_A$ and so $D_A \subseteq D_X$. Furthermore, since A admits a quadratic variation, Proposition 3.5 implies that A has a.s. only jumps on the countable set

$D_A \subseteq D_X$. Using moreover that A^d is a pure jump process of bounded variation and A^c is continuous we have that

$$A_t^d = \sum_{0<s\leq t} \Delta A_s^d = \sum_{0<s\leq t} \Delta A_s = \sum_{s\in D_X\cap(0,t]} \Delta A_s,$$

and we conclude that $A_t^d \in \mathcal{PC}$. The rest of the proof is now a straightforward consequence of Theorem 2.1. $\square$

Remark 4.3.

(i) X is $(\mathcal{F}, q)$-stable if

$$X_t = \int_0^T f(t,s)\, dM_s, \qquad t \in [0,T],$$

where M is a càdlàg $\mathcal{F}$-martingale being also a chaos process satisfying C_q for some $q \in [1,\infty]$, and $f(t,\cdot)$ are deterministic functions for which the integrals exist. The $(\mathcal{F}, q)$-stability follows easily since for $u, t \in [0,T]$

$$E[X_t|\mathcal{F}_u] = \int_0^u f(t,s)\, dM_s \in \overline{\operatorname{span}}_{L^0} \{M_s : s \in [0,T]\}.$$

(ii) The $(\mathcal{F}, q)$-stability of X is not automatic even when X is a Gaussian chaos process of order d. However, if $\mathcal{G}$ is given by (2.2) then X is $(\mathcal{F}^W, \infty)$-stable and more generally this is true if each $\mathcal{F}_s$ is generated by elements in $\mathcal{G}$; see Nualart et al. (1990) for related results. Thus for $d = 1$ X is always $(\mathcal{F}^X, \infty)$-stable, but when $d \geq 2$ this may fail as the following example shows.

Example 4.4. Assume $\mathcal{G}$ is given by (2.2) for some Wiener process $(W_t)_{t\in[0,3]}$. Let $X = (X_t)_{t\in[0,3]}$ be the second-order Gaussian chaos process

$$X_t = \big(W_1^2 + W_1\big)1_{[1,2)}(t) + W_2 1_{[2,3]}(t), \qquad t \in [0,3].$$

Then $(E[X_t|\mathcal{F}_s^X])_{s,t\in[0,3]}$ is not a Gaussian chaos process. In fact, X is a special $\mathcal{F}^X$-semimartingale but the $\mathcal{F}^X$-bounded variation component of X is not a Gaussian chaos process.

To see this, note that X is a special $\mathcal{F}^X$-semimartingale since it is of integrable variation. Moreover, the $\mathcal{F}^X$-bounded variation component of X is

$$\begin{aligned} A_t &= E[\Delta X_1|\mathcal{F}_{1-}^X]1_{[1,3]}(t) + E[\Delta X_2|\mathcal{F}_{2-}^X]1_{[2,3]}(t) \\ &= 1_{[1,3]}(t) + \Big(W_1^2 + W_1 - E[W_1|W_1^2 + W_1]\Big)1_{[2,3]}(t). \end{aligned}$$

So to show that A is not a Gaussian chaos process it is enough to show $Y := E[W_1|W_1^2 + W_1] \notin \cup_{d=1}^\infty \overline{\mathcal{P}}_{\mathcal{G}}^d$. For each integrable random variable U, which is absolutely continuous with density $f > 0$, we have

$$E\big[U\big||U|\big] = |U|\frac{f(|U|) - f(-|U|)}{f(|U|) + f(-|U|)}. \tag{4.4}$$

Applying (4.4) with $U = W_1 + 1/2$, we get

$$\begin{aligned} Y &= -1/2 + E[W_1 + 1/2 \big| |W_1 + 1/2|] \\ &= -1/2 + |W_1 + 1/2| \tanh\Big(|W_1 + 1/2|/2\Big), \end{aligned} \tag{4.5}$$

where $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$. Since $x \mapsto e^{x^2/4}$ is convex we have

$$E[e^{Y^2/4}] \le E[E[e^{W_1^2/4}|W_1^2 + W_1]] = E[e^{W_1^2/4}] < \infty. \tag{4.6}$$

For contradiction assume $Y \in \cup_{d=1}^{\infty} \overline{\mathcal{P}}_{\mathcal{G}}^{d}$. By (4.6) and Janson (1997, Theorem 6.12) this implies $Y \in \overline{\mathcal{P}}_{\mathcal{G}}^{1} = \mathcal{G} + \mathbb{R}$. Moreover, (4.5) shows that $Y \ge -1/2$ and hence Y is constant. This contradict (4.5) and gives $Y \notin \cup_{d=1}^{\infty} \overline{\mathcal{P}}_{\mathcal{G}}^{d}$. $\diamondsuit$

5 The semimartingale property of moving averages

This section is concerned with the semimartingale property of moving averages (also known as stochastic convolutions). In Subsection 5.1 we treat the one-sided case and in Subsection 5.2 the two-sided case is considered.

5.1 The one-sided case

In this subsection $(\mathcal{F}_t)_{t\ge0}$ denotes a filtration and $(M_t)_{t\ge0}$ a square-integrable càdlàg $(\mathcal{F}_t)_{t\ge0}$-martingale. Set $\gamma_M(t) = E[M_t^2]$ for $t \ge 0$ and note that γ_M is càdlàg and increasing and hence γ_M' exists Lebesgue a.s. Let $X = (X_t)_{t\ge0}$ be given by

$$X_t = \int_0^t \varphi(t-s)\, dM_s, \qquad t \ge 0, \tag{5.1}$$

where φ is a measurable deterministic function for which all the integrals exist, i.e. $\varphi(t-\cdot) \in L^2(\gamma_M)$ for all $t \ge 0$. In this set up we have the following theorem where all locally integrability conditions are with respect to the Lebesgue measure λ.

Theorem 5.1. *Assume that M is a chaos process satisfying C_q for some $q \in [2,\infty]$ such that γ_M' is bounded away from zero on some non-empty open interval. Then X defined by* (5.1) *is an $\mathcal{F}$-semimartingale if and only if φ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*

Extensions to $q < 2$ is not possible. To see this let M denote an α-stable motion with $\alpha \in (1,2)$. Then M is an $\mathcal{F}^M$-martingale satisfying C_q for all $q < \alpha$, but Basse and Pedersen (2009, Theorem 3.1) yields that X given by (5.1) is an $\mathcal{F}^M$-semimartingale if and only if φ is absolutely continuous with an α-integrable density.

The proof of Theorem 5.1 relies on two lemmas. Here for each $f\colon \mathbb{R} \to \mathbb{R}$ and $h > 0$ $\Delta_h f$ denotes the function $t \mapsto (f(t+h) - f(t))/h$.

Lemma 5.2 (Hardy and Littlewood)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a locally integrable function. Then $(\Delta_{\frac{1}{n}} f)_{n\ge1}$ is bounded in $L^2([a,b],\lambda)$ for all $0 \le a < b$ if and only if f is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*

For every $a \ge 0$ $(\Delta_{\frac{1}{n}} f)_{n\ge1}$ is bounded in $L^2([a,\infty),\lambda)$ if and only if f is absolutely continuous on $[a,\infty)$ with a square-integrable density.

Lemma 5.3. *Let $\mathcal{F}$ denote a filtration, Y an $\mathcal{F}$-semimartingale and X be given by*

$$X_t = \int_0^t \varphi(t-s)\, dY_s, \qquad t \geq 0,$$

where φ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density. Then X is an $\mathcal{F}$-semimartingale.

Proof. For fixed $t > 0$ we have

$$\begin{aligned} X_t &= \varphi(0)Y_t + \int_0^t \Big(\int_0^{t-s} \varphi'(u)\, du \Big)\, dY_s \\ &= \varphi(0)Y_t + \int_0^t \Big(\int_0^t 1_{[s,t]}(u)\varphi'(u-s)\, du \Big)\, dY_s. \end{aligned}$$

Since

$$\mathbb{R}_+ \ni s \mapsto \sqrt{\int_s^t |\varphi'(u-s)|^2\, du} = \sqrt{\int_0^{t-s} |\varphi'(u)|^2\, du}$$

is locally bounded, Protter (2004, Chapter IV, Theorem 65) shows that

$$\begin{aligned} X_t &= \varphi(0)Y_t + \int_0^t \Big(\int_0^t 1_{[s,t]}(u)\varphi'(u-s)\, dY_s \Big)\, du \\ &= \varphi(0)Y_t + \int_0^t \Big(\int_0^u \varphi'(u-s)\, dY_s \Big)\, du, \qquad \text{a.s.} \end{aligned}$$

Thus X has a modification which is an $\mathcal{F}$-semimartingale. □

Proof of Theorem 5.1. Assume X is an $\mathcal{F}$-semimartingale. By assumption there exists an interval $(a,b) \subseteq \mathbb{R}_+$ and an $\varepsilon > 0$ such that $\gamma_M' \geq \varepsilon$ λ-a.s. on (a,b). By Remark 4.3(i) X is $(\mathcal{F},q)$-stable and since $q \geq 1$ it follows by Theorem 4.1 that X is an $\mathcal{F}$-quasimartingale on each compact interval and in particular

$$\sup_{n\geq 1} \sum_{i=1}^{Nn} E[|E[X_{i/n} - X_{(i-1)/n}|\mathcal{F}_{(i-1)/n}]|] < \infty, \qquad \text{for all } N \geq 1. \tag{5.2}$$

By Theorem 2.1 there exists a constant $C > 0$ such that $C\|U\|_2 \leq \|U\|_1 < \infty$ for all $U \in \mathcal{PC}$. Moreover, for all $a < u \leq t$ we have

$$\begin{aligned} E[|E[X_t - X_u|\mathcal{F}_u]|] &= E\Big[\Big| \int_0^u (\varphi(t-s) - \varphi(u-s))\, dM_s \Big|\Big] \\ &\geq C\Big\| \int_0^u (\varphi(t-s) - \varphi(u-s))\, dM_s \Big\|_2 = C \int_0^u \big(\varphi(t-s) - \varphi(u-s)\big)^2 \gamma_M(ds) \\ &\geq C \int_0^u \big(\varphi(t-s) - \varphi(u-s)\big)^2 \gamma_M'(s)\, ds \\ &= C \int_0^u \big(\varphi(t-u+s) - \varphi(s)\big)^2 \gamma_M'(u-s)\, ds \\ &\geq C\varepsilon \int_{(u-b)\vee 0}^{u-a} \big(\varphi(t-u+s) - \varphi(s)\big)^2\, ds. \end{aligned} \tag{5.3}$$

Put $\delta = (b-a)/4$ and set $l_x = x + (b+3a)/4$ and $r_x = x + (5b-a)/4$ for $x > 0$. By (5.2) and (5.3) we have

$$\sup_{n\geq 1} \sum_{i=[l_x n]+2}^{[r_x n]+1} \sqrt{\int_{(x-\delta)\vee 0}^{x+\delta} (\varphi(1/n+s) - \varphi(s))^2 \, ds} < \infty,$$

showing that

$$\sup_{n\geq 1} n \sqrt{\int_{(x-\delta)\vee 0}^{x+\delta} (\varphi(1/n+s) - \varphi(s))^2 \, ds} < \infty.$$

Thus $\{\Delta_{\frac{1}{n}}\varphi : n \geq 1\}$ is bounded in $L^2([(x-\delta)\vee 0, x+\delta], \lambda)$ and so by Lemma 5.2 we need only show that φ is locally integrable. But this follows immediately from $\varphi(t-\cdot) \in L^2([0,t], \gamma_M)$ for all $t \geq 0$ and $\gamma_M' \geq \varepsilon$ λ-a.s. on (a,b). The reverse implication follows by Lemma 5.3. □

Let us rewrite Theorem 5.1 in the Gaussian chaos case. Define $\mathcal{G}$ by

$$\mathcal{G} = \left\{ \int_0^\infty h(s) \, dW_s : h \in L^2(\mathbb{R}_+, \lambda) \right\},$$

for some Wiener process W and let X be given by

$$X_t = \int_0^t \varphi(t-s)\sigma_s \, dW_s, \qquad t \geq 0, \tag{5.4}$$

where σ is $\mathcal{F}^W$-progressively measurable and not the zero-process, and φ is a measurable deterministic function such that all the integrals exist. We have the following corollary to Theorem 5.1:

Corollary 5.4. *Let X be given by (5.4), where σ is a Gaussian chaos process which is right- or left-continuous in probability. Then X is an $\mathcal{F}^W$-semimartingale if and only if φ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*

5.2 Two-sided case

Let now $M = (M_t)_{t\in\mathbb{R}}$ denote a two-sided square-integrable $\mathcal{F}$-martingale, in the sense that $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ is an increasing family of σ-algebras, M is a square-integrable càdlàg process such that for all $-\infty < u \leq t$ we have $E[M_t - M_u | \mathcal{F}_u] = 0$ and $M_t - M_u$ is $\mathcal{F}_t$-measurable. Let $\gamma_M(t) = \operatorname{sign}(t) E[(M_t - M_0)^2]$ for all $t \in \mathbb{R}$ and note that γ_M is increasing and càdlàg. Let X be given by

$$X_t = \int_{-\infty}^t \big(\varphi(t-s) - \psi(-s)\big) \, dM_s, \qquad t \geq 0, \tag{5.5}$$

where φ and ψ are deterministic functions for which all the integrals are well-defined, that is $\varphi(t-\cdot) - \psi(-\cdot)$ is square-integrable with respect to the measure γ_M. Assume there exists an interval $(-\infty, c)$ on which γ_M is absolutely continuous with

$$0 < \liminf_{t\to-\infty} \gamma_M'(t) \leq \limsup_{t\to-\infty} \gamma_M'(t) < \infty \quad \text{and} \quad \inf_{t\in(a,b)} \gamma_M'(t) > 0,$$

for some $0 \leq a < b$. Note that when M has stationary increments, and therefore $\gamma_M(t) = \kappa t$ for some $\kappa > 0$, the conditions are trivially satisfied.

Theorem 5.5. *Let the setting be as just described and assume that M is a chaos process satisfying C_q for some $q \in [2,\infty]$. Then X given by (5.5) is an $\mathcal{F}$-semimartingale if and only if φ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density.*

Proof. Assume that X is an $\mathcal{F}$-semimartingale. Since γ_M' is bounded away from 0 on some interval of $\mathbb{R}_+$, it follows (just as in the proof of Theorem 5.1) that φ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density. Choose $\varepsilon > 0$ and $\tilde{c} < 0$ such that $\varepsilon \le \gamma_M'$ on $(-\infty, \tilde{c}]$. As in the proof of Theorem 5.1 $\{\Delta_{\frac{1}{n}}\varphi : n \ge 1\}$ is bounded in $L^2([-\tilde{c}+1,\infty),\lambda)$ which by Lemma 5.2 implies that φ is absolutely continuous on $[-\tilde{c}+1,\infty)$ with a square-integrable density. This completes the proof of the *only if*-implication.

Assume now φ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density and choose $C > 0$ and $\tilde{c} < 0$ such that $\gamma_M' \le C$ on $(-\infty,\tilde{c}]$. Let

$$Y_t = \int_{\tilde{c}}^{t} (\varphi(t-s) - \psi(-s))\, dM_s, \qquad t \ge 0.$$

By the same argument as in Lemma 5.3 it follows that Y is an $\mathcal{F}$-semimartingale. Thus it is enough to show that

$$U_t = \int_{-\infty}^{\tilde{c}} (\varphi(t-s) - \psi(-s))\, dM_s, \qquad t \ge 0,$$

is of bounded variation. For $0 \le u \le t$ we have

$$\begin{aligned} E[|U_t - U_u|] \le \|U_t - U_u\|_2 = \Big(\int_{-\infty}^{\tilde{c}} (\varphi(t-s) - \varphi(u-s))^2\, \gamma_M(ds)\Big)^{1/2} \\ \le C\Big(\int_{-\infty}^{\tilde{c}} (\varphi(t-s) - \varphi(u-s))^2\, ds\Big)^{1/2} = C\Big(\int_{-\tilde{c}+u}^{\infty} (\varphi(t-u+s) - \varphi(s))^2\, ds\Big)^{1/2}. \end{aligned}$$

According to Lemma 5.2 this shows that U is of integrable variation on each compact interval and the proof is complete. □

Again we rewrite the result in a Gaussian the setting. More precisely consider the following: Let $\mathcal{G} = \{\int_{\mathbb{R}} h(s)\, dW_s : h \in L^2(\mathbb{R},\lambda)\}$, where $W = (W_t)_{t\in\mathbb{R}}$ is a two-sided Wiener process with $W_0 = 0$. Let

$$\mathcal{F}_t^W = \begin{cases} \sigma(W_s : s \in (-\infty, t]) & t \ge 0 \\ \sigma(W_t - W_s : s \in (-\infty, t]) & t < 0. \end{cases}$$

Consider a process X of the form

$$X_t = \int_{-\infty}^{t} \big(\varphi(t-s) - \psi(-s)\big)\sigma_s\, dW_s, \qquad t \ge 0,$$

where σ is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-progressively measurable Gaussian chaos process satisfying

$$0 < \liminf_{t\to-\infty} E[\sigma_t^2] \le \limsup_{t\to-\infty} E[\sigma_t^2] < \infty \quad \text{and} \quad \inf_{t\in(a,b)} E[\sigma_t^2] > 0,$$

for some $0 \le a < b$. Theorem 5.5 now gives the following corollary:

Theorem 5.6. *X is an $\mathcal{F}^W$-semimartingale if and only if φ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density.*

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