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Abstract

Some classes of increment martingales, and the corresponding localised classes, are studied. An increment martingale is indexed by $\mathbb{R}$ and its increment processes are martingales. We focus primarily on the behavior as time goes to $-\infty$ in relation to the quadratic variation or the predictable quadratic variation, and we relate the limiting behaviour to the martingale property. Finally, integration with respect to an increment martingale is studied.

Keywords: Martingales; increments; integration; compensators.

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1 Introduction

Stationary processes are widely used in many areas, and the key example is a moving average. That is, a process X of the form

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}M_s, \qquad t \in \mathbb{R}, \tag{1.1}$$

where $M = (M_t)_{t\in\mathbb{R}}$ is a process with stationary increments. One example is a stationary Ornstein-Uhlenbeck process which corresponds to the case $\psi(t) = e^{-\lambda t}1_{[0,\infty)}(t)$ and M is a Brownian motion indexed by $\mathbb{R}$. See Doob (1990) for second order properties of moving averages and Barndorff-Nielsen and Schmiegel (2007) for applications of them in turbulence.

Integration with respect to a local martingale indexed by $\mathbb{R}_+$ is well-developed and in this case one can even allow the integrand to be random. However, when trying to define a stochastic integral from $-\infty$ as in (1.1) with random integrands, the class of local martingales indexed by $\mathbb{R}$ does not provide the right framework for $M = (M_t)_{t\in\mathbb{R}}$; indeed, in simple cases, such as when M is a Brownian motion, M is not a martingale in any filtration. Rather, it seems better to think of M as a process for which the increment $(M_{t+s} - M_s)_{t\geq 0}$ is a martingale for all $s \in \mathbb{R}$. It is natural to call such a process an *increment martingale*. Another interesting example within this framework is a diffusion on natural scale started in ∞ (cf. Example 3.17); indeed, if ∞ is an entrance boundary then all increments are local martingales but the diffusion itself is not. Thus, the class of increment (local) martingales indexed

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by $\mathbb{R}$ is strictly larger than the class of (local) martingales indexed by $\mathbb{R}$ and it contains several interesting examples. We refer to Subsection 1.1 for a discussion of the relations to other kinds of martingale-type processes indexed by $\mathbb{R}$.

In the present paper we introduce and study basic properties of some classes of increment martingales $M = (M_t)_{t\in\mathbb{R}}$ and the corresponding localised classes. Some of the problems studied are the following. Necessary and sufficient conditions for M to be a local martingale up to addition of a random variable will be given when M is either an increment martingale or an increment square integrable martingale. In addition, we give various necessary and sufficient conditions for $M_{-\infty} = \lim_{t\to-\infty} M_t$ to exist P-a.s. and $M - M_{-\infty}$ to be a local martingale expressed in terms of either the predictable quadratic variation $\langle M\rangle$ or the quadratic variation $[M]$ for M, where the latter two quantities will be defined below for increment martingales. These conditions rely on a convenient decomposition of increment martingales, and are particularly simple when M is continuous. We define two kinds of integrals with respect to M; the first of these is an increment integral $\phi \overset{\mathrm{in}}{\bullet} M$, which we can think of as process satisfying $\phi \overset{\mathrm{in}}{\bullet} M_t - \phi \overset{\mathrm{in}}{\bullet} M_s = \int_{(s,t]} \phi_u \,\mathrm{d}M_u$; i.e. increments in $\phi \overset{\mathrm{in}}{\bullet} M$ correspond to integrals over finite intervals. The second integral, $\phi \bullet M$, is a usual stochastic integral with respect to M which we can think of as an integral from $-\infty$. The integral $\phi \bullet M$ exists if and only if the increment integral $\phi \overset{\mathrm{in}}{\bullet} M$ has an a.s. limit, $\phi \overset{\mathrm{in}}{\bullet} M_{-\infty}$, at $-\infty$ and $\phi \overset{\mathrm{in}}{\bullet} M - \phi \overset{\mathrm{in}}{\bullet} M_{-\infty}$ is a local martingale. Thus, $\phi \overset{\mathrm{in}}{\bullet} M_{-\infty}$ may exists without $\phi \bullet M$ being defined and in this case we may think of $\phi \overset{\mathrm{in}}{\bullet} M_{-\infty}$ as an improper integral. In special cases we give necessary and sufficient conditions for $\phi \overset{\mathrm{in}}{\bullet} M_{-\infty}$ to exist.

The present paper relies only on standard martingale results and martingale integration as developed in many textbooks, see e.g. Jacod and Shiryaev (2003) and Jacod (1979). While we focus primarily on the behaviour at $-\infty$, it is also of interest to consider the behaviour at ∞; we refer to Cherny and Shiryaev (2005), and references therein, for a study of this case for semimartingales, and to Sato (2006), and references therein, for a study of improper integrals with respect to Lévy processes when the integrand is deterministic.

1.1 Relations to other martingale-type processes

Let us briefly discuss how to define processes with some kind of martingale structure when processes are indexed by $\mathbb{R}$. There are at least three natural definitions:

(i) $E[M_t|\mathcal{F}^M_s] = M_s$ for all $s \le t$, where $\mathcal{F}^M_s = \sigma(M_u : u \in (-\infty, s])$.

(ii) $E[M_t - M_u|\mathcal{F}^{IM}_{v,s}] = M_s - M_v$ for all $u \le v \le s \le t$, where $\mathcal{F}^M_{v,s} = \sigma(M_s - M_u : v \le u \le t \le s)$.

(iii) $E[M_t - M_s|\mathcal{F}^{IM}_s] = 0$ for all $s \le t$, where $\mathcal{F}^{\mathcal{I}M}_s = \sigma(M_t - M_u : u \le t \le s)$.

(The first definition is the usual martingale definition and the third one corresponds to increment martingales). Both (i) and (iii) generalise the usual notion of martingales indexed by $\mathbb{R}_+$, in the sense that if $(M_t)_{t\in\mathbb{R}}$ is a process with $M_t = 0$ for $t \in (-\infty, 0]$, then $(M_t)_{t\ge0}$ is a martingale (in the usually sense) if and only if $(M_t)_{t\in\mathbb{R}}$ is a martingale in the sense of (i), or equivalently in the sense of (iii). Definition (ii) does not generalise martingales indexed by $\mathbb{R}_+$ in this manner. Note moreover that a centered Lévy process indexed by $\mathbb{R}$ (cf. Example 3.3) is a martingale in the sense of (ii) and (iii) but not in the sense of (i). Thus, (iii) is the only one of the above definitions which generalise the usual notion of martingales on $\mathbb{R}_+$ and is general enough to allow centered Lévy processes to be martingales. Note also that both (i) and (ii) imply (iii).

The general theory of martingales indexed by partially ordered sets (for short, posets) does not seem to give us much insight about increment martingales since the research in this field mainly has a different focus; indeed, one of the main problems has been to study martingales $M = (M_t)_{t\in I}$ in the case where $I = [0,1]^2$; see e.g. Cairoli and Walsh (1975,1977). However, below we recall some of the basic definitions and relate them to the above (i)–(iii).

Consider a poset $(I, \leq)$ and a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in I}$, that is, for all $s, t \in I$ with $s \leq t$ we have that $\mathcal{F}_s \subseteq \mathcal{F}_t$. Then, $(M_t)_{t\in I}$ is called a martingale with respect to $\leq$ and $\mathcal{F}$, if for all $s, t \in I$ with $s \leq t$ we have that $E[M_t|\mathcal{F}_s] = M_s$. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a stochastic process. Then, definition (i) corresponds to $I = \mathbb{R}$ with the usually order. To cover (ii) and (iii) let $I = \{(a_1, a_2] : a_1, a_2, \in \mathbb{R},\ a_1 < a_2\}$, and for $A = (a_1, a_2] \in I$ let $M_A = M_{a_2} - M_{a_1}$, $\mathcal{F}^M_A = \sigma(M_B : B \in I,\ B \subseteq A)$. Furthermore, for all $A = (a_1, a_2], B = (b_1, b_2] \in I$ we will write $A \leq_2 B$ if $A \subseteq B$, and $A \leq_3 B$ if $a_1 = b_1$ and $a_2 \leq b_2$. Clearly, $\leq_2$ and $\leq_3$ are two partial orders on I. Moreover, it is easily seen that $(M_t)_{t\in\mathbb{R}}$ satisfies (ii)/(iii) if and only if $(M_A)_{A\in I}$ is a martingale with respect to $\leq_2/\leq_3$ and $\mathcal{F}^M$. Recall that a poset $(I, \leq)$ is called directed if for all $s, t \in I$ there exists an element $u \in I$ such that $s \leq u$ and $t \leq u$. Note that $(I, \leq_2)$ is directed, but $(I, \leq_3)$ is not; and in particular $(I, \leq_3)$ is not a lattice. We refer to Kurtz (1980) for some nice considerations about martingales indexed by directed posets.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ denote a complete probability space on which all random variables appearing in the following are defined. Let $\mathcal{F}_. = (\mathcal{F}_t)_{t\in\mathbb{R}}$ denote a filtration in $\mathcal{F}$, i.e. a right-continuous increasing family of sub σ-algebras in $\mathcal{F}$ satisfying $\mathcal{N} \subseteq \mathcal{F}_t$ for all t, where $\mathcal{N}$ is the collection of all P-null sets. Set $\mathcal{F}_{-\infty} := \cap_{t\in\mathbb{R}}\mathcal{F}_t$ and $\mathcal{F}_\infty := \cup_{t\in\mathbb{R}}\mathcal{F}_t$. The notation $\overset{\mathcal{D}}{=}$ will be used to denote identity in distribution. Similarly, $\overset{P}{=}$ will denote equality up to P-indistinguishability of stochastic processes. When $X = (X_t)_{t\in\mathbb{R}}$ is a real-valued stochastic process we say that $\lim_{s\to-\infty} X_s$ exists P-a.s. if X_s converges almost surely as $s \to -\infty$, to a finite limit.

Definition 2.1. A *stopping time* σ is a mapping $\sigma : \Omega \to (-\infty, \infty]$ satisfying $\{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}$. A *localising sequence* $(\sigma_n)_{n\geq1}$ is a sequence of stopping times satisfying $\sigma_1(\omega) \leq \sigma_2(\omega) \leq \cdots$ for all ω, and $\sigma_n \to \infty$ P-a.s.

Let $\mathcal{P}(\mathcal{F}_.)$ denote the *predictable σ-algebra* on $\mathbb{R} \times \Omega$. That is, the σ-algebra generated by the set of *simple predictable sets*, where a subset of $\mathbb{R} \times \Omega$ is said to be simple predictable if it is of the form $B \times C$ where, for some $t \in \mathbb{R}$, C is in $\mathcal{F}_t$ and B is a bounded Borel set in $]t, \infty[$. Note that the set of simple predictable sets is closed under finite intersections.

Any left-continuous and adapted process is predictable. Moreover, the set of predictable processes is stable under stopping in the sense that whenever $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ is predictable and σ is a stopping time, the stopped process $\alpha^\sigma := (\alpha_{t\wedge\sigma})_{t\in\mathbb{R}}$ is also predictable.

By an *increasing process* we mean a process $V = (V_t)_{t\in\mathbb{R}}$ (not necessarily adapted) for which $t \mapsto V_t(\omega)$ is nondecreasing for all $\omega \in \Omega$. Similarly, a process V is said to be càdlàg if $t \mapsto V_t(\omega)$ is right-continuous and has left limits in $\mathbb{R}$ for all $\omega \in \Omega$.

In what follows increments of processes play an important role. Whenever $X = (X_t)_{t\in\mathbb{R}}$ is a process and $s, t \in \mathbb{R}$ define the *increment of* X *over the interval* $(s, t]$, to be denoted ${}^sX_t$, as

$$
{}^sX_t := X_t - X_{t\wedge s} = \begin{cases} 0 & \text{if } t \le s \\ X_t - X_s & \text{if } t \ge s. \end{cases} \tag{2.1}
$$

Set furthermore ${}^sX = ({}^sX_t)_{t\in\mathbb{R}}$. Note that

$$
({}^sX)^\sigma = {}^s(X^\sigma) \quad \text{for } s \in \mathbb{R} \text{ and } \sigma \text{ a stopping time.} \tag{2.2}
$$

Moreover, for $s \le t \le u$ we have

$$
{}^t({}^sX)_u = {}^tX_u. \tag{2.3}
$$

Definition 2.2. Let $\mathcal{A}(\mathcal{F}_\cdot)$ denote the class of increasing adapted càdlàg processes.

Let $\mathcal{A}^1(\mathcal{F}_\cdot)$ denote the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of integrable increasing càdlàg adapted processes; $\mathcal{LA}^1(\mathcal{F}_\cdot)$ denotes the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of càdlàg increasing adapted processes $V = (V_t)_{t\in\mathbb{R}}$ for which there exists a localising sequence $(\sigma_n)_{n\ge1}$ such that $V^{\sigma_n} \in \mathcal{A}^1(\mathcal{F}_\cdot)$ for all n.

Let $\mathcal{A}_0(\mathcal{F}_\cdot)$ denote the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of increasing càdlàg adapted processes $V = (V_t)_{t\in\mathbb{R}}$ for which $\lim_{t\to-\infty} V_t = 0$ P-a.s. Set $\mathcal{A}_0^1(\mathcal{F}_\cdot) := \mathcal{A}_0(\mathcal{F}_\cdot) \cap \mathcal{A}^1(\mathcal{F}_\cdot)$ and $\mathcal{LA}_0^1(\mathcal{F}_\cdot) := \mathcal{A}_0(\mathcal{F}_\cdot) \cap \mathcal{LA}^1(\mathcal{F}_\cdot)$.

Let $\mathcal{IA}(\mathcal{F}_\cdot)$ (resp. $\mathcal{IA}^1(\mathcal{F}_\cdot)$, $\mathcal{ILA}^1(\mathcal{F}_\cdot)$) denote the class of càdlàg increasing processes V for which ${}^sV \in \mathcal{A}(\mathcal{F}_\cdot)$ (resp. ${}^sV \in \mathcal{A}^1(\mathcal{F}_\cdot)$, ${}^sV \in \mathcal{LA}^1(\mathcal{F}_\cdot)$) for all $s \in \mathbb{R}$. We emphasize that V is not assumed adapted.

Motivated by our interest in increments we say that two càdlàg processes $X = (X_t)_{t\in\mathbb{R}}$ and $Y = (Y_t)_{t\in\mathbb{R}}$ have *identical increments*, and write $X \overset{\text{in}}{=} Y$, if ${}^sX \overset{P}{=} {}^sY$ for all $s \in \mathbb{R}$. In this case also $X^\sigma \overset{\text{in}}{=} Y^\sigma$ whenever σ is a stopping time.

Remark 2.3. Assume X and Y are càdlàg processes with $X \overset{\text{in}}{=} Y$. Then by definition $X_t - X_s = Y_t - Y_s$ for all $s \le t$ P-a.s. for all t and so by the càdlàg property $X_t - X_s = Y_t - Y_s$ for all $s, t \in \mathbb{R}$ P-a.s. This shows that there exists a random variable Z such that $X_t = Y_t + Z$ for all $t \in \mathbb{R}$ P-a.s., and thus ${}^sX_t = {}^sY_t$ for all $s, t \in \mathbb{R}$ P-a.s.

For any stochastic process $X = (X_t)_{t\in\mathbb{R}}$ we have

$$
{}^sX_t + {}^tX_u = {}^sX_u \quad \text{for } s \le t \le u. \tag{2.4}
$$

This leads us to consider increment processes, defined as follows. Let $I = \{{}^sI\}_{s\in\mathbb{R}}$ with ${}^sI = ({}^sI_t)_{t\in\mathbb{R}}$ be a family of stochastic processes. We say that I is *a consistent family of increment processes* if the following three conditions are satisfied:

(1) ${}^sI$ is an adapted process for all $s \in \mathbb{R}$, and ${}^sI_t = 0$ P-a.s. for all $t \leq s$.

(2) For all $s \in \mathbb{R}$ and $\omega \in \Omega$ the mapping $t \mapsto {}^sI_t(\omega)$ is càdlàg.

(3) For all $s \leq t \leq u$ we have ${}^sI_t + {}^tI_u = {}^sI_u$ P-a.s.

Whenever X is a càdlàg process such that ${}^sX$ is adapted for all $s \in \mathbb{R}$, the family $\{{}^sX\}_{s\in\mathbb{R}}$ of increment processes is then consistent by equation (2.4). Conversely, let I be a consistent family of increment processes. A càdlàg process $X = (X_t)_{t\in\mathbb{R}}$ is said to be *associated with* I if ${}^sX \overset{P}{=} {}^sI$ for all $s \in \mathbb{R}$. It is easily seen that there exists such a process; for example, let

$$X_t = \begin{cases} {}^0I_t & \text{for } t \geq 0 \\ -{}^tI_0 & \text{for } t = -1, -2, \dots, \\ X_{-n} + {}^{-n}I_t & \text{for } t \in (-n, -n+1) \text{ and } n = 1, 2, \dots \end{cases}$$

Thus, consistent families of increment processes correspond to increments in càdlàg processes with adapted increments. If $X = (X_t)_{t\in\mathbb{R}}$ and $Y = (Y_t)_{t\in\mathbb{R}}$ are càdlàg processes associated with I then $X \overset{\text{in}}{=} Y$ and hence by Remark 2.3 there is a random variable Z such that $X_t = Y_t + Z$ for all t P-a.s.

Remark 2.4. Let I be a consistent family of increment processes, and assume X is a càdlàg process associated with I such that $X_{-\infty} := \lim_{t\to-\infty} X_t$ exists in probability. Then, $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is adapted and associated with I. Indeed, $X_t - X_{-\infty} = \lim_{s\to-\infty} {}^sX_t$ in probability for $t \in \mathbb{R}$ and since ${}^sX_t = {}^sI_t$ (P-a.s.) is $\mathcal{F}_t$-measurable, it follows that $X_t - X_{-\infty}$ is $\mathcal{F}_t$-measurable. In this case, $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is the unique (up to P-indistinguishability) càdlàg process associated with I which converges to 0 in probability as time goes to $-\infty$. If, in addition, ${}^sI$ is predictable for all $s \in \mathbb{R}$ then $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is also predictable. To see this, choose a P-null set N and a sequence $(s_n)_{n\geq1}$ decreasing to $-\infty$ such that $X_{s_n}(\omega) \to X_{-\infty}(\omega)$ as $n \to \infty$ for all $\omega \in N^c$. For $\omega \in N^c$ and $t \in \mathbb{R}$ we then have $X_t(\omega) - X_{-\infty}(\omega) = \lim_{n\to\infty} {}^{s_n}X_t(\omega)$, implying the result due to inheritance of predictability under pointwise limits.

3 Martingales and increment martingales

Let us now introduce the classes of (square integrable) martingales and the corresponding localised classes.

Definition 3.1. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a càdlàg adapted process.

We call M an *$\mathcal{F}_\cdot$-martingale* if it is integrable and for all $s < t$, $E[M_t|\mathcal{F}_s] = M_s$ P-a.s. If in addition M_t is square integrable for all $t \in \mathbb{R}$ then M is called a *square integrable martingale*. Let $\mathcal{M}(\mathcal{F}_\cdot)$ resp. $\mathcal{M}^2(\mathcal{F}_\cdot)$ denote the class of $\mathcal{F}_\cdot$-martingales resp. square integrable $\mathcal{F}_\cdot$-martingales. Note that these classes are both stable under stopping.

We call M a *local $\mathcal{F}_\cdot$-martingale* if there exists a localising sequence $(\sigma_n)_{n\geq1}$ such that $M^{\sigma_n} \in \mathcal{M}(\mathcal{F}_\cdot)$ for all n. The definition of a *locally square integrable martingale* is similar. Let $\mathcal{LM}(\mathcal{F}_\cdot)$ resp. $\mathcal{LM}^2(\mathcal{F}_\cdot)$ denote the class of local martingales resp. locally square integrable martingales. These classes are stable under stopping.

Remark 3.2. (1) The backward martingale convergence theorem shows that if $M \in \mathcal{M}(\mathcal{F}_\cdot)$ then M_t converges P-a.s. and in $L^1(P)$ to an $\mathcal{F}_{-\infty}$-measurable integrable random variable $M_{-\infty}$ as $t \to -\infty$ (cf. Doob (1990, Chapter II, Theorem 2.3)). In this case we may consider $(M_t)_{t\in[-\infty,\infty)}$ as a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[-\infty,\infty)}$. If $M \in \mathcal{M}^2(\mathcal{F}_\cdot)$ then M_t converges in $L^2(P)$ to $M_{-\infty}$.

(2) Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ and choose a localising sequence $(\sigma_n)_{n\geq 1}$ such that $M^{\sigma_n} \in \mathcal{M}(\mathcal{F}_\cdot)$ for all n. From (1) follows that there exists an $\mathcal{F}_{-\infty}$-measurable integrable random variable $M_{-\infty}$ (which does not depend on n) such that for all n we have $M_t^{\sigma_n} \to M_{-\infty}$ P-a.s. and in $L^1(P)$ as $t \to -\infty$, and $M_t \to M_{-\infty}$ P-a.s. Thus, defining $M_{-\infty}^{\sigma_n} := M_{-\infty}$ it follows that for all n the process $(M_t)^{\sigma_n}_{t\in[-\infty,\infty)}$ can be considered a martingale with respect to $(\mathcal{F}_t)_{t\in[-\infty,\infty)}$, and consequently $(M_t)_{t\in[-\infty,\infty)}$ is a local martingale. (Note, though, that σ_n is not allowed to take on the value $-\infty$.) In the case $M \in \mathcal{LM}^2(\mathcal{F}_\cdot)$ assume $(\sigma_n)_{n\geq 1}$ is chosen such that $M^{\sigma_n} \in \mathcal{M}^2(\mathcal{F}_\cdot)$ for all n; then $M_t^{\sigma_n} \to M_{-\infty}$ in $L^2(P)$.

(3) The preceding shows that a local martingale indexed by $\mathbb{R}$ can also be regarded as a local martingale indexed by $[-\infty,\infty)$, where localising stopping times, however, are not allowed to take on the value $-\infty$. Let us argue that the latter restriction is of minor importance. Thus, call $\sigma : \Omega \to [-\infty,\infty]$ an $\bar{\mathbb{R}}$*-valued stopping time* if $\{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in [-\infty,\infty)$, and call a sequence of nondecreasing $\bar{\mathbb{R}}$-valued stopping times $\sigma_1 \leq \sigma_2 \leq \cdots$ an $\bar{\mathbb{R}}$*-valued localising sequence* if $\sigma_n \to \infty$ P-a.s. as $n \to \infty$. Then we claim that a càdlàg adapted process $M = (M_t)_{t\in\mathbb{R}}$ is a local martingale if and only if $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s and there is an $\bar{\mathbb{R}}$-valued localising sequence $(\sigma_n)_{n\geq 1}$ such that $(M_t^{\sigma_n})_{t\in[-\infty,\infty)}$ is a martingale. We emphasize that the latter characterisation is the most natural one when considering the index set $[-\infty,\infty)$, while the former is better when considering $\mathbb{R}$. Note that the *only if* part follows from (2). Conversely, assume $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s and let $(\sigma_n)_{n\geq 1}$ be an $\bar{\mathbb{R}}$-valued localising sequence such that $(M_t^{\sigma_n})_{t\in[-\infty,\infty)}$ is a martingale, and let us prove the existence of a localising sequence $(\tau_n)_{n\geq 1}$ such that M^{τ_n} is a martingale for all n. Since $M_{-\infty}$ is integrable it suffices to consider $M_t - M_{-\infty}$ instead of M_t; consequently we may and do assume $M_{-\infty} = 0$. In this case, $(\tau_n)_{n\geq 1} = (\tau \vee \sigma_n)_{n\geq 1}$ will do if τ is a stopping time such that M^τ is a martingale. To construct this τ set $Z_t^n = E[|M_t^{\sigma_n}||\mathcal{F}_{-\infty}]$ for $t \in [-\infty,\infty)$. Then Z^n is $\mathcal{F}_{-\infty}$-measurable and can be chosen non-decreasing, càdlàg and 0 at $-\infty$. Therefore

$$\rho_n = \inf\{t \in \mathbb{R} : Z^n_{\frac{t}{2}} > 1\} \wedge 0$$

is real-valued, $\mathcal{F}_{-\infty}$-measurable and $Z^n_{\rho_n} \leq 1$. Define

$$\tau = \rho_n \wedge \sigma_n \text{ on } A_n = \{\sigma_1 = \cdots = \sigma_{n-1} = -\infty \text{ and } \sigma_n > -\infty\}$$

and set $\tau = 0$ on $(\cup_{n\geq 1} A_n)^c$. Then τ is a stopping time since the A_n's are disjoint and $\mathcal{F}_{-\infty}$-measurable. Furthermore, $\cup_{n\geq 1} A_n = \Omega$ P-a.s. Thus, for all $t > -\infty$,

$$E[|M_{t\wedge\tau}|] = \sum_{n=1}^{\infty} E[|M_{\sigma_n\wedge\rho_n\wedge t}|1_{A_n}] = \sum_{n=1}^{\infty} E[|Z^n_{\rho_n\wedge\sigma_n\wedge t}|1_{A_n}] \leq 1,$$

implying

$$E[M_{\tau\wedge t}|\mathcal{F}_s] = \sum_{n=1}^{\infty} E[M_{\sigma_n\wedge\rho_n\wedge t}|\mathcal{F}_s]1_{A_n} = \sum_{n=1}^{\infty} M_{\sigma_n\wedge\tau_n\wedge s}1_{A_n} = M_{\tau\wedge s}$$

for all $-\infty < s < t$; thus, M^τ is a martingale.

Example 3.3. A càdlàg process $X = (X_t)_{t\in\mathbb{R}}$ is called a *Lévy process indexed by* $\mathbb{R}$ if it has stationary independent increments; that is, whenever $n \geq 1$ and $t_0 < t_1 < \cdots < t_n$, the increments ${}^{t_0}X_{t_1}, {}^{t_1}X_{t_2}, \ldots, {}^{t_{n-1}}X_{t_n}$ are independent and ${}^{s}X_t \overset{\mathcal{D}}{=} {}^{u}X_v$ whenever $s < t$ and $u < v$ satisfy $t - s = v - u$. In this case $({}^{s}X_{s+t})_{t\geq 0}$ is an ordinary Lévy process indexed by $\mathbb{R}_+$ for all $s \in \mathbb{R}$.

Let X be a Lévy process indexed by $\mathbb{R}$. There is a unique infinitely divisible distribution μ on $\mathbb{R}$ associated with X in the sense that for all $s < t$, ${}^{s}X_t \overset{\mathcal{D}}{=} \mu^{t-s}$. When $\mu = N(0,1)$, the standard normal distribution, X is called a (standard) Brownian motion indexed by $\mathbb{R}$. If Y is a càdlàg process with $X \overset{\text{in}}{=} Y$, it is a Lévy process as well and μ is also associated with Y; that is, Lévy processes indexed by $\mathbb{R}$ are determined by the infinitely divisible μ only up to addition of a random variable.

Note that $(X_{(-t)-})_{t\in\mathbb{R}}$ (where, for $s \in \mathbb{R}$, X_{s-} denotes the left limit at s) is again a Lévy process indexed by $\mathbb{R}$ and the distribution associated with it is μ^- given by $\mu^-(B) := \mu(-B)$ for $B \in \mathcal{B}(\mathbb{R})$. Since this process appears by time reversion of X, the behaviour of X at $-\infty$ corresponds to the behaviour of $(X_{(-t)-})_{t\in\mathbb{R}}$ at ∞, which is well understood, cf. e.g. Sato (1999); in particular, $\lim_{s\to-\infty} X_s$ does not exist (in any reasonable sense) except when X is constant. Thus, except in nontrivial cases X is not a local martingale in any filtration.

This example clearly indicates that we need to generalise the concept of a martingale.

Definition 3.4. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a càdlàg process, in general not assumed adapted.

We say that M is an *increment martingale* if for all $s \in \mathbb{R}$, ${}^{s}M \in \mathcal{M}(\mathcal{F}_\cdot)$. This is equivalent to saying that for all $s < t$, ${}^{s}M_t$ is $\mathcal{F}_t$-measurable, integrable and satisfies $E[{}^{s}M_t|\mathcal{F}_s] = 0$ P-a.s. If in addition all increments are square integrable, then M is called a *increment square integrable martingale*. Let $\mathcal{IM}(\mathcal{F}_\cdot)$ and $\mathcal{IM}^2(\mathcal{F}_\cdot)$ denote the corresponding classes.

M is called an *increment local martingale* if for all s, ${}^{s}M$ is an adapted process and there exists a localising sequence $(\sigma_n)_{n\geq 1}$ (which may depend on s) such that $({}^{s}M)^{\sigma_n} \in \mathcal{M}(\mathcal{F}_\cdot)$ for all n. Define an *increment locally square integrable martingale* in the obvious way. Denote the corresponding classes by $\mathcal{ILM}(\mathcal{F}_\cdot)$ and $\mathcal{ILM}^2(\mathcal{F}_\cdot)$.

Obviously the four classes of increment processes are $\overset{\text{in}}{=}$-stable and by (2.2) stable under stopping. Moreover, $\mathcal{M}(\mathcal{F}_\cdot) \subseteq \mathcal{IM}(\mathcal{F}_\cdot)$ and $\mathcal{M}^2(\mathcal{F}_\cdot) \subseteq \mathcal{IM}^2(\mathcal{F}_\cdot)$ with the following characterizations

$$\mathcal{M}(\mathcal{F}_\cdot) = \{M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot) : M \text{ is adapted and integrable}\} \tag{3.1}$$

$$\mathcal{M}^2(\mathcal{F}_\cdot) = \{M \in \mathcal{IM}^2(\mathcal{F}_\cdot) : M \text{ is adapted and square integrable}\}. \tag{3.2}$$

Likewise, $\mathcal{LM}(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}(\mathcal{F}_\cdot)$ and $\mathcal{LM}^2(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}^2(\mathcal{F}_\cdot)$. But no similar simple characterizations as in (3.1)–(3.2) of the localised classes seem to be valid. Note that $\mathcal{LIM}(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}(\mathcal{F}_\cdot)$, where the former is the set of *local increment martingales*, i.e. the localising sequence can be chosen independent of s. A similar statement holds for $\mathcal{ILM}^2(\mathcal{F}_\cdot)$.

When τ is a stopping time, we define ${}^{\tau}M$ in the obvious way as ${}^{\tau}M_t = M_t - M_{t\wedge\tau}$ for $t \in \mathbb{R}$.

Proposition 3.5. *Let $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$ and τ be a stopping time. Then ${}^{\tau}M \in \mathcal{M}(\mathcal{F}_\cdot)$ if $\{M_0 - M_{\tau\vee(-n)\wedge 0} : n \geq 1\}$ is uniformly integrable.*

If τ is bounded from below then the above set is always uniformly integrable.

Proof. Assume first that τ is bounded from below, that is, there exists an $s_0 \in (0, -\infty)$ such that $\tau \geq s_0$. Then, since $({}^{\tau}M_t)_{t\in\mathbb{R}} = ({}^{s_0}M_t - {}^{s_0}M_{\tau\wedge t})_{t\in\mathbb{R}}$, ${}^{\tau}M$ is a sum of two martingales and hence a martingale. Assume now that $\{M_0 - M_{\tau\vee(-n)\wedge 0} : n \geq 1\}$ is uniformly integrable. Then, with $\tau_n = \tau \vee (-n)$ we have

$$\{{}^{\tau_n}M_t : n \geq 1\} \text{ is uniformly integrable for all } t \in \mathbb{R}. \tag{3.3}$$

Moreover, ${}^{\tau_n}M_t \to {}^{\tau}M_t$ a.s. and hence in $L^1(P)$ by (3.3). For all $n \geq 1$, τ_n is bounded from below and hence ${}^{\tau_n}M$ is a martingale, implying that ${}^{\tau}M$ is an $L^1(P)$-limit of martingales and hence a martingale. □

Example 3.6. Let $X = (X_t)_{t\in\mathbb{R}}$ denote a Lévy process indexed by $\mathbb{R}$. *The filtration generated by the increments of* X is $\mathcal{F}^{\mathcal{I}X}_\cdot = (\mathcal{F}^{\mathcal{I}X}_t)_{t\in\mathbb{R}}$, where

$$\mathcal{F}^{\mathcal{I}X}_t = \sigma({}^{s}X_t : s \leq t) \vee \mathcal{N} = \sigma({}^{s}X_u : s \leq u \leq t) \vee \mathcal{N}, \quad \text{for } t \in \mathbb{R},$$

and we recall that $\mathcal{N}$ is the set of P-null sets. Using a standard technique it can be verified that $\mathcal{F}^{\mathcal{I}X}_\cdot$ is a filtration. Indeed, we only have to verify right-continuity of $\mathcal{F}^{\mathcal{I}X}_\cdot$. For this, fix $t \in \mathbb{R}$ and consider random variables Z_1 and Z_2 where Z_1 is bounded and $\mathcal{F}^{\mathcal{I}X}_t$-measurable, and Z_2 is bounded and measurable with respect to $\sigma({}^{s}X_u : t+\epsilon < s < u)$ for some $\epsilon > 0$. Then

$$E[Z_1 Z_2 | \mathcal{F}^{\mathcal{I}X}_{t+}] = Z_1 E[Z_2] = E[Z_1 Z_2 | \mathcal{F}^{\mathcal{I}X}_t] \quad P\text{-a.s.}$$

by independence of Z_2 and $\mathcal{F}^{\mathcal{I}X}_{t+}$. Applying the monotone class lemma it follows that whenever Z is bounded and measurable with respect to $\mathcal{F}^{\mathcal{I}X}_\infty$ we have $E[Z|\mathcal{F}^{\mathcal{I}X}_{t+}] = E[Z|\mathcal{F}^{\mathcal{I}X}_t]$ P-a.s., which in turn implies right-continuity of $\mathcal{F}^{\mathcal{I}X}_\cdot$. It is readily seen that $X \in \mathcal{IM}(\mathcal{F}^{\mathcal{I}X}_\cdot)$ if X has integrable centered increments.

Increment martingales are not necessarily integrable. But for $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$, $M_t \in L^1(P)$ for all $t \in \mathbb{R}$ if and only if $M_t \in L^1(P)$ for some $t \in \mathbb{R}$. Likewise $(M_s)_{s\leq t}$ is uniformly integrable for all t if and only if $(M_s)_{s\leq t}$ is uniformly integrable for some t. Similarly, for $M \in \mathcal{IM}^2(\mathcal{F}_\cdot)$ we have $M_t \in L^2(P)$ for all $t \in \mathbb{R}$ if and only if $M_t \in L^2(P)$ for some $t \in \mathbb{R}$, and $(M_s)_{s\leq t}$ is $L^2(P)$-bounded for some t if and only if $(M_s)_{s\leq t}$ is $L^2(P)$-bounded for some t. For integrable elements of $\mathcal{IM}(\mathcal{F}_\cdot)$ we have the following decomposition.

Proposition 3.7. *Let $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$ be integrable. Then M can be decomposed uniquely up to P-indistinguishability as $M = K + N$ where $K = (K_t)_{t\in\mathbb{R}} \in \mathcal{M}(\mathcal{F}_\cdot)$ and $N = (N_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$ is an integrable process satisfying*

$$E[N_t|\mathcal{F}_t] = 0 \textit{ for all } t \in \mathbb{R} \quad \textit{and} \quad \lim_{t\to\infty} N_t = 0 \ P\textit{-a.s. and in } L^1(P). \tag{3.4}$$

If M is square integrable then so are K and N, and $E[K_t N_t] = 0$ for all $t \in \mathbb{R}$. Thus $E[M_t^2] = E[K_t^2] + E[N_t^2]$ for all t and moreover $t \mapsto E[N_t^2]$ is decreasing.

Proof. The uniqueness is evident. To get the existence set $K_t = E[M_t|\mathcal{F}_t]$. Then K is integrable and adapted and for $s < t$ we have

$$E[K_t|\mathcal{F}_s] = E[M_t|\mathcal{F}_s] = E[M_s|\mathcal{F}_s] + E[{}^sM_t|\mathcal{F}_s] = K_s.$$

Thus, $K \in \mathcal{M}(\mathcal{F}_\cdot)$ and therefore $N := M - K \in \mathcal{IM}(\mathcal{F}_\cdot)$. Clearly, N is integrable and $E[N_t|\mathcal{F}_t] = 0$ for all $t \in \mathbb{R}$. Take $s \le t$. Then ${}^sN_t = E[{}^sN_t|\mathcal{F}_t]$, giving

$${}^sN_t = E[N_t - N_s|\mathcal{F}_t] = -E[N_s|\mathcal{F}_t], \tag{3.5}$$

that is $N_t = N_s - E[N_s|\mathcal{F}_t]$, proving that $\lim_{t\to\infty} N_t = 0$ P-a.s. and in $L^1(P)$. If M is square integrable then so are K and N and they are orthogonal. Furthermore for $s \le t$

$$\begin{aligned} E[N_s(N_t - N_s)] &= E[(N_t - N_s)E[N_s|\mathcal{F}_t]] \\ &= E[(N_t - N_s)E[(N_s - N_t)|\mathcal{F}_t]] = -E[(N_t - N_s)^2] \end{aligned}$$

implying

$$E[N_t^2] = E[N_s^2] - E[(N_t - N_s)^2]. \tag{3.6}$$

□

As a corollary we may deduce the following convergence result for integrable increment martingales.

Corollary 3.8. *Let $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$ be integrable.*

(a) If $(M_s)_{s\le 0}$ is uniformly integrable then $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. and in $L^1(P)$ and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}(\mathcal{F}_\cdot)$.

(b) If $(M_s)_{s\le 0}$ is bounded in $L^2(P)$ then $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. and in $L^2(P)$ and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}^2(\mathcal{F}_\cdot)$.

Proof. Write $M = K + N$ as in Proposition 3.7. As noticed in Remark 3.2 the conclusion holds for K. Furthermore $(N_s)_{s\le 0}$ is uniformly integrable when this is true for M so we may and will assume $M = N$. That is, M satisfies (3.4). By uniform integrability we can find a sequence s_n decreasing to $-\infty$ and an $\tilde{M} \in L^1(P)$ such that $M_{s_n} \to \tilde{M}$ in $\sigma(L^1, L^\infty)$. For all t we have by (3.5)

$$M_t = M_{s_n} - E[M_{s_n}|\mathcal{F}_t] \quad \text{for } s_n < t$$

and thus

$$M_t = \tilde{M} - E[\tilde{M}|\mathcal{F}_t] \quad \text{for all } t,$$

proving part (a). In (b) the martingale part K again has the right behaviour at $-\infty$. Likewise, $(N_s)_{s\le 0}$ is bounded in $L^2(P)$ if this is true for M. Thus we may assume that M satisfies (3.4). The a.s. convergence is already proved and the $L^2(P)$-convergence follows from (3.6) since $t \mapsto E[M_t]$ is decreasing and $\sup_{s<0} E[M_s^2] < \infty$. □

Observe that $(M_t - M_{t_0})_{t\in\mathbb{R}}$ is in $\mathcal{IM}(\mathcal{F}_\cdot)$ and is integrable for every $t_0 \in \mathbb{R}$ and every $M \in \mathcal{IM}(\mathcal{F}_\cdot)$. Since a similar result holds in the square integrable case, Corollary 3.8 implies the following result relating convergence of an increment martingale to the martingale property.

Proposition 3.9. *Let $M = (M_t)_{t\in\mathbb{R}}$ be a given càdlàg process. The following are equivalent:*

(a) $M_{-\infty} := \lim_{s\to-\infty} M_s$ *exists P-a.s. and* $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ *is in* $\mathcal{M}(\mathcal{F}_\cdot)$.

(b) $M \in \mathcal{IM}(\mathcal{F}_\cdot)$ *and* $({}^sM_0)_{s<0}$ *is uniformly integrable.*

Likewise, the following are equivalent:

(c) $M_{-\infty} := \lim_{s\to-\infty} M_s$ *exists P-a.s. and* $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ *is in* $\mathcal{M}^2(\mathcal{F}_\cdot)$

(d) $M \in \mathcal{IM}^2(\mathcal{F}_\cdot)$ *and* $\sup_{s:s\le 0} E[({}^sM_0)^2] < \infty$.

Proof. Assuming $M \in \mathcal{IM}(\mathcal{F}_\cdot)/\mathcal{IM}^2(\mathcal{F}_\cdot)$, (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) follow by using Corollary 3.8 on $(M_t - M_0)_{t\in\mathbb{R}}$. The remaining two implications follow from standard martingale theory and the identity ${}^sM_0 = (M_0 - M_{-\infty}) - (M_s - M_{-\infty})$. □

Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ with $M_{-\infty} = 0$. It is well-known that there exists a unique (up to P-indistinguishability) process $[M]$ called *the quadratic variation for M* satisfying $[M] \in \mathcal{A}_0(\mathcal{F}_\cdot)$, $(\Delta M)_t^2 = \Delta[M]_t$ for all $t \in \mathbb{R}$ P-a.s., and $M^2 - [M] \in \mathcal{LM}(\mathcal{F}_\cdot)$. We have

$$ {}^s[M] \overset{P}{=} [{}^sM] \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad [M]^\sigma \overset{P}{=} [M^\sigma] \text{ when } \sigma \text{ is a stopping time.} \tag{3.7}$$

If, in addition, $M \in \mathcal{LM}^2(\mathcal{F}_\cdot)$, there is a unique predictable process $\langle M\rangle \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$ satisfying $M^2 - \langle M\rangle \in \mathcal{LM}(\mathcal{F}_\cdot)$, and we shall call this process the *predictable quadratic variation for M*. In this case,

$$ {}^s\langle M\rangle \overset{P}{=} \langle {}^sM\rangle \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \langle M\rangle^\sigma \overset{P}{=} \langle M^\sigma\rangle \text{ when } \sigma \text{ is a stopping time.} \tag{3.8}$$

Definition 3.10. Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. We say that an increasing process $V = (V_t)_{t\in\mathbb{R}}$ is a *generalised quadratic variation* for M if

$$ V \in \mathcal{IA}(\mathcal{F}_\cdot) \tag{3.9}$$
$$ (\Delta M)_t^2 = \Delta V_t \quad \text{for all } t \in \mathbb{R},\ P\text{-a.s.} \tag{3.10}$$
$$ ({}^sM)^2 - {}^sV \in \mathcal{LM}(\mathcal{F}_\cdot) \quad \text{for all } s \in \mathbb{R}. \tag{3.11}$$

We say that V is *quadratic variation for M* if, instead of (3.9), $V \in \mathcal{A}_0(\mathcal{F}_\cdot)$.

Let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$. We say that an increasing process $V = (V_t)_{t\in\mathbb{R}}$ is a *generalised predictable quadratic variation* for M if

$$ V \in \mathcal{ILA}^1(\mathcal{F}_\cdot) \tag{3.12}$$
$$ {}^sV \text{ is predictable for all } s \in \mathbb{R} \tag{3.13}$$
$$ ({}^sM)^2 - {}^sV \in \mathcal{LM}(\mathcal{F}_\cdot) \quad \text{for all } s \in \mathbb{R}. \tag{3.14}$$

We say that V is a *predictable quadratic variation* for V if, instead of (3.12), $V \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$.

Remark 3.11. (1) Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ and V denote a generalised quadratic variation for M such that $V_{-\infty} := \lim_{s \to -\infty} V_s$ exists P-a.s. From Remark 2.4 it follows that $(V_t - V_{-\infty})_{t \in \mathbb{R}}$ is a quadratic variation for M.

Similarly, let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ and V denote a generalised predictable quadratic variation for M such that $V_{-\infty} := \lim_{s \to -\infty} V_s$ exists P-a.s. Then $(V_t - V_{-\infty})_{t \in \mathbb{R}}$ is a predictable quadratic variation for M. Indeed, by Jacod and Shiryaev (2003), Lemma I.3.10, $(V_t - V_{-\infty})_{t \in \mathbb{R}}$ is a predictable process in $\mathcal{LA}_0^1(\mathcal{F}_\cdot)$. (Strictly speaking, this lemma only ensures the existence of an $\bar{\mathbb{R}}$-value localising sequence $(\sigma_n)_{n \geq 1}$ (cf. Remark 3.2 (3)) such that $(V_t - V_{-\infty})^{\sigma_n}$ is in $\mathcal{A}_0^1(\mathcal{F}_\cdot)$; this problem can, however, be dealt with as described in Remark 3.2).

(2) If $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ with $M_{-\infty} = 0$ then the usual quadratic variation $[M]$ for M is, by (3.7), also a quadratic variation in the sense of Definition 3.10, and similarly, if $M \in \mathcal{LM}^2(\mathcal{F}_\cdot)$ then the usual predictable quadratic variation $\langle M \rangle$ is a predictable quadratic variation also in the sense defined above.

(3) (Existence of generalised quadratic variation). Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Then V is a generalised quadratic variation for M if and only if we have (3.9)–(3.10) and V is associated with the family $\{[{}^s M]\}_{s \in \mathbb{R}}$. By Section 2, existence and uniqueness (up to addition of random variables) of the generalised quadratic variation is thus ensured once we have shown that the latter family is consistent. In other words, we must show for $s \leq t \leq u$ that $[{}^s M]_u = [{}^s M]_t + [{}^t M]_u$ P-a.s. Equivalently, ${}^t([{}^s M])_u = [{}^t M]_u$ P-a.s. This follows, however, from (3.7) and (2.2).

(4) (Existence of generalised predictable quadratic variation). Similarly, let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$. Then V is a generalised predictable quadratic variation for M if and only if we have (3.12)–(3.13) and V is associated with $\{\langle {}^s M \rangle\}_{s \in \mathbb{R}}$. Moreover, the latter family is consistent, ensuring existence and uniqueness of the generalised predictable quadratic variation up to addition of random variables.

(5) By Remark 2.4, the quadratic variation and the predictable quadratic variation are unique up to P-indistinguishability when they exist.

(6) Generalised compensators and predictable compensators are $\overset{\text{in}}{=}$-invariant, i.e. if for example $M, N \in \mathcal{IM}(\mathcal{F}_\cdot)$ with $M \overset{\text{in}}{=} N$ then V is a generalised compensator for M if and only if it is a generalised compensator for N.

When $M \in \mathcal{ILM}(\mathcal{F}.)$ we use $[M]^{\mathrm{g}}$ to denote a generalised quadratic variation for M, and $[M]$ denotes the quadratic variation when it exists. For $M \in \mathcal{ILM}^2(\mathcal{F}.)$, $\langle M \rangle^{\mathrm{g}}$ denotes a generalised quadratic variation for M, and $\langle M \rangle$ denotes the predictable quadratic variation when it exists. Generalising (3.7)–(3.8) we have the following.

Lemma 3.12. *Let σ denote a stopping time and $s \in \mathbb{R}$. If $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ then*

$$([M]^{\mathrm{g}})^\sigma \overset{\text{in}}{=} [M^\sigma]^{\mathrm{g}} \quad \textit{and} \quad {}^s([M]^{\mathrm{g}}) \overset{P}{=} [{}^s M]. \tag{3.15}$$

If $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ then

$$(\langle M \rangle^{\mathrm{g}})^\sigma \overset{\text{in}}{=} \langle M^\sigma \rangle^{\mathrm{g}} \quad \textit{and} \quad {}^s(\langle M \rangle^{\mathrm{g}}) \overset{P}{=} \langle {}^s M \rangle.$$

Proof. We only prove the part concerning the quadratic variation. As seen above, $[M]^{\mathrm{g}}$ is associated with $\{[{}^s M]\}_{s \in \mathbb{R}}$, which implies the second statement in (3.15).

To prove the first statement in (3.15) it suffices to show that $([M]^{\mathrm{g}})^{\sigma}$ is associated with $\{[{}^sM^{\sigma}]\}_{s\in\mathbb{R}}$. Note that, by (2.2) and (3.7),

$$^s(([M]^{\mathrm{g}})^{\sigma}) \overset{P}{=} ({}^s[M]^{\mathrm{g}})^{\sigma} \overset{P}{=} [{}^sM]^{\sigma} \overset{P}{=} [{}^sM^{\sigma}].$$

□

Example 3.13. Let τ_1 and τ_2 denote independent absolutely continuous random variables with densities f_1 and f_2 and distribution functions F_1 and F_2 satisfying $F_i(t) < 1$ for all t and $i = 1, 2$. Set

$$N_t^i = 1_{[\tau_i,\infty)}(t),\ A_t^i = \int_{-\infty}^{t\wedge\tau_i} \tfrac{f_i(u)}{1-F_i(u)}\,\mathrm{d}u,\ N_t = (N_t^1, N_t^1) \text{ and } \mathcal{F}_t = \sigma(N_s : s \le t) \vee \mathcal{N}$$

for $t \in \mathbb{R}$. From Brémaud (1981), A2 T26, follows that $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is right-continuous and hence a filtration in the sense defined in the present paper. It is well-known that M^i defined by $M_t^i = N_t^i - A_t^i$ is a square integrable martingale with $\langle M^i\rangle_t = A_t^i$, and M^1M^2 is a martingale. Assume, in addition,

$$\int_{-\infty}^{t} \tfrac{uf_i(u)}{1-F_i(u)}\,\mathrm{d}u = -\infty \quad \text{for all } t \in \mathbb{R}.$$

(This is satisfied if, for example, $F_i(s)$ equals a constant times $(1 + |s|\log(|s|))^{-1}$ when s is small.) Let $B^i \in \mathcal{IA}^1(\mathcal{F}_{\cdot})$ satisfy

$$^sB_t^i = \int_{(s,t]} u\,\mathrm{d}A_u^i = \int_{s\wedge\tau_i}^{t\wedge\tau_i} \frac{uf_i(u)}{1-F_i(u)}\,\mathrm{d}u$$

for $s < t$ and set $X_t^i = \tau_i N_t^i - B_t^i$. Then

$$\lim_{s\to-\infty} X_s^i = -\lim_{s\to-\infty} B_s^i = \infty \quad \text{pointwise,}$$

implying that X^i is not a local martingale. However, since for $s < t$,

$$^sX_t^i = \int_{(s,t]} u\,\mathrm{d}M_u^i$$

it follows that ${}^sX^i$ is a square integrable martingale. That is, $X^i \in \mathcal{ILM}^2(\mathcal{F}_{\cdot})$.

The quadratic variations, $[X^i]$ resp. $[X^1 - X^2]$, of X^i resp. $X^1 - X^2$ do exist and are $[X^i]_t = (\tau_i)^2N_t^i$ resp. $[X^1 - X^2]_t = (\tau_1)^2N_t^1 + (\tau_2)^2N_t^2$. Moreover, up to addition of random variables,

$$\liminf_{s\to-\infty}(X_s^1 - X_s^2) = \liminf_{s\to-\infty}(B_s^2 - B_s^1) = \liminf_{s\to-\infty} \int_s^0 u\big(\tfrac{f_2(u)}{1-F_2(u)} - \tfrac{f_1(u)}{1-F_1(u)}\big)\,\mathrm{d}u$$

$$\limsup_{s\to-\infty}(X_s^1 - X_s^2) = \liminf_{s\to-\infty}(B_s^2 - B_s^1) = \limsup_{s\to-\infty} \int_s^0 u\big(\tfrac{f_2(u)}{1-F_2(u)} - \tfrac{f_1(u)}{1-F_1(u)}\big)\,\mathrm{d}u.$$

If τ_1 and τ_2 are identically distributed then $X_s^1 - X_s^2$ converges pointwise. In other cases we may have $\limsup_{s\to-\infty}(X_s^1 - X_s^2) = -\liminf_{s\to-\infty}(X_s^1 - X_s^2) = \infty$ pointwise.

To sum up, we have seen that even if the quadratic variation exists, the process may or may not converge as time goes to $-\infty$.

The next result shows in particular that for increment local martingales with bounded jumps, a.s. convergence in $-\infty$ is closely related to the local martingale property.

Theorem 3.14. *Let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$. The following are equivalent.*

(a) There is a predictable quadratic variation $\langle M \rangle$ for M.

(b) $M_{-\infty} = \lim_{s\to-\infty} M_s$ exists P-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}^2(\mathcal{F}_\cdot)$.

Remark 3.15. Let M in $\mathcal{ILM}(\mathcal{F}_\cdot)$ have bounded jumps; then, $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ as well. In this case (b) is satisfied if and only if $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. Indeed, if the limit exists we define

$$\sigma_n = \inf\{t \in \mathbb{R} : |M_t - M_{-\infty}| > n\}.$$

Then $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is a bounded and adapted process in $\mathcal{ILM}(\mathcal{F}_\cdot)$ and hence in $\mathcal{IM}^2(\mathcal{F}_\cdot)$. By Proposition 3.9, $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}^2(\mathcal{F}_\cdot)$.

Proof. (a) implies (b): Choose a localising sequence $(\sigma_n)_{n\geq 1}$ such that

$$E[\langle M \rangle_t^{\sigma_n}] < \infty, \quad \text{for all } t \in \mathbb{R} \text{ and all } n \geq 1.$$

Since ${}^s\langle M \rangle^{\sigma_n} = \langle {}^sM^{\sigma_n} \rangle$, it follows in particular that

$$E[\langle {}^sM^{\sigma_n} \rangle_t] \leq E[\langle M \rangle_t^{\sigma_n}] < \infty$$

for all $s \leq t$ and n. Therefore, for all s and n we have ${}^sM^{\sigma_n} \in \mathcal{M}^2(\mathcal{F}_\cdot)$, and

$$E[({}^sM_t^{\sigma_n})^2] \leq E[\langle M \rangle_t^{\sigma_n}] < \infty$$

for all $s \leq t$. Using Proposition 3.9 on M^{σ_n} it follows that $M_{-\infty} := \lim_{s\to-\infty} M_s^{\sigma_n}$ exists P-a.s. (this limit does not depend on n) and $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is a square integrable martingale.

(b) implies (a): Let $\langle M - M_{-\infty} \rangle$ denote the predictable quadratic variation for $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ which exists since this process is a locally square integrable martingale. Since $M \overset{\text{in}}{=} (M_t - M_{-\infty})_{t\in\mathbb{R}}$, $\langle M - M_{-\infty} \rangle$ is a predictable quadratic variation for M as well.

□

We have seen that a continuous increment local martingale is a local martingale if it converges almost surely as time goes to $-\infty$. A main purpose of the next examples is to study the behaviour in $-\infty$ when this is not the case.

Example 3.16. In (2) below we give an example of a continuous increment local martingale which converges to zero in probability as time goes to $-\infty$ without being a local martingale. As a building block for this construction we first consider a simple example of a continuous local martingale which is nonzero only on a finite interval.

(1) Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion and τ be the first visit to zero after a visit to k, i.e.

$$\tau = \inf\{t > 0 : B_t = 0 \text{ and there is an } s < t \text{ such that } B_s > k\}, \tag{3.16}$$

where $k > 0$ is some fixed level. Then τ is finite with probability one, the stopped process $(B_{t\wedge\tau})_{t\geq 0}$ is a square integrable martingale, and $B_{t\wedge\tau} = 0$ when $t \geq \tau$. Let $a < b$ be real numbers and $\phi : [a, b) \to [0, \infty)$ be a surjective, continuous and strictly increasing mapping and define $Y = (Y_t)_{t\in\mathbb{R}}$ as

$$Y_t = \begin{cases} 0 & \text{if } t < a \\ B_{\phi(t)\wedge\tau} & \text{if } t \in [a, b) \\ 0 & \text{if } t \geq b. \end{cases} \tag{3.17}$$

Note that $t \mapsto Y_t$ is continuous P-a.s. and that with probability one $Y_t = 0$ for $t \notin [a, b]$. Define, with $\mathcal{N}$ denoting the P-null sets,

$$\mathcal{F}_t = \sigma(B_u : u \leq \phi(t)) \vee \mathcal{N} \quad \text{for } t \in \mathbb{R}, \tag{3.18}$$

where we let $\phi(t) = 0$ for $t \leq a$ and $\phi(t) = \infty$ for $t \geq b$. Interestingly, Y is a local martingale. To see this, define the "canonical" localising sequence $(\sigma_n)_{n\geq 1}$ as $\sigma_n = \inf\{t \in \mathbb{R} : |Y_t| > n\}$. Since $(Y_t^{\sigma_n})_{t\in[a,b)}$ is a deterministic time change of $(B_{t\wedge\tau})_{t\geq 0}$ stopped at σ_n, it is a bounded, and hence uniformly integrable, martingale. By continuity of the paths and the property $Y_t^{\sigma_n} = Y_b^{\sigma_n}$ for $t \geq b$ it thus follows that $(Y_t^{\sigma_n})_{t\in\mathbb{R}}$ is a bounded martingale.

(2) For $n = 1, 2, \ldots$ let $B^n = (B_t^n)_{t\geq 0}$ denote independent standard Brownian motions, and define $Y^n = (Y_t^n)_{t\in\mathbb{R}}$ as in (3.17) with $a = -n$ and $b = -n + 1$, and Y resp. B replaced by Y^n resp. B^n. Let $(\mathcal{F}_t^n)_{t\in\mathbb{R}}$ be the corresponding filtration defined as in (3.18), and $(\theta_n)_{n\geq 1}$ denote a sequence of independent Bernoulli variables that are independent of the Brownian motions as well and satisfy $P(\theta_n = 1) = 1 - P(\theta_n = 0) = \frac{1}{n}$ for all n. Let $X_t^n = \theta_n Y_t^n$ for $t \in \mathbb{R}$.

Define $X_t = \sum_{n=1}^{\infty} X_t^n$ for $t \in \mathbb{R}$, which is well-defined since $X_t^n = 0$ for $t \notin [-n, -n + 1]$, and set $\mathcal{F}_t = \vee_{n=1}^{\infty}(\mathcal{F}_t^n \vee \sigma(\theta_n))$ for $t \in \mathbb{R}$. For $s \in [-n, -n + 1]$ and $n = 1, 2, \ldots$, ${}^sX_t = \sum_{m=1}^{n} {}^sX_t^m$, and since it is easily seen that each $(X_t^m)_{t\in\mathbb{R}}$ is a local martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}}$, it follows that ${}^sX$ is a local martingale as well; that is, X is an increment local martingale. By Borel-Cantelli, infinitely many of the θ_n's are 1 P-a.s., implying that X_s does not converge P-a.s. as $s \to -\infty$. On the other hand, $P(X_t = 0) \geq \frac{n-1}{n}$ for $t \in [-n, -n + 1]$, which means that $X_s \to 0$ in probability as $s \to -\infty$.

From (3.1) it follows that if a process in $\mathcal{IM}(\mathcal{F}_\cdot)$ is adapted and integrable then it is in $\mathcal{M}(\mathcal{F}_\cdot)$. By the above there is no such result for $\mathcal{ILM}(\mathcal{F}_\cdot)$; indeed, X is both adapted and p-integrable for all $p > 0$ but it is not in $\mathcal{LM}(\mathcal{F}_\cdot)$.

Example 3.17. Let $X = (X_t)_{t\geq 0}$ denote the inverse of BES(3), the three-dimensional Bessel process. It is well-known (see e.g. Rogers and Williams (2000)) that X is a diffusion on natural scale and hence for all $s > 0$ the increment process $({}^sX_t)_{t\geq 0}$ is a local martingale. That is, we may consider X as an increment martingale indexed by $[0, \infty)$. By Rogers and Williams (2000), ∞ is an entrance boundary, which means that if the process is started in ∞, it immediately leaves this state and never returns. Since we can obviously stretch $(0, \infty)$ into $\mathbb{R}$, this shows that there are interesting examples of continuous increment local martingales $(X_t)_{t\in\mathbb{R}}$ for which $\lim_{t\to-\infty} X_t = \pm\infty$ almost surely.

Using the Dambis-Dubins-Schwartz theorem it follows easily that any continuous local martingale indexed by $\mathbb{R}$ is a time change of a Brownian motion indexed by $\mathbb{R}_+$. It is not clear to us whether there is some analogue of this result for continuous increment local martingales but there are indications that this it not the case; indeed, above we saw that a continuous increment local martingale may converge to ∞ as time goes to $-\infty$; in particular this limiting behaviour does not resemble that of a Brownian motion indexed by $\mathbb{R}_+$ as time goes to 0 or of a Brownian motion indexed by $\mathbb{R}$ as time goes to $-\infty$.

Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$. It is well-known that M can be decomposed uniquely up to P-indistinguishability as $M_t = M_{-\infty} + M_t^c + M_t^d$ where $M^c = (M_t^c)_{t\in\mathbb{R}}$, the *continuous part* of M, is a continuous local martingale with $M_{-\infty} = 0$, and M^d, the *purely discontinuous part* of M, is a purely discontinuous local martingale with $M_{-\infty}^d = 0$, which means that $M^d N$ is a local martingale for all continuous local martingales N. Note that for $s \in \mathbb{R}$,

$$({}^s\!M)^c = {}^s\!(M^c) \quad \text{ and } \quad ({}^s\!M)^d = {}^s\!(M^d). \tag{3.19}$$

We need a further decomposition of M^d so let $\mu^M = \{\mu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$ denote the random measure on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ induced by the jumps of M; that is,

$$\mu^M(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{s\in\mathbb{R}} \delta_{(s,\Delta M_s(\omega))}(\mathrm{d}t, \mathrm{d}x),$$

and let $\nu^M = \{\nu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$ denote the compensator of μ^M in the sense of Jacod and Shiryaev (2003), II.1.8. From Proposition II.2.29 and Corollary II.2.38 in Jacod and Shiryaev (2003) it follows that $(|x| \wedge |x|^2) * \nu^M \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$ and $M^d \overset{P}{=} x * (\mu^M - \nu^M)$, implying that for arbitrary $\epsilon > 0$, M can be decomposed as

$$\begin{aligned} M_t &= M_{-\infty} + M_t^c + M_t^d = M_{-\infty} + M_t^c + x * (\mu^M - \nu^M)_t \\ &= M_{-\infty} + M_t^c + (x1_{\{|x|\le\epsilon\}}) * (\mu^M - \nu^M)_t + (x1_{\{|x|>\epsilon\}}) * \mu_t^M - (x1_{\{|x|>\epsilon\}}) * \nu_t^M. \end{aligned}$$

Recall that when M is quasi-left continuous we have

$$\nu^M(\cdot; \{t\} \times (\mathbb{R} \setminus \{0\})) = 0 \quad \text{for all } t \in \mathbb{R} \ P\text{-a.s.} \tag{3.20}$$

Finally, for $s \in \mathbb{R}$, $\mu^{{}^s\!M}(\cdot; \mathrm{d}t, \mathrm{d}x) = 1_{(s,\infty)}(\mathrm{d}t)\mu^M(\cdot; \mathrm{d}t, \mathrm{d}x)$ and thus

$$\nu^{{}^s\!M}(\cdot; \mathrm{d}t, \mathrm{d}x) = 1_{(s,\infty)}(\mathrm{d}t)\nu^M(\cdot; \mathrm{d}t, \mathrm{d}x). \tag{3.21}$$

Now consider the case $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Denote the continuous resp. purely discontinuous part of ${}^s\!M$ by ${}^s\!M^c$ resp. ${}^s\!M^d$. By (3.19), $\{{}^s\!M^c\}_{s\in\mathbb{R}}$ and $\{{}^s\!M^d\}_{s\in\mathbb{R}}$ are consistent families of increment processes, and M is associated with $\{{}^s\!M^c + {}^s\!M^d\}_{s\in\mathbb{R}}$. Thus, there exist two processes, which we call the *continuous* resp. *purely discontinuous* part of M, and denote M^{cg} and M^{dg}, such that M^{cg} is associated with $\{{}^s\!M^c\}_{s\in\mathbb{R}}$ and M^{dg} is associated with $\{{}^s\!M^d\}_{s\in\mathbb{R}}$, and

$$M_t = M_t^{cg} + M_t^{dg} \quad \text{for all } t \in \mathbb{R},\ P\text{-a.s.} \tag{3.22}$$

Once again these processes are unique only up to addition of random variables. In view of (3.21) we define the compensator of μ^M, to be denoted $\{\nu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$, as the random measure on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ satisfying that for all $s \in \mathbb{R}$,

$$1_{(s,\infty)}(\mathrm{d}t)\nu(\omega; \mathrm{d}t, \mathrm{d}x) = \nu^{{}^sM}(\omega; \mathrm{d}t, \mathrm{d}x),$$

where, noticing that ${}^sM$ is a local martingale, the right-hand side is the compensator of $\mu^{{}^sM}$ in the sense of Jacod and Shiryaev (2003), II.1.8.

Theorem 3.18. *Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$.*

(1) The quadratic variation $[M]$ for M exists if and only if there is a continuous martingale component M^{cg} with $M^{\mathrm{cg}} \in \mathcal{LM}(\mathcal{F}_\cdot)$ and $M^{\mathrm{cg}}_{-\infty} = 0$, and for all $t \in \mathbb{R}$, $\sum_{s\le t}(\Delta M_s)^2 < \infty$ P-a.s. In this case

$$[M]_t = \langle M^{\mathrm{cg}}\rangle_t + \sum_{s\le t}(\Delta M_s)^2.$$

(2) We have that $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$ if and only if the quadratic variation $[M]$ for M exists and $[M]^{\frac{1}{2}} \in \mathcal{LA}^1_0(\mathcal{F}_\cdot)$.

(3) Assume (3.20) is satisfied and there is an $\epsilon > 0$ such that

$$\lim_{s\to-\infty} \int_{(s,0]} \int_{|x|>\epsilon} x\nu^M(\cdot; \mathrm{d}u, \mathrm{d}x) \tag{3.23}$$

exists P-a.s. Then, $\lim_{s\to-\infty} M_s$ exists P-a.s. if and only if $[M]$ exists.

Note that the conditions in (3) are satisfied if ν^M can be decomposed as $\nu^M(\cdot; \mathrm{d}t\times \mathrm{d}x) = F(\cdot; t, \mathrm{d}x)\,\mu(\mathrm{d}t)$ where $F(\cdot; t, \mathrm{d}x)$ is a symmetric measure for all $t \in \mathbb{R}$ and μ does not have positive point masses.

Proof. (1) For $s \le t$ we have

$$\begin{aligned} {}^s[M]^{\mathrm{g}}_t = [{}^sM]_t &= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle {}^sM^c\rangle_t \\ &= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle {}^s(M^{\mathrm{cg}})\rangle_t \\ &= \sum_{u:s<u\le t} (\Delta M_u)^2 + {}^s\langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_t \\ &= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_t - \langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_s, \end{aligned} \tag{3.24}$$

where the first equality is due to the fact that $[M]^{\mathrm{g}}$ is associated with $\{[{}^sM]\}_{s\in\mathbb{R}}$, the second is a well-known decomposition of the quadratic variation of a local martingale, the third equality is due to M^{cg} being associated with $\{{}^sM^c\}_{s\in\mathbb{R}}$ and the fourth is due to $\langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}$ being associated with $\{\langle {}^sM^{\mathrm{cg}}\rangle\}_{s\in\mathbb{R}}$. By Remark 3.11 (1), the quadratic variation $[M]$ exists if and only if $[M]^{\mathrm{g}}_s$ converges P-a.s. as $s \to -\infty$, which, by the above, is equivalent to convergence almost surely of both terms in (3.24). By Theorem 3.14, $\langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_s$ converges P-a.s. as $s\to-\infty$ if and only if $M^{\mathrm{cg}}_{-\infty}$ exists P-a.s. and $(M^{\mathrm{cg}}_t - M^{\mathrm{cg}}_{-\infty})_{t\in\mathbb{R}}$ is a continuous local martingale. If the quadratic variation exists, we may replace M^{cg} by $(M^{\mathrm{cg}}_t - M^{\mathrm{cg}}_{-\infty})_{t\in\mathbb{R}}$ and $M^{d\mathrm{g}}$ by $(M^{d\mathrm{g}}_t + M^{\mathrm{cg}}_{-\infty})_{t\in\mathbb{R}}$, thus obtaining a continuous part of M which starts at 0.

(2) First assume that $M_{-\infty}$ exists and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$. Since $M \overset{\text{in}}{=} (M_t - M_{-\infty})_{t\in\mathbb{R}}$, the quadratic variation for M exists and equals the quadratic variation for $(M_t - M_{-\infty})_{t\in\mathbb{R}}$. It is well-known that since the latter is a local martingale, $[M]^{\frac{1}{2}} \in \mathcal{LA}^1_0(\mathcal{F}_\cdot)$.

Conversely assume that $[M]$ exists and $[M]^{\frac{1}{2}} \in \mathcal{LA}^1_0(\mathcal{F}_\cdot)$. Choose a localising sequence $(\sigma_n)_{n\geq1}$ such that $[M^{\sigma_n}]^{\frac{1}{2}} \in \mathcal{A}^1_0(\mathcal{F}_\cdot)$. Since ${}^s[M^{\sigma_n}]_0 \leq [M^{\sigma_n}]_0$ if follows from Davis' inequality that for some constant $c>0$,

$$E[\sup_{u:s\leq u\leq 0} |{}^sM^{\sigma_n}_0|] \leq cE[[M^{\sigma_n}]_0^{\frac{1}{2}}] < \infty$$

for all $s\leq 0$, implying that $({}^sM^{\sigma_n}_0)_{s<0}$ is uniformly integrable. The result now follows from Proposition 3.9.

(3) By (3.21), the three families of increment processes $\{(x1_{\{|x|\leq\epsilon\}})*(\mu^{{}^sM}-\nu^{{}^sM}\}_{s\in\mathbb{R}}$, $\{(x1_{\{|x|>\epsilon\}})*\mu^{{}^sM}\}_{s\in\mathbb{R}}$ and $\{(x1_{\{|x|>\epsilon\}})*\nu^{{}^sM}\}_{s\in\mathbb{R}}$ are all consistent. Choose $X=(X_t)_{t\in\mathbb{R}}$, $Y=(Y_t)_{t\in\mathbb{R}}$ and $Z=(Z_t)_{t\in\mathbb{R}}$ associated with these families such that $X_t+Y_t-Z_t = M^{d\mathrm{g}}_t$; in particular we then have $M \overset{P}{=} M^{\mathrm{cg}}+X+Y-Z$. Since Z is associated with $\{(x1_{\{|x|>\epsilon\}})*\nu^{{}^sM}\}_{s\in\mathbb{R}}$ we have

$$Z_0 - Z_s = \int_s^0\int_{|x|>\epsilon} x\,\nu^M(\cdot;\mathrm{d}u,\mathrm{d}x) \quad \text{for all } s\in\mathbb{R} \text{ with probability one,}$$

implying that $s\mapsto Z_s$ is continuous by (3.20) and $\lim_{s\to-\infty} Z_s$ exists P-a.s. by (3.23). By (3.20) it also follows that $(\Delta X_s)_{s\in\mathbb{R}} \overset{P}{=} (\Delta M_s 1_{\{|\Delta M_s|\leq\epsilon\}})_{s\in\mathbb{R}}$, implying that X is an increment local martingale with jumps bounded by ϵ in absolute value and

$$\sum_{s:s\leq t}(\Delta M_s)^2 = \sum_{s:s\leq t}(\Delta X_s)^2 + \sum_{s:s\leq t}(\Delta Y_s)^2 \quad \text{for all } t\in\mathbb{R} \text{ with probability one.} \tag{3.25}$$

If $[M]$ exists then by (1) $M^{\mathrm{cg}}_{-\infty}$ exists P-a.s. and (3.25) is finite for all t with probability one. Since Y is piecewise constant with jumps of magnitude at least ϵ, it follows that Y_s is constant when s is small enough almost surely. In addition, since the quadratic variation of the increment local martingale X exists and X has bounded jumps it follows from (2) that, up to addition of a random variable, X is a local martingale and thus $\lim_{s\to-\infty} X_s$ exists as well; that is, $\lim_{s\to-\infty} M_s$ exists P-a.s.

If, conversely, $\lim_{s\to-\infty} M_s$ exists P-a.s., there are no jumps of magnitude at least ϵ in M when s is small enough; thus there are no jumps in Y_s when s is sufficiently small P-a.s., implying that $\lim_{s\to-\infty}(M^{\mathrm{cg}}_s + X_s)$ exists P-a.s. Combining Theorem 3.14, (3.25) and (1) it follows that $[M]$ exists. □

4 Stochastic integration

In the following we define a stochastic integral with respect to an increment local martingale. Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ and set

$$\begin{aligned}&\mathcal{L}L^1(M)\\ &\quad := \{\phi = (\phi_t)_{t\in\mathbb{R}} : \phi \text{ is predictable and } \Big(\big(\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}[M]_s\big)^{\frac{1}{2}}\Big)_{t\in\mathbb{R}} \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)\}.\end{aligned}$$

Since in this case the index set set can be taken to be $[-\infty,\infty)$, it is well-known, e.g. from Jacod (1979), that the stochastic integral of $\phi \in \mathcal{L}L^1(M)$ with respect to M, which we denote $(\int_{(-\infty,t]} \phi_s \,\mathrm{d}M_s)_{t\in\mathbb{R}}$ or $\phi \bullet M = (\phi \bullet M_t)_{t\in\mathbb{R}}$, does exist. All fundamental properties of the integral are well-known so let us just explicitly mention the following two results that we are going to use in the following: For σ a stopping time, $s \in \mathbb{R}$ and $\phi \in \mathcal{L}L^1(M)$ we have

$$(\phi \bullet M)^\sigma \overset{P}{=} (\phi 1_{(-\infty,\sigma]}) \bullet M \overset{P}{=} \phi \bullet (M^\sigma) \tag{4.1}$$

and

$$^s(\phi \bullet M) \overset{P}{=} \phi \bullet ({}^sM) \overset{P}{=} (\phi 1_{(s,\infty)}) \bullet M. \tag{4.2}$$

Next we define and study a *stochastic increment integral* with respect an increment local martingale. For $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ set

$$\begin{aligned}\mathcal{L}L^1(M) &:= \{\phi : \phi \text{ is predictable and } \Big(\big(\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}[M]_s^{\mathrm{g}}\big)^{\frac{1}{2}}\Big)_{t\in\mathbb{R}} \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)\}\\ \mathcal{IL}L^1(M) &:= \{\phi : \phi \in \mathcal{L}L^1({}^sM) \text{ for all } s \in \mathbb{R}\}.\end{aligned}$$

As an example, if $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ then a predictable ϕ is in $\mathcal{L}L^1(M)$ resp. in $\mathcal{IL}L^1(M)$ if (but in general not only if) $\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}\langle M\rangle_s^{\mathrm{g}} < \infty$ for all $t \in \mathbb{R}$ P-a.s. resp. $\int_{(s,t]} \phi_u^2 \,\mathrm{d}\langle M\rangle_u^{\mathrm{g}} < \infty$ for all $s < t$ P-a.s. If $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ is continuous then

$$\begin{aligned}\mathcal{L}L^1(M) &= \{\phi : \phi \text{ is predictable and } \int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}\langle M\rangle_s^{\mathrm{g}} < \infty \ P\text{-}a.s. \text{ for all } t\}\\ \mathcal{IL}L^1(M) &= \{\phi : \phi \text{ is predictable and } \int_{(s,t]} \phi_u^2 \,\mathrm{d}\langle M\rangle_u^{\mathrm{g}} < \infty \ P\text{-}a.s. \text{ for all } s < t\}.\end{aligned}$$

Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. The stochastic integral $\phi \bullet ({}^sM)$ of ϕ in $\mathcal{IL}L^1(M)$ exists for all $s \in \mathbb{R}$; in addition, $\{\phi \bullet ({}^sM)\}_{s\in\mathbb{R}}$ is a consistent family of increment processes. Indeed, for $s \le t \le u$ we must verify

$$(\phi \bullet ({}^sM))_u = (\phi \bullet ({}^sM))_t + (\phi \bullet ({}^tM))_u, \quad P\text{-a.s.}$$

or equivalently

$$^t(\phi \bullet ({}^sM))_u = (\phi \bullet ({}^tM))_u \quad P\text{-a.s.},$$

which follows from (2.3) and (4.2). Based on this, we define the *stochastic increment integral of* ϕ *with respect to* M, to be denoted $\phi \overset{\text{in}}{\bullet} M$, as a càdlàg process associated with the the family $\{\phi \bullet ({}^sM)\}_{s\in\mathbb{R}}$. Note that the increment integral $\phi \overset{\text{in}}{\bullet} M$ is uniquely determined only up to addition of a random variable and it is an increment local martingale. For $s < t$ and $\phi \in \mathcal{ILL}^1(M)$ we think of $\phi \overset{\text{in}}{\bullet} M_t - \phi \overset{\text{in}}{\bullet} M_s$ as the integral of ϕ with respect to M over the interval $(s,t]$ and hence use the notation

$$\int_{(s,t]} \phi_u \, \mathrm{d}M_u := \phi \overset{\text{in}}{\bullet} M_t - \phi \overset{\text{in}}{\bullet} M_s \quad \text{for } s < t. \tag{4.3}$$

When $\phi \overset{\text{in}}{\bullet} M_{-\infty} := \lim_{s \to -\infty} \phi \overset{\text{in}}{\bullet} M$ exists P-a.s. we define the *improper integral of ϕ with respect to M from $-\infty$ to t* for $t \in \mathbb{R}$ as

$$\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u := \phi \overset{\text{in}}{\bullet} M_t - \phi \overset{\text{in}}{\bullet} M_{-\infty}. \tag{4.4}$$

Put differently, the improper integral $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t \in \mathbb{R}}$ is, when it exists, the unique, up to P-indistinguishability, increment integral of ϕ with respect to M which is 0 in $-\infty$. Moreover, it is an adapted process.

The following summarises some fundamental properties.

Theorem 4.1. *Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$.*

(1) Whenever $\phi \in \mathcal{ILL}^1(M)$ and $s < t$ we have ${}^s(\phi \overset{\text{in}}{\bullet} M)_t = (\phi \bullet ({}^sM))_t$ P-a.s.

(2) $\phi \overset{\text{in}}{\bullet} M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ for all $\phi \in \mathcal{ILL}^1(M)$.

(3) If $\phi, \psi \in \mathcal{ILL}^1(M)$ and $a, b \in \mathbb{R}$ then $(a\phi + b\psi) \overset{\text{in}}{\bullet} M \overset{\text{in}}{=} a(\phi \overset{\text{in}}{\bullet} M) + b(\psi \overset{\text{in}}{\bullet} M)$.

(4) For $\phi \in \mathcal{ILL}^1(M)$ we have

$$\Delta \phi \overset{\text{in}}{\bullet} M_t = \phi_t \Delta M_t, \quad \text{for } t \in \mathbb{R},\ P\text{-a.s.} \tag{4.5}$$

$${}^s[\phi \overset{\text{in}}{\bullet} M]^{\mathrm{g}}_t = \int_{(s,t]} \phi_u^2 \, \mathrm{d}[M]^{\mathrm{g}}_s \quad \text{for } s \leq t \ P\text{-a.s.} \tag{4.6}$$

In particular $[\phi \overset{\text{in}}{\bullet} M]$ exists if and only if $\int_{(-\infty,t]} \phi_s^2 \, \mathrm{d}[M]^{\mathrm{g}}_s < \infty$ for all $t \in \mathbb{R}$ P-a.s.

(5) If σ a stopping time and $\phi \in \mathcal{ILL}^1(M)$ then

$$(\phi \overset{\text{in}}{\bullet} M)^\sigma \overset{\text{in}}{=} (\phi 1_{(-\infty,\sigma]}) \overset{\text{in}}{\bullet} M \overset{\text{in}}{=} \phi \overset{\text{in}}{\bullet} (M^\sigma).$$

(6) Let $\phi \in \mathcal{ILL}^1(M)$ and $\psi = (\psi_t)_{t \in \mathbb{R}}$ be predictable. Then $\psi \in \mathcal{ILL}^1(\phi \overset{\text{in}}{\bullet} M)$ if and only if $\phi\psi \in \mathcal{ILL}^1(M)$, and in this case $\psi \overset{\text{in}}{\bullet} (\phi \overset{\text{in}}{\bullet} M) \overset{\text{in}}{=} (\psi\phi) \overset{\text{in}}{\bullet} M$.

(7) Let $\phi \in \mathcal{ILL}^1(M)$. Then $\phi \overset{\text{in}}{\bullet} M_{-\infty} := \lim_{s \to -\infty} \phi \overset{\text{in}}{\bullet} M_s$ exists P-a.s. and $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t \in \mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$ if and only if $\phi \in \mathcal{LL}^1(M)$.

Remark 4.2. (a) When M is continuous it follows from Theorem 3.14 that (7) can be simplified to the statement that $\phi \overset{\text{in}}{\bullet} M_{-\infty} = \lim_{s \to -\infty} \phi \overset{\text{in}}{\bullet} M_s$ exists P-a.s. if and only if $\phi \in \mathcal{LL}^1(M)$, and in this case $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t \in \mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$.

(b) Result (7) above gives a necessary and sufficient condition for the improper integral to exist and be a local martingale; however, improper integrals may exist without being a local martingale (but as noted above they are always increment local martingales). For example, assume M is purely discontinuous and that the compensator ν^M of the jump measure ν^M can be decomposed as $\nu^M(\cdot;\mathrm{d}t\times\mathrm{d}x) = F(\cdot;t,\mathrm{d}x)\mu(\mathrm{d}t)$ where $F(\cdot;t,\mathrm{d}x)$ is a symmetric measure and $\mu(\{t\}) = 0$ for all $t\in\mathbb{R}$. Then by Theorem 3.18 (3), $\phi \overset{\text{in}}{\bullet} M_{-\infty}$ exists P-a.s. if and only if the quadratic variation $[\phi \overset{\text{in}}{\bullet} M]$ exists; that is,

$$\sum_{s\leq 0}\phi_s^2(\Delta M_s)^2 < \infty \quad P\text{-}a.s.$$

Proof. Property (1) is merely by definition, and (2) is due to the fact that ${}^s(\phi \overset{\text{in}}{\bullet} M) \overset{P}{=} \phi\bullet {}^sM$, which is a local martingale.

(3) We must show that $a(\phi \overset{\text{in}}{\bullet} M) + b(\psi \overset{\text{in}}{\bullet} M)$ is associated with $\{(a\phi+b\psi)\bullet({}^sM)\}_{s\in\mathbb{R}}$, i.e. that ${}^s\big(a(\phi \overset{\text{in}}{\bullet} M) + b(\psi \overset{\text{in}}{\bullet} M)\big) \overset{P}{=} (a\phi+b\psi)\bullet({}^sM)$. However, by definition of the stochastic increment integral and linearity of the stochastic integral we have

$$a\,{}^s\big(\phi \overset{\text{in}}{\bullet} M\big) + b\,{}^s\big(\psi \overset{\text{in}}{\bullet} M\big) \overset{P}{=} a\big(\phi\bullet({}^sM)\big) + b\big(\psi\bullet({}^sM)\big) \overset{P}{=} (a\phi+b\psi)\bullet({}^sM).$$

(4) Using that ${}^s(\phi \overset{\text{in}}{\bullet} M) = \phi\bullet({}^sM)$ and $\Delta\phi\bullet({}^sM) \overset{P}{=} \phi\Delta({}^sM)$, the result in (4.5) follows. By definition, $[\phi \overset{\text{in}}{\bullet} M]^{\mathrm{g}}$ is associated with $\{[{}^s(\phi \overset{\text{in}}{\bullet} M)]\}_{s\in\mathbb{R}} = \{[\phi\bullet({}^sM)]\}_{s\in\mathbb{R}}$. That is, for $s\in\mathbb{R}$ we have, using that $[M]^{\mathrm{g}}$ is associated with $\{[{}^sM]_s\}_{s\in\mathbb{R}}$,

$$\begin{aligned}{}^s[\phi \overset{\text{in}}{\bullet} M]^{\mathrm{g}}_t &= [\phi\bullet({}^sM)]_t = \int_{(s,t]}\phi_u^2\,\mathrm{d}[{}^sM]_u \\ &= \int_{(s,t]}\phi_u^2\,\mathrm{d}({}^s[M]^{\mathrm{g}})_u = \int_{(s,t]}\phi_u^2\,\mathrm{d}[M]^{\mathrm{g}}_u \quad \text{for } s\leq t \quad P\text{-a.s.},\end{aligned}$$

which yields (4.6). The last statement in (4) follows from Remark 3.11 (1).

The proofs of (5) and (6) are left to the reader.

(7) Using (4) the result follows immediately from Theorem 3.18. □

Let us turn to the definition of a stochastic integral $\phi\bullet M$ of a predictable ϕ with respect to an increment local martingale M. Thinking of $\phi\bullet M_t$ as an integral from $-\infty$ to t it seems reasonable to say that $\phi\bullet M$ (defined for a suitable class of predictable processes ϕ) is a stochastic integral with respect to M if the following is satisfied:

(1) $\lim_{t\to-\infty}\phi\bullet M_t = 0$ P-a.s.

(2) $\phi_t\bullet M_t - \phi\bullet M_s = \int_{(s,t]}\phi_u\,\mathrm{d}M_u$ P-a.s. for all $s<t$

(3) $\phi\bullet M$ is a local martingale.

By definition of $\int_{(s,t]}\phi_u\,\mathrm{d}M_u$, (2) implies that $\phi\bullet M$ must be an increment integral of ϕ with respect to M. Moreover, since we assume $\phi\bullet M_{-\infty} = 0$, $\phi\bullet M$ is uniquely determined as $(\phi\bullet M_t)_{t\in\mathbb{R}} \overset{P}{=} (\int_{(-\infty,t]}\phi_u\,\mathrm{d}M_u)_{t\in\mathbb{R}}$, i.e. the improper integral of ϕ. Since we also insist that $\phi\bullet M$ is a local martingale, Theorem 4.1 (7) shows that $\mathcal{L}L^1(M)$ is the largest possible set on which $\phi\bullet M$ can be defined. We summarise these findings as follows.

Theorem 4.3. *Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Then there exists a unique stochastic integral $\phi \bullet M$ defined for $\phi \in \mathcal{L}L^1(M)$. This integral is given by*

$$\phi \bullet M_t = \int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u \quad \textit{for } t \in \mathbb{R} \tag{4.7}$$

and it satisfied the following.

(1) $\phi \bullet M \in \mathcal{LM}(\mathcal{F}_\cdot)$ and $\phi \bullet M_{-\infty} = 0$ for $\phi \in \mathcal{L}L^1(M)$.

(2) The mapping $\phi \mapsto \phi \bullet M$ is, up to P-indistinguishability, linear in $\phi \in \mathcal{L}L^1(M)$.

(3) For $\phi \in \mathcal{L}L^1(M)$ we have

$$\Delta\phi \bullet M_t = \phi_t \Delta M_t, \quad \textit{for } t \in \mathbb{R},\ P\textit{-a.s.}$$
$$[\phi \bullet M]_t = \int_{(-\infty,t]} \phi_s^2 \, \mathrm{d}[M]_s^{\mathrm{g}} \quad \textit{for } t \in \mathbb{R},\ P\textit{-a.s.}$$

(4) For σ a stopping time, $s \in \mathbb{R}$ and $\phi \in \mathcal{L}L^1(M)$ we have

$$(\phi \bullet M)^\sigma \overset{P}{=} (\phi 1_{(-\infty,\sigma]}) \bullet M \overset{P}{=} \phi \bullet (M^\sigma)$$

and ${}^s(\phi \bullet M) \overset{P}{=} \phi \bullet ({}^sM)$.

Example 4.4. Let $X \in \mathcal{ILM}(\mathcal{F}_\cdot)$ be continuous and assume there is a positive continuous predictable process $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ such that for all $s < t$, ${}^s[X]_t^{\mathrm{g}} = \int_s^t \sigma_u^2 \, \mathrm{d}u$. Set $B = \sigma^{-1} \overset{\text{in}}{\bullet} X$ and note that by Lévy's theorem B is a standard Brownian motion indexed by $\mathbb{R}$, and X is given by $X \overset{\text{in}}{=} \sigma \overset{\text{in}}{\bullet} B$.

Example 4.5. As a last example assume $B = (B_t)_{t\in\mathbb{R}}$ is a Brownian motion indexed by $\mathbb{R}$ and consider the filtration $\mathcal{F}_\cdot^{\mathcal{I}B}$ generated by the increments of B cf. Example 3.6. In this case a predictable ϕ is in $\mathcal{L}L^1(B)$ resp. $\mathcal{ILL}^1(B)$ if and only if $\int_{-\infty}^t \phi_u^2 \, \mathrm{d}u < \infty$ for all t P-a.s. resp. $\int_s^t \phi_u^2 \, \mathrm{d}u < \infty$ for all $s < t$ P-a.s. Moreover, if $M \in \mathcal{ILM}(\mathcal{F}_\cdot^{\mathcal{I}B})$ then there is a $\phi \in \mathcal{ILL}^1(B)$ such that

$$M \overset{\text{in}}{=} \phi \overset{\text{in}}{\bullet} B \tag{4.8}$$

and if $M \in \mathcal{LM}(\mathcal{F}_\cdot^{\mathcal{I}B})$ then there is a $\phi \in \mathcal{L}L^1(B)$ such that

$$M \overset{P}{=} M_{-\infty} + \phi \bullet B. \tag{4.9}$$

That is, we have a *martingale representation result* in the filtration $\mathcal{F}_\cdot^{\mathcal{I}B}$. To see that this is the case, it suffices to prove (4.8). Let $s \in \mathbb{R}$ and set $\mathcal{H} = \mathcal{F}_s^{\mathcal{I}B}$. Since $\mathcal{F}_t^{\mathcal{I}B} = \mathcal{H} \vee \sigma(B_u - B_s : s \le u \le t)$ for $t \ge s$ it follows from Jacod and Shiryaev (2003), Theorem III.4.34, that there is a ϕ^s in $\mathcal{L}L^1({}^sB)$ such that ${}^sM \overset{P}{=} \phi^s \bullet ({}^sB)$. If $u < s$ then by (2.3) and (4.2) we have ${}^sM = \phi^u \bullet ({}^sB)$; thus, there is a ϕ in $\mathcal{ILL}^1(B)$ such that ${}^sM \overset{P}{=} \phi \bullet ({}^sB)$ for all s and hence $M \overset{\text{in}}{=} \phi \overset{\text{in}}{\bullet} B$ by definition of the increment integral.

The above generalises in an obvious way to the case where instead of a Brownian motion B we have, say, a Lévy process X with integrable centred increments. In this case, we have to add an integral with respect to $\mu^X - \nu^X$ on the right-hand sides of (4.8) and (4.9).

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