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Abstract

The question of existence and properties of stationary solutions to Langevin equations driven by noise processes with stationary increments is discussed, with particular focus on noise processes of pseudo moving average type. On account of the Wold-Karhunen decomposition theorem such solutions are in principle representable as a moving average (plus a drift like term) but the kernel in the moving average is generally not available in explicit form. A class of cases is determined where an explicit expression of the kernel can be given, and this is used to obtain information on the asymptotic behavior of the associated autocorrelation functions, both for small and large lags. Applications to Gaussian and Lévy driven fractional Ornstein-Uhlenbeck processes are presented. As an element in the derivations a Fubini theorem for Lévy bases is established.

Keywords: fractional Ornstein-Uhlenbeck processes; Fubini theorem for Lévy bases; Langevin equations; stationary processes

AMS Subject Classification (2010): 60G22; 60G10; 60G15; 60G52; 60G57

1 Introduction

This paper studies existence and properties of stationary solutions to Langevin equations driven by a noise process with, in general, stationary dependent increments. We shall refer to such solutions as quasi Ornstein-Uhlenbeck (QOU) processes. Of particular interest are the cases where the noise process is of the pseudo moving average (PMA) type. In wide generality the stationary solutions can, in principle, be written in the form of a Wold-Karhunen type representation, but it is relatively rare that an explicit expression for the kernel of such a representation can be given. When this is possible it often provides a more direct and simpler access to the character and properties of the process, for instance concerning the autocovariance function.

The structure of the paper is as follows. Section 2 defines the concept of quasi Ornstein-Uhlenbeck processes and provides conditions for existence and uniqueness of stationary solutions to the Langevin equation. The form of the autocovariance function of the solutions is given and its asymptotic behavior for $t \to \infty$ is discussed. As a next, intermediate, step a Fubini theorem for Lévy bases is established in Section 3. In Section 4 explicit forms of Wold-Karhunen representations are derived and used to analyze the asymptotics, under more specialized assumptions, of the autocovariance functions, both for $t \to \infty$ and for $t \to 0$. The results are applied in particular to the case of Gaussian and Lévy driven fractional Ornstein-Uhlenbeck processes. Section 5 concludes.

2 Langevin equations and QOU processes

Let $N = (N_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and let $\lambda > 0$ be a positive number. By a quasi Ornstein-Uhlenbeck (QOU) process X driven by N and with parameter λ, we mean a stationary solution to the Langevin equation $\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t$, that is, $X = (X_t)_{t\in\mathbb{R}}$ is a stationary process which satisfies

$$X_t = X_0 - \lambda \int_0^t X_s \,\mathrm{d}s + N_t, \qquad t \in \mathbb{R}, \tag{2.1}$$

where the integral is a pathwise Lebesgue integral. For all $a < b$ we use the notation $\int_b^a := -\int_a^b$. Recall that a process $Z = (Z_t)_{t\in\mathbb{R}}$ is measurable if $(t,\omega) \mapsto Z_t(\omega)$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, and that Z has stationary increments if for all $s \in \mathbb{R}$, $(Z_t - Z_0)_{t\in\mathbb{R}}$ has the same finite distributions as $(Z_{t+s} - Z_s)_{t\in\mathbb{R}}$. For $p \geq 0$ we will say that a process Z has finite p-moments if $\mathrm{E}[|Z_t|^p] < \infty$ for all $t \in \mathbb{R}$. Moreover for $t \to 0$ or ∞, we will write $f(t) \sim g(t)$, $f(t) = o(g(t))$ or $f(t) = O(g(t))$ provided that $f(t)/g(t) \to 1$, $f(t)/g(t) \to 0$ or $\limsup_t |f(t)/g(t)| < \infty$, respectively. For each process Z with finite second-moments, let $\mathrm{V}_Z(t) = \mathrm{Var}(Z_t)$ denote its variance function. When Z, in addition, is stationary, let $\mathrm{R}_Z(t) = \mathrm{Cov}(Z_t, Z_0)$ denote its autocovariance function, and $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$ its complementary autocovariance function.

Before discussing the general setting further we recall some well known cases. The stationary solution X to (2.1) where $N_t = \mu t + \sigma B_t$, and B is a Brownian motion is of particular interest in finance; here X is the Gaussian Ornstein-Uhlenbeck process, μ/λ is the mean level, λ is the speed of reversion and σ is the volatility. When N is a Lévy process the corresponding QOU process, X, exists if and only if $\mathrm{E}[\log^+|N_1|] < \infty$ or, equivalently, if $\int_{\{|x|>1\}} \log|x| \,\nu(dx) < \infty$ where ν is the Lévy measure of N; see Rocha-Arteaga and Sato (2003). In this case X is called an Ornstein-Uhlenbeck type process; for applications of such processes in financial economics see Barndorff-Nielsen and Shephard (2001, 2010).

2.1 Auxiliary continuity result

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space, and $\phi\colon \mathbb{R} \to \mathbb{R}_+$ an even and continuous function which is non-decreasing on $\mathbb{R}_+$, with $\phi(0) = 0$. Assume there exists a constant $C > 0$ such that $\phi(2x) \leq C\phi(x)$ for all $x \in \mathbb{R}$ (that is, ϕ satisfies the Δ_2-condition). Let $L^0 = L^0(E, \mathcal{E}, \mu)$ denote the space of all measurable functions from E into $\mathbb{R}$, and let Φ denote the modular on L^0 given by

$$\Phi(g) = \int_E \phi(g)\,\mathrm{d}\mu, \qquad g \in L^0,$$

and $L^\phi = \{g \in L^0 : \Phi(g) < \infty\}$ the corresponding modular space. Furthermore, for $g \in L^0$ define

$$\rho(g) = \inf\{c > 0 : \Phi(g/c) \le c\}, \qquad \text{and} \qquad \|g\|_\phi = \inf\{c > 0 : \Phi(g/c) \le 1\}.$$

Then ρ is an F-norm on L^ϕ, and when ϕ is convex, the *Luxemburg norm* $\|\cdot\|_\phi$ is a norm on L^ϕ; see e.g. Khamsi (1996). If not explicitly said otherwise, L^ϕ will be equipped with the metric $d_\phi(f,g) = \rho(f-g)$.

Theorem 2.1. *Let $f\colon \mathbb{R}\times E \to \mathbb{R}$ denote a measurable function satisfying that $f_t = f(t,\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and*

$$d_\phi(f_{t+u}, f_{v+u}) = d_\phi(f_t, f_v), \qquad \text{for all } t,u,v \in \mathbb{R}. \tag{2.2}$$

Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is continuous. Moreover, if ϕ is convex, then there exist $\alpha, \beta > 0$ such that $\|f_t\|_\phi \le \alpha + \beta|t|$ for all $t \in \mathbb{R}$.

To prove Theorem 2.1 we shall need the following lemma.

Lemma 2.2. *Let $f\colon \mathbb{R}\times E \to \mathbb{R}$ denote a measurable function, such that $f_t \in L^\phi$ for all $t \in \mathbb{R}$. Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is Borel measurable and has a separable range.*

Recall that $f\colon E \to F$ has a separable range, if $f(E)$ is a separable subset of F.

Proof. We will use a Monotone Class Lemma argument to prove this result, so let $\mathcal{M}_2$ be the set of all functions f for which Lemma 2.2 holds, and $\mathcal{M}_1$ the set of all functions f of the form

$$f_t(s) = \sum_{i=1}^n \alpha_i 1_{A_i}(t) 1_{B_i}(s), \qquad t \in \mathbb{R},\ s \in E,$$

where for $n \ge 1$, $A_1, \dots, A_n$ are measurable subsets of $\mathbb{R}$, $B_1, \dots, B_n$ are measurable subsets of E of finite μ-measure, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then, $\Psi_f : (t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ has separable range, and since $t \mapsto d_\phi(f_t, g)$ is measurable for all $g \in L^\phi$, Ψ_f is measurable. This shows that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Note that the set $\mathrm{b}\mathcal{M}_2$ of bounded elements from $\mathcal{M}_2$ is a vector space with $1 \in \mathrm{b}\mathcal{M}_2$, and that $(f_n)_{n\ge1} \subseteq \mathrm{b}\mathcal{M}_2$ with $0 \le f_n \uparrow f \le K$ implies that $f \in \mathrm{b}\mathcal{M}_2$. Moreover, since $\mathcal{M}_1$ is stable under pointwise multiplication the Monotone Class Lemma, see e.g. Chapter II, Theorem 3.2 in Rogers and Williams (2000), shows that

$$\mathrm{bM}(\mathcal{B}(\mathbb{R}) \times \mathcal{F}) = \mathrm{bM}(\sigma(\mathcal{M}_1)) \subseteq \mathrm{b}\mathcal{M}_2.$$

(For a family of functions $\mathcal{M}$, $\sigma(\mathcal{M})$ denotes the least σ-algebra for which all the functions are measurable, and for each σ-algebra $\mathcal{E}$, $\mathrm{bM}(\mathcal{E})$ denotes the space of all bounded $\mathcal{E}$-measurable functions). For a general function f define $f^{(n)}$ by $f^{(n)}_t = f_t 1_{\{|f_t| \le n\}}$. For all $n \ge 1$, $f^{(n)}$ is a bounded measurable function and hence $\Psi_{f^{(n)}}$ is a measurable map with a separable range. Moreover, $\lim_n \Psi_{f^{(n)}} = \Psi_f$ pointwise in L^ϕ, showing that Ψ_f is measurable and has a separable range. $\square$

Proof of Theorem 2.1. Let Ψ_f denote the map $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$, and for fixed $\epsilon > 0$ and arbitrary $t \in \mathbb{R}$, consider the ball $B_t = \{s \in \mathbb{R} : d_\phi(f_t, f_s) < \epsilon\}$. By Lemma 2.2, Ψ_f is measurable, and hence B_t is a measurable subset of $\mathbb{R}$ for all $t \in \mathbb{R}$. According to Lemma 2.2 Ψ_f has a separable range, and therefore there exists a countable set $(t_n)_{n\geq 1} \subseteq \mathbb{R}$ such that the range of Ψ_f is included in $\cup_{n\geq 1} B(f_{t_n}, \epsilon)$, implying that $\mathbb{R} = \cup_{n\geq 1} B_{t_n}$. (Here, $B(g, r) = \{h \in L^\phi : d_\phi(g, h) < r\}$). In particular, there exists an $n \geq 1$ such that B_{t_n} has strictly positive Lebesgue measure. By the Steinhaus Lemma, see Theorem 1.1.1 in Bingham et al. (1989), there exists a $\delta > 0$ such that $(-\delta, \delta) \subseteq B_{t_n} - B_{t_n}$. Note that by (2.2) it is enough to show continuity of Ψ_f at $t = 0$. For $|t| < \delta$ there exists, by definition, $s_1, s_2 \in \mathbb{R}$ such that $d_\phi(f_{t_n}, f_{s_i}) < \epsilon$ for $i = 1, 2$, showing that

$$d_\phi(f_t, f_0) \leq d_\phi(f_t, f_{s_1}) + d_\phi(f_t, f_{s_2}) < 2\epsilon,$$

which completes the proof of the continuity part.

To show the last part of the theorem assume that ϕ is convex. For each $t > 0$ choose $n = 0, 1, 2, \ldots$ such that $n \leq t < n + 1$. Then,

$$\begin{aligned} \|f_t - f_0\|_\phi &\leq \sum_{i=1}^{n} \|f_i - f_{i-1}\|_\phi + \|f_t - f_n\|_\phi \\ &\leq n\|f_1 - f_0\|_\phi + \|f_{t-n} - f_0\|_\phi \leq t\beta + a, \end{aligned} \tag{2.3}$$

where $\beta = \|f_1 - f_0\|_\phi$ and $a = \sup_{s\in[0,1]} \|f_s - f_0\|_\phi$. We have already shown that $t \mapsto f_t$ is continuous, and hence $a < \infty$. Since $\|f_{-t} - f_0\|_\phi = \|f_t - f_0\|_\phi$ for all $t \in \mathbb{R}$, (2.3) shows that $\|f_t - f_0\|_\phi \leq a + \beta|t|$ for all $t \in \mathbb{R}$, implying that $\|f_t\|_\phi \leq \alpha + \beta|t|$ where $\alpha = a + \|f_0\|_\phi$. □

For $(E, \mathcal{E}, \mu) = (\Omega, \mathcal{F}, P)$ and $\phi(t) = |t|^p$ for $p > 0$ or $\phi(t) = |t| \wedge 1$ for $p = 0$, we have the following corollary to Theorem 2.1.

Corollary 2.3. *Let $p \geq 0$ and $X = (X_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and finite p-moments. Then, X is continuous in L^p. Moreover if $p \geq 1$, then there exist $\alpha, \beta > 0$ such that $\|X_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.*

Note that in Corollary 2.3 the reversed implication is also true; in fact, all stochastic processes $X = (X_t)_{t\in\mathbb{R}}$ that are continuous in L^0 have a measurable modification according to Theorem 2 in Cohn (1972).

The idea by using the Steinhaus Lemma to prove Theorem 2.1 is borrowed from Surgailis et al. (1998), where Corollary 2.3 is shown for $p = 0$. Furthermore, when μ is a probability measure and $\phi(t) = |t| \wedge 1$, Lemma 2.2 is known from Cohn (1972).

2.2 Existence and uniqueness of QOU processes

The next result shows existence and uniqueness for the stationary solution X to the Langevin equation $\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t$, in the case where the the noise N is integrable. That is, we show existence and uniqueness of QOU processes X, and moreover provide an explicit form of the solution which is used to calculate the mean and variance of X.

Theorem 2.4. *Let N be a measurable process with stationary increments and finite first-moments, and let $\lambda > 0$ be a positive real number. Then, $X = (X_t)_{t\in\mathbb{R}}$ given by*

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{2.4}$$

is a QOU process driven by N with parameter λ (the integral is a pathwise Lebesgue integral). Furthermore, any other QOU process driven by N and with parameter λ equals X in law. Finally, if N has finite p-moments, $p \geq 1$, then X has also finite p-moments and is continuous in L^p.

Remark 2.5. It is an open problem to relax the integrability of N in Theorem 2.4, e.g. is it enough that N has finite log-moments? Recall that when N is a Lévy process, finite log-moments is a necessary and sufficient condition for the existence of the corresponding QOU process.

Proof. Existence: Let $p \geq 1$ and assume that N has finite p-moments. Choose $\alpha, \beta > 0$, according to Corollary 2.3, such that $\|N_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$. By Jensen's inequality,

$$\begin{aligned} E\left[\left(\int_{-\infty}^{t} e^{\lambda s}|N_s| \,\mathrm{d}s\right)^p\right] &\leq (e^{\lambda t}/\lambda)^{p-1} \int_{-\infty}^{t} e^{\lambda s} \mathrm{E}[|N_s|^p] \,\mathrm{d}s \\ &\leq (e^{\lambda t}/\lambda)^{p-1} \int_{-\infty}^{t} e^{\lambda s} (\alpha + \beta|s|)^p \,\mathrm{d}s < \infty, \end{aligned}$$

which shows that the integral in (2.4) exists almost surely as a Lebesgue integral and that X_t, given by (2.4), is p-integrable. Using substitution we obtain from (2.4),

$$X_t = \lambda \int_{-\infty}^{0} e^{\lambda u}(N_t - N_{t+u}) \,\mathrm{d}u, \qquad t \in \mathbb{R}. \tag{2.5}$$

By Corollary 2.3 N is L^p-continuous and therefore it follows that the right-hand side of (2.5) exists as a limit of Riemann sums in L^p. Hence the stationarity of the increments of N implies that X is stationary. Moreover, using integration by parts on $t \mapsto \int_{-\infty}^{t} e^{\lambda s} N_s(\omega) \,\mathrm{d}s$, we get

$$\int_0^t X_s \,\mathrm{d}s = e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \,\mathrm{d}s - \int_{-\infty}^{0} e^{\lambda s} N_s \,\mathrm{d}s,$$

which shows that X satisfies (2.1), and hence X is a QOU process driven by N with parameter λ.

Since X is a measurable process with stationary increments and finite p-moments, Proposition 2.3 shows that it is continuous in L^p.

To show uniqueness in law, let $\mathcal{L}(V)$ denote law of a random vector V, and by $\lim_k \mathcal{L}(V_k) = \mathcal{L}(V)$ we mean that, $(V_k)_{k\geq 1}$ are random vectors converging in law to V. Let Y be a QOU process driven by N with parameter $\lambda > 0$, that is, Y is a stationary process which satisfies (2.1). For all $t_0 \in \mathbb{R}$ we have with $Z_t = N_t - N_{t_0} + Y_{t_0}$ that

$$Y_t = Z_t - \lambda \int_{t_0}^{t} Y_s \,\mathrm{d}s, \qquad t \geq t_0. \tag{2.6}$$

Solving (2.6) pathwise, it follows that for all $t \geq t_0$,

$$
\begin{aligned}
Y_t &= Z_t - \lambda e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} Z_s \, \mathrm{d}s \\
&= N_t - \lambda e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} N_s \, \mathrm{d}s + (Y_{t_0} - N_{t_0}) e^{-\lambda(t-t_0)}.
\end{aligned}
$$

Note that $\lim_{t\to\infty}(Y_{t_0} - N_{t_0})e^{-\lambda(t-t_0)} = 0$ a.s., thus for all $n \geq 1$ and $t_0 < t_1 < \cdots < t_n$, the stationarity of Y implies that

$$
\begin{aligned}
\mathcal{L}(Y_{t_1}, \ldots, Y_{t_n}) &= \lim_{k\to\infty} \mathcal{L}(Y_{t_1+k}, \ldots, Y_{t_n+k}) \\
&= \lim_{k\to\infty} \mathcal{L}\Big(N_{t_1+k} - \lambda e^{-\lambda(t_1+k)} \int_{t_0}^{t_1+k} e^{\lambda s} N_s \, \mathrm{d}s, \\
&\qquad\qquad \ldots, N_{t_n+k} - \lambda e^{-\lambda(t_n+k)} \int_{t_0}^{t_n+k} e^{\lambda s} N_s \, \mathrm{d}s \Big).
\end{aligned}
$$

This shows that the distribution of Y only depends on N and λ, and completes the proof. □

Proposition 2.1 in Surgailis et al. (1998) and Proposition 2.1 in Maejima and Yamamoto (2003) provide also existence results for stationary solutions to Langevin equations. However, these results do not cover Theorem 2.4. The first result considers only Bochner type integrals and the second result requires, in particular, that the sample paths of N are Riemann integrable.

Let $B = (B_t)_{t\in\mathbb{R}}$ denote an $\mathcal{F}$-Brownian motion indexed by $\mathbb{R}$ and $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ be a predictable process, that is, σ is measurable with respect to

$$
\mathcal{P} = \sigma((s,t] \times A : s, t \in \mathbb{R},\ s < t,\ A \in \mathcal{F}_s).
$$

Next assume that for all $u \in \mathbb{R}$, $(\sigma_t, B_t)_{t\in\mathbb{R}}$ has the same finite distributions as $(\sigma_{t+u}, B_{t+u} - B_u)_{t\in\mathbb{R}}$ and that $\sigma_0 \in L^2$. Then N given by

$$
N_t = \int_0^t \sigma_s \, \mathrm{d}B_s, \qquad t \in \mathbb{R}, \tag{2.7}
$$

is a well-defined continuous process with stationary increments and finite second-moments. (Recall that for $t < 0$, $\int_0^t := -\int_t^0$).

Corollary 2.6. *Let N be given by* (2.7). *Then, there exists a unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is given by*

$$
X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \sigma_s \, \mathrm{d}B_s, \qquad t \in \mathbb{R}. \tag{2.8}
$$

Proof. Since N is a measurable process with finite second-moments it follows by Theorem 2.4 that there exists a unique in law QOU process X, and it is given by

$$
\begin{aligned}
X_t &= N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \, \mathrm{d}s = \lambda \int_{-\infty}^{0} e^{\lambda s} (N_t - N_{t+s}) \, \mathrm{d}s \\
&= \lambda \int_{-\infty}^{0} \left(\int_{\mathbb{R}} 1_{(t+s,t]}(u) e^{\lambda s} \sigma_u \, \mathrm{d}B_u \right) \mathrm{d}s.
\end{aligned} \tag{2.9}
$$

By a minor extension of Theorem 65, Chapter IV in Protter (2004) we may switch the order of integration in (2.9) and hence we obtain (2.8). $\square$

Let us conclude this section with formulas for the mean and variance of a QOU process X. In the rest of this section let N be a measurable process with stationary increments and finite first-moments, and let X be a QOU process driven by N with parameter $\lambda > 0$ (which exists by Theorem 2.4). Since X is unique in law it makes sense to consider the mean and variance function of X. Let us assume for simplicity that $N_0 = 0$ a.s. The following proposition gives the mean and variance of X.

Proposition 2.7. *Let N and X be given as above. Then,*

$$\mathrm{E}[X_0] = \frac{\mathrm{E}[N_1]}{\lambda}, \qquad \textit{and} \qquad \mathrm{Var}(X_0) = \frac{\lambda}{2}\int_0^\infty e^{-\lambda s}\mathrm{V}_N(s)\,\mathrm{d}s.$$

In the part concerning the variance of X_0, we assume moreover that N has finite second-moments.

Note that Proposition 2.7 shows that the variance of X_0 is $\lambda/2$ times the Laplace transform of V_N. In particular, if $N_t = \mu t + \sigma B_t^H$ where B^H is a fractional Brownian motion (fBm) of index $H \in (0,1)$, then $\mathrm{E}[N_1] = \mu$ and $\mathrm{V}_N(s) = \sigma^2|s|^{2H}$, and hence by Proposition 2.7 we have that

$$\mathrm{E}[X_0] = \frac{\mu}{\lambda}, \qquad \text{and} \qquad \mathrm{Var}(X_0) = \frac{\sigma^2\Gamma(1+2H)}{2\lambda^{2H}}. \tag{2.10}$$

For $H = 1/2$, (2.10) is well-known, and in this case $\mathrm{Var}(X_0) = \sigma^2/(2\lambda)$.

Before proving Proposition 2.7 let us note that $\mathrm{E}[N_t] = \mathrm{E}[N_1]t$ for all $t \in \mathbb{R}$. Indeed, this follows by the continuity of $t \mapsto \mathrm{E}[N_t]$ (see Corollary 2.3) and the stationarity of the increments of N.

Proof. Recall that by Corollary 2.3, we have that $\mathrm{E}[|N_t|] \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence by (2.4) and Fubini's theorem we have that

$$\begin{aligned}\mathrm{E}[X_0] &= \mathrm{E}\left[-\lambda\int_{-\infty}^0 e^{\lambda s}N_s\,\mathrm{d}s\right] = -\lambda\int_{-\infty}^0 e^{\lambda s}\mathrm{E}[N_s]\,\mathrm{d}s \\ &= -\lambda\mathrm{E}[N_1]\int_{-\infty}^0 e^{\lambda s}s\,\mathrm{d}s = \mathrm{E}[N_1]/\lambda,\end{aligned}$$

where in the third equality we have used that $\mathrm{E}[N_s] = \mathrm{E}[N_1]s$. This shows the part concerning the mean of X_0.

To show the last part assume that N has finite second-moments. By using $\mathrm{E}[X_0] = \mathrm{E}[N_1]/\lambda$, (2.4) shows that with $\tilde{N}_t := N_t - \mathrm{E}[N_1]t$, we have

$$\mathrm{Var}(X_0) = \mathrm{E}[(X_0 - \mathrm{E}[X_0])^2] = \mathrm{E}\left[\left(\lambda\int_{-\infty}^0 e^{\lambda s}\tilde{N}_s\,\mathrm{d}s\right)^2\right].$$

Since $\|\tilde{N}_t\|_2 \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$ by Corollary 2.3, Fubini's theorem shows

$$\mathrm{Var}(X_0) = \lambda^2\int_{-\infty}^0\int_{-\infty}^0 \left(e^{\lambda s}e^{\lambda u}\mathrm{E}[\tilde{N}_s\tilde{N}_u]\right)\mathrm{d}s\,\mathrm{d}u,$$

and since $\mathrm{E}[\tilde{N}_s\tilde{N}_u] = \frac{1}{2}[\mathrm{V}_N(s) + \mathrm{V}_N(u) - \mathrm{V}_N(s-u)]$ we have

$$\begin{aligned}\mathrm{Var}(X_0) &= \frac{\lambda^2}{2}\int_{-\infty}^0\int_{-\infty}^0 \left(e^{\lambda s}e^{\lambda u}(\mathrm{V}_N(s)+\mathrm{V}_N(u)-\mathrm{V}_N(s-u))\right)\mathrm{d}s\,\mathrm{d}u \\ &= \lambda\int_{-\infty}^0 e^{\lambda s}\mathrm{V}_N(s)\,\mathrm{d}s - \frac{\lambda^2}{2}\int_{-\infty}^0 e^{\lambda u}\left(\int_{-\infty}^{-u} e^{\lambda(s+u)}\mathrm{V}_N(s)\,\mathrm{d}s\right)\mathrm{d}u. \quad (2.11)\end{aligned}$$

Moreover,

$$\begin{aligned}&\frac{\lambda^2}{2}\int_{-\infty}^0 e^{\lambda u}\left(\int_{-\infty}^{-u} e^{\lambda(s+u)}\mathrm{V}_N(s)\,\mathrm{d}s\right)\mathrm{d}u \\ &= \frac{\lambda^2}{2}\int_{\mathbb{R}}\mathrm{V}_N(s)e^{\lambda s}\left(\int_{-\infty}^{(-s)\wedge 0} e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s \\ &= \frac{\lambda^2}{2}\left(\int_{-\infty}^0 \mathrm{V}_N(s)e^{\lambda s}\left(\int_{-\infty}^0 e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s + \int_0^\infty \mathrm{V}_N(s)e^{\lambda s}\left(\int_{-\infty}^{-s} e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s\right) \\ &= \frac{\lambda}{4}\left(\int_{-\infty}^0 \mathrm{V}_N(s)e^{\lambda s}\mathrm{d}s + \int_0^\infty \mathrm{V}_N(s)e^{\lambda s}\left(e^{-2\lambda s}\right)\mathrm{d}s\right) \\ &= \frac{\lambda}{2}\int_0^\infty e^{-\lambda s}\mathrm{V}_N(s)\,\mathrm{d}s,\end{aligned}$$

which by (2.11) gives the expression for the variance of X_0. □

2.3 Asymptotic behavior of the autocovariance function

The next result shows that the autocovariance function of a QOU process X driven by N with parameter λ has the same asymptotic behavior at infinity as the second derivative of the variance function of N divided by $2\lambda^2$.

Proposition 2.8. *Let N be a measurable process with stationary increments, $N_0 = 0$ a.s., and finite second-moments, and let X be a QOU process driven by N with parameter $\lambda > 0$.*

(i) Assume there exists a $\beta > 0$ such that $\mathrm{V}_N \in C^3((\beta,\infty);\mathbb{R})$, and for $t \to \infty$ we have that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$, $e^{-\lambda t} = o(\mathrm{V}_N''(t))$ and $\mathrm{V}_N'''(t) = o(\mathrm{V}_N''(t))$. Then, for $t \to \infty$, we have $\mathrm{R}_X(t) \sim (\frac{1}{2\lambda^2})\mathrm{V}_N''(t)$.

(ii) Assume for $t \to 0$ that $t^2 = o(\mathrm{V}_N(t))$, then for $t \to 0$ we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}\mathrm{V}_N(t)$. More generally, let $p \geq 1$ and assume that N has finite p-moments and $t = o(\|N_t\|_p)$ as $t \to 0$. Then, for $t \to 0$, we have $\|X_t - X_0\|_p \sim \|N_t\|_p$.

Note that by Proposition 2.8(ii) the short term asymptotic behavior of $\bar{\mathrm{R}}_X$ is not influenced by λ.

Proof. (i): Let $t_0 = \beta + 1$, and let us show that $t \geq t_0$ and for $t \to \infty$,

$$\mathrm{R}_X(t) = \frac{e^{-\lambda t}}{4\lambda}\int_{t_0}^t e^{\lambda u}\mathrm{V}_N''(u)\,\mathrm{d}u + \frac{e^{\lambda t}}{4\lambda}\int_t^\infty e^{-\lambda u}\mathrm{V}_N''(u)\,\mathrm{d}u + O(e^{-\lambda t}). \quad (2.12)$$

If we have shown (2.12), then by using that $e^{-\lambda t} = o(\mathrm{V}_N''(t))$, $\mathrm{V}_N'''(t) = o(\mathrm{V}_N''(t))$ and l'Hôpital's rule, (i) follows.

Similar to the proof of Proposition 2.7 let $\tilde{N}_t = N_t - E[N_1]t$. To show (2.12), recall that by Corollary 2.3 we have $\|\tilde{N}_t\|_2 \leq \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence by (2.4) and Fubini's theorem, we find that

$$\mathrm{R}_X(t) = \mathrm{E}[(X_t - \mathrm{E}[X_t])(X_0 - \mathrm{E}[X_0])] = g(t) - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} g(s)\,\mathrm{d}s, \quad (2.13)$$

where

$$g(t) = -\lambda \int_{-\infty}^{0} e^{\lambda s} \mathrm{E}[\tilde{N}_s \tilde{N}_t]\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

Since $\mathrm{E}[\tilde{N}_s \tilde{N}_t] = \frac{1}{2}[\mathrm{V}_N(t) + \mathrm{V}_N(s) - \mathrm{V}_N(s-t)]$ we have that

$$\begin{aligned} g(t) &= -\frac{\lambda}{2} \int_{-\infty}^{0} e^{\lambda s} [\mathrm{V}_N(t) + \mathrm{V}_N(s) - \mathrm{V}_N(t-s)]\,\mathrm{d}s \\ &= -\frac{1}{2}\left(\mathrm{V}_N(t) - \lambda e^{\lambda t} \int_t^\infty e^{-\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s\right) - \frac{\lambda}{2} \int_{-\infty}^{0} e^{\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s. \end{aligned} \quad (2.14)$$

From (2.14) it follows that $g \in C^1((\beta, \infty); \mathbb{R})$ and hence, using partial integration on (2.13), we have for $t \geq t_0$,

$$\mathrm{R}_X(t) = e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} g'(s)\,\mathrm{d}s + e^{-\lambda t}\left(e^{\lambda t_0} g(t_0) - \lambda \int_{-\infty}^{t_0} e^{\lambda s} g(s)\,\mathrm{d}s\right). \quad (2.15)$$

Moreover, from (2.14) and for $t \geq t_0$ we find

$$g'(t) = -\frac{1}{2}\left(\mathrm{V}_N'(t) - \lambda^2 e^{\lambda t} \int_t^\infty e^{-\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s + \lambda \mathrm{V}_N(t)\right). \quad (2.16)$$

For $t \to \infty$ we have, by assumption, that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$, and hence also $\mathrm{V}_N'(t) = O(e^{(\lambda/2)t})$. Thus, from (2.16) and a double use of partial integration we obtain that

$$g'(t) = \frac{e^{\lambda t}}{2} \int_t^\infty e^{-\lambda s} \mathrm{V}_N''(s)\,\mathrm{d}s, \qquad t \geq t_0. \quad (2.17)$$

Using (2.17), Fubini's theorem and that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$ we have for $t \geq t_0$,

$$\begin{aligned} e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} g'(s)\,\mathrm{d}s &= e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \left(\frac{e^{\lambda s}}{2} \int_s^\infty e^{-\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u\right)\mathrm{d}s \\ &= e^{-\lambda t} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \left(\int_{t_0}^{t \wedge u} \frac{1}{2} e^{2\lambda s}\,\mathrm{d}s\right)\mathrm{d}u \\ &= e^{-\lambda t} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \left(\frac{1}{4\lambda}(e^{2\lambda(t \wedge u)} - e^{2\lambda t_0})\right)\mathrm{d}u \\ &= \frac{e^{-\lambda t}}{4\lambda} \int_{t_0}^{t} e^{\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u + \frac{e^{\lambda t}}{4\lambda} \int_t^\infty e^{-\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u \\ &\quad - e^{-\lambda t}\left(\frac{e^{2\lambda t_0}}{4\lambda} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u\right). \end{aligned}$$

Combining this with (2.15) we obtain (2.12), and the proof of (i) is complete.

(ii): Using (2.1) we have for all for $t>0$ that

$$\|X_t - X_0\|_p \leq \|N_t\|_p + \lambda \int_0^t \|X_s\|_p \,\mathrm{d}s = \|N_t\|_p + \lambda t \|X_0\|_p.$$

On the other hand,

$$\|X_t - X_0\|_p \geq \|N_t\|_p - \lambda \int_0^t \|X_s\|_p \,\mathrm{d}s = \|N_t\|_p - \lambda t \|X_0\|_p,$$

which shows that

$$1 - \lambda \|X_0\|_p \frac{t}{\|N_t\|_p} \leq \frac{\|X_t - X_0\|_p}{\|N_t\|_p} \leq 1 + \lambda \|X_0\|_p \frac{t}{\|N_t\|_p}.$$

A similar inequality is available when $t<0$, and hence for $t \to 0$ we have that $\|X_t - X_0\|_p \sim \|N_t\|_p$ if $\lim_{t\to 0}(t/\|N_t\|_p) = 0$. □

When N is a fBm of index $H \in (0,1)$ then $\mathrm{V}_N(t) = |t|^{2H}$, and hence

$$\mathrm{V}_N''(t) = 2H(2H-1)t^{2H-2}, \qquad t>0.$$

The conditions in Proposition 2.8 are clearly fulfilled and thus we have the following corollary.

Corollary 2.9. *Let N be a fBm of index $H \in (0,1)$, and let X be a QOU process driven by N with parameter $\lambda > 0$. For $H \in (0,1) \setminus \{\frac{1}{2}\}$ and $t \to \infty$, we have $\mathrm{R}_X(t) \sim (H(2H-1)/\lambda^2)t^{2H-2}$. For $H \in (0,1)$ and $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}|t|^{2H}$.*

The above result concerning the behavior of R_X for $t \to \infty$ when N is a fBm has been obtained previously, via a different approach, by Cheridito et al. (2003), see their Theorem 2.3.

A square-integrable stationary process $Y = (Y_t)_{t\in\mathbb{R}}$ is said to have *long-range dependence* of order $\alpha \in (0,1)$ if R_Y is regulary varying at ∞ of index $-\alpha$. Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is regulary varying at ∞ of index $\beta \in \mathbb{R}$, if for $t \to \infty$, $f(t) \sim t^\beta l(t)$ where l is slowly varying, which means that for all $a>0$, $\lim_{t\to\infty} l(at)/l(t) = 1$. Many empirical observations have shown evidence for long-range dependence in various fields, such as finance, telecommunication and hydrology; see e.g. Doukhan et al. (2003). Let X be a QOU process driven by N, then Proposition 2.8(i) shows that X has long-range dependence of order $\alpha \in (0,1)$ if and only if V_N'' is regulary varying at ∞ of order $-\alpha$.

3 A Fubini theorem for Lévy bases

Let $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ denote a centered Lévy basis on a non-empty space S equipped with a δ-ring $\mathcal{S}$. (A Lévy basis is an infinitely divisible independently scattered random measure. Recall also that a δ-ring on S is a family of subsets of S which is closed under union, countable intersection and set difference). As usual we assume that $\mathcal{S}$ is σ-finite, meaning that there exists $(S_n)_{n\geq 1} \subseteq \mathcal{S}$ such that $\cup_{n\geq 1} S_n = S$. All integrals $\int_S f(s)\,\Lambda(\mathrm{d}s)$ will be defined in the sense of Rajput and Rosiński (1989). We can now find a measurable parameterization of Lévy measures $\nu(du,s)$ on $\mathbb{R}$, a σ-finite measure m on S, and a positive measurable function $\sigma^2 : S \to \mathbb{R}_+$, such that for all $A \in \mathcal{S}$,

$$\mathrm{E}[e^{iy\Lambda(A)}] = \exp\Big(\int_A \Big[-\sigma^2(s)y^2/2 + \int_{\mathbb{R}} (e^{iyu} - 1 - iyu)\,\nu(\mathrm{d}u,s)\Big] m(\mathrm{d}s)\Big), \quad y \in \mathbb{R}, \tag{3.1}$$

see Rajput and Rosiński (1989). Let $\phi : \mathbb{R} \times S \mapsto \mathbb{R}$ be given by

$$\phi(y,s) = y^2\sigma^2(s) + \int_{\mathbb{R}} [(uy)^2 1_{\{|uy|\leq 1\}} + (2|uy|-1)1_{\{|uy|>1\}}]\,\nu(\mathrm{d}u,s),$$

and for all measurable functions $g\colon S \to \mathbb{R}$ define

$$\|g\|_\phi = \inf\Big\{c > 0 : \int_S \phi(c^{-1}g(s),s)\,m(\mathrm{d}s) \leq 1\Big\} \in [0,\infty].$$

Moreover, let $L^\phi = L^\phi(S,\sigma(\mathcal{S}),m)$ denote the *Musielak-Orlicz space* of measurable functions g with

$$\int_S \Big[g(s)^2\sigma^2(s) + \int_{\mathbb{R}} \Big(|ug(s)|^2 \wedge |ug(s)|\Big)\,\nu(\mathrm{d}u,s)\Big] m(\mathrm{d}s) < \infty,$$

equipped with the Luxemburg norm $\|g\|_\phi$. Note that $g \in L^\phi$ if and only if $\|g\|_\phi < \infty$, since $\phi(2x,s) \leq C\phi(x,s)$ for some $C > 0$ and all $s \in S,\ x \in \mathbb{R}$. We refer to Musielak (1983) for the basic properties of Musielak-Orlicz spaces. When $\sigma^2 \equiv 0$ and $g \in L^\phi$, Theorem 2.1 in Marcus and Rosiński (2001) shows that $\int_S g(s)\,\Lambda(\mathrm{d}s)$ is well-defined, integrable and centered and

$$c_1\|g\|_\phi \leq E\Big[\Big|\int_S g(s)\,\Lambda(\mathrm{d}s)\Big|\Big] \leq c_2\|g\|_\phi,$$

and we may choose $c_1 = 1/8$ and $c_2 = 17/8$. Hence for general σ^2 it is easily seen that for all $g \in L^\phi$, $\int_S g(s)\,\Lambda(\mathrm{d}s)$ is well-defined, integrable and centered and

$$E\Big[\Big|\int_S g(s)\,\Lambda(\mathrm{d}s)\Big|\Big] \leq 2c_2\|g\|_\phi. \tag{3.2}$$

Let T denote a complete separable metric space, and $Y = (Y_t)_{t\in T}$ be given by

$$Y_t = \int_S f(t,s)\,\Lambda(\mathrm{d}s), \qquad t \in T,$$

for some measurable function $f(\cdot,\cdot)$ for which the integrals are well-defined. Then we can and will choose a measurable modification of Y. Indeed, the existence of a measurable modification of Y is equivalent to measurability of $(t \in T) \mapsto (Y_t \in L^0)$ according to Theorem 3 and the Remark in Cohn (1972). Hence, since f is measurable, the maps $(t \in T) \mapsto (\|f(t,\cdot) - g(\cdot)\|_\phi \in \mathbb{R})$ for all $g \in L^\phi$, are measurable. This shows that $(t \in \mathbb{R}) \mapsto (f(t,\cdot) \in L^\phi)$ is measurable since L^ϕ is a separable Banach space. Hence by continuity of $(f(t,\cdot) \in L^\phi) \mapsto (Y_t \in L^0)$, see Rajput and Rosiński (1989), it follows that $(t \in T) \mapsto (Y_t \in L^0)$ is measurable.

Assume that μ is a σ-finite measure on a complete and separable metric space T, then we arrive at the following stochastic Fubini result extending Rosiński (1986, Lemma 7.1), Pérez-Abreu and Rocha-Arteaga (1997, Lemma 5) and Basse and Pedersen (2009, Lemma 4.9). Stochastic Fubini type results for semimartingales can be founded in Protter (2004) and Ikeda and Watanabe (1981), however the assumptions in these results are too strong for our purpose.

Theorem 3.1 (Fubini). *Let $f : T \times S \mapsto \mathbb{R}$ be an $\mathcal{B}(T) \otimes \sigma(\mathcal{S})$-measurable function such that*

$$f_x = f(x,\cdot) \in L^\phi, \text{ for } x \in T, \qquad \text{and} \qquad \int_E \|f_x\|_\phi \, \mu(\mathrm{d}x) < \infty. \tag{3.3}$$

Then $f(\cdot, s) \in L^1(\mu)$ for m-a.a. $s \in S$ and $s \mapsto \int_T f(x,s)\,\mu(\mathrm{d}x)$ belongs to L^ϕ, all of the below integrals exist and

$$\int_T \Big(\int_S f(x,s)\,\Lambda(\mathrm{d}s) \Big)\,\mu(\mathrm{d}x) = \int_S \Big(\int_T f(x,s)\,\mu(\mathrm{d}x) \Big)\,\Lambda(\mathrm{d}s) \qquad a.s. \tag{3.4}$$

Remark 3.2. If μ is a finite measure then the last condition in (3.3) is equivalent to

$$\int_T \Big[\int_S f(x,s)^2 \sigma^2(s) + \int_{\mathbb{R}} \Big(|uf(x,s)|^2 \wedge |uf(x,s)| \Big)\,\nu(\mathrm{d}u, s) \Big] m(\mathrm{d}s)\,\mu(\mathrm{d}x) < \infty.$$

Before proving Theorem 3.1 we will need the following observation:

Lemma 3.3. *For all measurable functions $f \colon T \times S \to \mathbb{R}$ we have*

$$\Big\| \int_T |f(x,\cdot)|\,\mu(\mathrm{d}x) \Big\|_\phi \le \int_T \|f(x,\cdot)\|_\phi \,\mu(\mathrm{d}x). \tag{3.5}$$

Moreover, if $f \colon T \times S \to \mathbb{R}$ is a measurable function such that $\int_T \|f(x,\cdot)\|_\phi \,\mu(\mathrm{d}x) < \infty$, then for m-a.a. $s \in S$, $f(\cdot,s) \in L^1(\mu)$ and $s \mapsto \int_T f(x,s)\,\mu(\mathrm{d}x)$ is a well-defined function which belongs to L^ϕ.

Proof. Let us sketch the proof of (3.5). For f of the form

$$f(x,s) = \sum_{i=1}^k g_i(s) 1_{A_i}(x),$$

where $k \ge 1$, $g_1, \dots, g_k \in L^\phi$ and $A_1, \dots, A_k$ are disjoint measurable subsets of T of finite μ-measure, (3.5) easily follows. Hence by a Monotone Class Lemma argument it is possible to show (3.5) for all measurable f. The second statement is a consequence of (3.5). $\square$

Recall that if $(F, \|\cdot\|)$ is a separable Banach space, μ is a measure on T, and $f \colon T \to F$ is a measurable map such that $\int_T \|f(x)\|\,\mu(\mathrm{d}x) < \infty$, then the Bochner integral $\mathrm{B}\!\int_T f(x)\,\mu(\mathrm{d}x)$ exists in F and $\|\mathrm{B}\!\int_T f(x)\,\mu(\mathrm{d}x)\| \le \int_T \|f(x)\|\,\mu(\mathrm{d}x)$. Even though $(L^\phi, \|\cdot\|_\phi)$ is a Banach space, this result does not cover Lemma 3.3.

Proof of Theorem 3.1. For f of the form

$$f(x,s)=\sum_{i=1}^{n}\alpha_i 1_{A_i}(x)1_{B_i}(s),\qquad x\in T,\ s\in S, \tag{3.6}$$

where $n\geq 1$, $A_1,\dots,A_n$ are measurable subsets of T of finite μ-measure, $B_1,\dots,B_n\in\mathcal{S}$, and $\alpha_1,\dots,\alpha_n\in\mathbb{R}$, the theorem is trivially true. Thus for a general f, as in the theorem, choose f_n for $n\geq 1$ of the form (3.6) in such a way such that $\int_T \|f_n(x,\cdot)-f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x)\to 0$. Indeed, the existence of such a sequence follows by an application of the Monotone Class Lemma. Let

$$X_n=\int_E\Big(\int_S f_n(x,s)\,\Lambda(\mathrm{d}s)\Big)\,\mu(\mathrm{d}x),\qquad X=\int_E\Big(\int_S f(x,s)\,\Lambda(\mathrm{d}s)\Big)\,\mu(\mathrm{d}x),$$

and let us show that X is well-defined and $X_n\to X$ in L^1. This follows since

$$E\left[\int_E\left|\int_S f(x,s)\,\Lambda(\mathrm{d}s)\right|\,\mu(\mathrm{d}x)\right]\leq 2c_2\int_E\|f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x)<\infty,$$

and

$$\mathrm{E}[|X_n-X|]\leq 2c_2\int_E\|f_n(x,\cdot)-f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x).$$

Similarly, let

$$Y_n=\int_S\Big(\int_E f_n(x,s)\,\mu(\mathrm{d}x)\Big)\,\Lambda(\mathrm{d}s),\qquad Y=\int_S\Big(\int_E f(x,s)\,\mu(\mathrm{d}x)\Big)\,\Lambda(\mathrm{d}s),$$

and let us show that Y is well-defined and $Y_n\to Y$ in L^1. By Remark 3.3, $s\mapsto\int_E f(x,s)\,\mu(\mathrm{d}x)$ is a well-defined function which belongs to L^ϕ, which shows that Y is well-defined. By (3.2) and (3.5) we have

$$\mathrm{E}[|Y_n-Y|]\leq 2c_2\int_E\|f_n(x,\cdot)-f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x),$$

which shows that $Y_n\to Y$ in L^1. We have therefore proved (3.4), since $Y_n=X_n$ a.s., $X_n\to X$ and $Y_n\to Y$ in L^1. $\square$

Let $Z=(Z_t)_{t\in\mathbb{R}}$ denote an integrable and centered Lévy process with Lévy measure ν and Gaussian component σ^2. Then Z induces a Lévy basis Λ on $S=\mathbb{R}$ and $\mathcal{S}=\mathcal{B}_b(\mathbb{R})$, the bounded Borel sets, which is uniquely determined by $\Lambda((a,b])=Z_b-Z_a$ for all $a,b\in\mathbb{R}$ with $a<b$. In this case m is the Lebesgue measure on $\mathbb{R}$ and

$$\phi(y,s)=\phi(y)=\sigma^2+\int_{\mathbb{R}}\big(|uy|^2 1_{\{|uy|\leq 1\}}+(2|uy|-1)1_{\{|uy|>1\}}\big)\,\nu(\mathrm{d}u).$$

We will often write $\int f(s)\,\mathrm{d}Z_s$ instead of $\int f(s)\,\Lambda(\mathrm{d}s)$. Note that, $\int_{\mathbb{R}} f(s)\,\mathrm{d}Z_s$ exists and is integrable if and only if $f\in L^\phi$, i.e.,

$$\int_{\mathbb{R}}\left(f(s)^2\sigma^2+\int_{\mathbb{R}}\big(|uf(s)|^2\wedge|uf(s)|\big)\,\nu(\mathrm{d}x)\right)\mathrm{d}s<\infty. \tag{3.7}$$

Moreover, if Z is a symmetric α-stable Lévy process, $\alpha\in(0,2]$, then $L^\phi=L^\alpha(\mathbb{R},\lambda)$, where $L^\alpha(\mathbb{R},\lambda)$ is the space of α-integrable functions with respect to the Lebesgue measure λ.

4 Moving average representations

In wide generality, if X is a continuous time stationary processes then it is representable, in principle, as a moving average (MA), i.e.

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s$$

where ϕ is a deterministic function and Ξ has stationary and orthogonal increments, at least in the second order sense. (For a precise statements, see the beginning of Subsection 4.1 below). However, an explicit expression for ϕ is seldom available.

We show in Subsection 4.2 below that an expression can be found in cases where the process X is the stationary solution to a Langevin equation for which the driving noise process N is a pseudo moving average (PMA), i.e.

$$N_t = \int_{\mathbb{R}} (f(t-s) - f(-s))\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.1}$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ is a suitable process specified later on and $f\colon \mathbb{R} \to \mathbb{R}$ a deterministic function for which the integrals exist.

In Subsection 4.3, continuing the discussion from Subsection 2.3, we use the MA representation to study the asymptotic behavior of the associated autocovariance functions. Subsection 4.4 comments on a notable cancellation effect. But first, in Subsection 4.1 we summarize known results concerning Wold-Karhunen type representations of stationary continuous time processes.

4.1 Wold-Karhunen type decompositions

Let $X = (X_t)_{t\in\mathbb{R}}$ be a second-order stationary process of mean zero and continuous in quadratic-mean. Let F_X denote the spectral measure of X, i.e., F_X is a finite and symmetric measure on $\mathbb{R}$ satisfying

$$\mathrm{E}[X_t X_u] = \int_{\mathbb{R}} e^{i(t-u)x}\,F_X(\mathrm{d}x), \qquad t, u \in \mathbb{R},$$

and let F'_X denote the density of the absolutely continuous part of F_X. For each $t \in \mathbb{R}$ let $\mathcal{X}_t = \overline{\mathrm{span}}\{X_s : s \le t\}$, $\mathcal{X}_{-\infty} = \cap_{t\in\mathbb{R}}\mathcal{X}_t$ and $\mathcal{X}_\infty = \overline{\mathrm{span}}\{X_s : s \in \mathbb{R}\}$ ($\overline{\mathrm{span}}$ denotes the L^2-closure of the linear span). Then X is called deterministic if $\mathcal{X}_{-\infty} = \mathcal{X}_\infty$ and purely non-deterministic if $\mathcal{X}_{-\infty} = \{0\}$. The following result, which is due to Satz 5–6 in Karhunen (1950) (cf. also Doob (1990)), provides a decomposition of stationary processes as a sum of a deterministic process and a purely non-deterministic process.

Theorem 4.1 (Karhunen). *Let X and F_X be given as above. If*

$$\int_{\mathbb{R}} \frac{|\log F'_X(x)|}{1+x^2}\,\mathrm{d}x < \infty \tag{4.2}$$

then there exists a unique decomposition of X as

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s + V_t, \qquad t \in \mathbb{R}, \tag{4.3}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a Lebesgue square-integrable deterministic function, and Ξ is a process with second-order stationary and orthogonal increments, $\mathrm{E}[|\Xi_u - \Xi_s|^2] = |u-s|$ *and for all $t \in \mathbb{R}$ $\mathcal{X}_t = \overline{\mathrm{span}}\{\Xi_s - \Xi_u : -\infty < u < s \le t\}$, and V is a deterministic second-order stationary process.*

Moreover, if F_X is absolutely continuous and (4.2) *is satisfied then $V \equiv 0$ and hence X is a backward moving average. Finally, the integral in* (4.2) *is infinite if and only if X is deterministic.*

The results in Karhunen (1950) are formulated for complex-valued processes, however if X is real-valued (as it is in our case) then one can show that all the above processes and functions are real-valued as well. Note also that if X is Gaussian then the process Ξ in (4.3) is a standard Brownian motion. If σ is a stationary process with $\mathrm{E}[\sigma_0^2] = 1$ and B is a Brownian motion, then $\mathrm{d}\Xi_s = \sigma_s \mathrm{d}B_s$ is of the above type.

A generalization of the classical Wold-Karhunen result to a broad range of non-Gaussian infinitely divisible processes was given in Rosiński (2007).

4.2 Explicit MA solutions of Langevin equations

Assume initially that Z is an integrable and centered Lévy process, and recall that L^ϕ is the space of all measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying (3.7). Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(t - \cdot) - f(-\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and let N be given by

$$N_t = \int_{\mathbb{R}} (f(t-s) - f(-s))\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}.$$

Proposition 4.2. *Let N be given as above. Then there exists an unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is a MA of the form*

$$X_t = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where $\psi_f\colon \mathbb{R} \to \mathbb{R}$ belongs to L^ϕ, and is given by

$$\psi_f(t) = \left(f(t) - \lambda e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} f(s)\,\mathrm{d}s \right), \qquad t \in \mathbb{R}. \tag{4.4}$$

Proof. Since $(t,s) \mapsto f(t-s) - f(-s)$ is measurable we may choose a measurable modification of N, see Section 3, and hence, by Theorem 2.4, there exists a unique in law QOU process X driven by N with parameter λ. For fixed $t \in \mathbb{R}$, we have by (2.4) and with $h_u(s) = f(t-s) - f(t+u-s)$ for all $u, s \in \mathbb{R}$ and $\mu(\mathrm{d}u) = 1_{\{u \le 0\}} e^{\lambda u}\,\mathrm{d}u$ that

$$X_t = \lambda \int_{-\infty}^0 e^{\lambda u}(N_t - N_{t+u})\,\mathrm{d}u = \int_{-\infty}^0 \left(\int_{\mathbb{R}} h_u(s)\,\mathrm{d}Z_s \right) \mu(\mathrm{d}u).$$

By Theorem 2.1 there exist $\alpha, \beta > 0$ such that $\|h_u\|_\phi \le \alpha + \beta|t|$ for all $u \in \mathbb{R}$, implying that $\int_{\mathbb{R}} \|h_u\|_\phi\,\mu(\mathrm{d}u) < \infty$. By Theorem 3.1, $(u \mapsto h_u(s)) \in L^1(\mu)$ for Lebesgue almost all $s \in \mathbb{R}$, which implies that $\int_{-\infty}^t |f(u)| e^{\lambda u}\,\mathrm{d}u < \infty$ for all $t > 0$, and hence ψ_f, defined in (4.4), is a well-defined function. Moreover by Theorem 3.1, $\psi_f \in L^\phi(\mathbb{R}, \lambda)$ and

$$X_t = \int_{\mathbb{R}} \left(\int_{-\infty}^{0} h(u,s)\, \mu(\mathrm{d}u) \right) \mathrm{d}Z_s = \int_{\mathbb{R}} \psi_f(t-s)\, \mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

which completes the proof. □

Note that for $f = 1_{\mathbb{R}_+}$, we have $N_t = Z_t$ and $\psi_f(t) = e^{-\lambda t} 1_{\mathbb{R}_+}(t)$. Thus, in this case we recover the well-known result that the QOU process X driven by Z with parameter $\lambda > 0$ is a MA of the form $X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)}\, \mathrm{d}Z_s$.

Let us use the notation $x_+ := x 1_{\{x \geq 0\}}$, and let c_H be given by

$$c_H = \frac{\sqrt{2H \sin(\pi H) \Gamma(2H)}}{\Gamma(H + 1/2)}.$$

A PMA N of the form (4.1), where Z is an α-stable Lévy process with $\alpha \in (0, 2]$ and f is given by $t \mapsto c_H t_+^{H-1/\alpha}$ is called a *linear fractional α-stable motion* of index $H \in (0, 1)$; see Samorodnitsky and Taqqu (1994). Moreover, PMAs with $f(t) = t^\alpha$ for $\alpha \in (0, \frac{1}{2})$ and where Z is a square-integrable and centered Lévy process is called *fractional Lévy processes* in Marquardt (2006).

A QOU process driven by a linear fractional α-stable motion is called a fractional Ornstein-Uhlenbeck process. For previous work on such processes see Maejima and Yamamoto (2003), where $\alpha \in (1, 2)$, and Cheridito et al. (2003), where $\alpha = 2$.

Corollary 4.3. *Let $\alpha \in (1, 2]$ and N be a linear fractional α-stable motion of index $H \in (0, 1)$. Then there exists a unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is a MA of the form*

$$X_t = \int_{-\infty}^{t} \psi_{\alpha,H}(t-s)\, \mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where $\psi_{\alpha,H} : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\psi_{\alpha,H}(t) = c_H \left(t^{H-1/\alpha} - \lambda e^{-\lambda t} \int_0^t e^{\lambda u} u^{H-1/\alpha}\, \mathrm{d}u \right), \qquad t \geq 0.$$

For $t \to \infty$, we have $\psi_{\alpha,H}(t) \sim (c_H (H - 1/\alpha)/\lambda) t^{H-1/\alpha-1}$, and $\psi_{\alpha,H}(t) \sim c_H t^{H-1/\alpha}$, for $t \to 0$.

Remark 4.4. For $H \in (0, 1/\alpha)$ the existence of the stationary solution to the Langevin equation is somewhat unexpected due to the fact that the sample paths of the linear fractional α-stable motion are unbounded on each compact interval, cf. page 4 in Maejima and Yamamoto (2003) where nonexistence is surmised.

In the next lemma we will show a special property of ψ_f, given by (4.4); namely that $\int_0^\infty \psi_f(s)\, \mathrm{d}s = 0$ whenever f tends to zero fast enough. This property has a great impact on the behavior of the autocovariance function of QOU processes. We will return to this point in Section 4.4.

Lemma 4.5. *Let $\alpha \in (-\infty, 0)$, $c \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function which is zero on $(-\infty, 0)$ and satisfies that $f(t) \sim ct^{\alpha}$ for $t \to \infty$. Then, $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$.*

Proof. For $t > 0$,

$$\int_0^t \left(\lambda e^{-\lambda s} \int_0^s e^{\lambda u} f(u)\,\mathrm{d}u\right) \mathrm{d}s$$
$$= \int_0^t \left(\int_u^t \lambda e^{-\lambda s}\,\mathrm{d}s\right) e^{\lambda u} f(u)\,\mathrm{d}u = \int_0^t f(u)\,du - e^{-\lambda t} \int_0^t e^{\lambda u} f(u)\,\mathrm{d}u,$$

and hence by l'Hôpital's rule we have that

$$\int_0^\infty \psi_f(s)\,\mathrm{d}s = \lim_{t\to\infty} \int_0^t \psi_f(s)\,\mathrm{d}s = \lim_{t\to\infty} \left(e^{-\lambda t} \int_0^t e^{\lambda u} f(u)\,\mathrm{d}u\right) = 0.$$

□

Proposition 4.2 carries over to a much more general setting. E.g. if N is of the form

$$N_t = \int_{\mathbb{R}\times V} [f(t-s,x) - f(-s,x)]\,\Lambda\,(\mathrm{d}s, \mathrm{d}x)\,, \qquad t \in \mathbb{R},$$

where Λ is a centered Lévy basis on $\mathbb{R} \times V$ (V is a non-empty space) with control measure $m(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\, n(\mathrm{d}x)$ and $a(s,x), \sigma^2(s,x)$ and $\nu(\mathrm{d}u, (s,x))$, from (3.1), do not depend on $s \in \mathbb{R}$, and $f(t - \cdot, \cdot) - f(-\cdot, \cdot) \in L^\phi$ for all $t \in \mathbb{R}$, then using Theorem 2.1, 2.4 and 3.1 the arguments from Proposition 4.2 show that there exists a unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is given by

$$X_t = \int_{\mathbb{R}\times V} \psi_f(t-s,x)\,\Lambda(\mathrm{d}s, \mathrm{d}x), \qquad t \in \mathbb{R},$$

where

$$\psi_f(s,x) = f(s,x) - \lambda e^{-\lambda s} \int_{-\infty}^s f(u,x) e^{\lambda u}\,\mathrm{d}u, \qquad s \in \mathbb{R},\ x \in V.$$

We recover Proposition 4.2 when $V = \{0\}$ and $n = \delta_0$ is the Dirac delta measure at 0.

4.3 Asymptotic behavior of the autocovariance function

The representation, from the previous section, of QOU processes as moving averages enables us to calculate the autocovariance function in case it exists. In Section 4.3.1 we calculate the autocovariance function for general MAs. By use of these results Section 4.3.2 relates the asymptotic behavior of the kernel of the noise N to the asymptotic behavior of the autocovariance function of the QOU process X driven by N.

4.3.1 Autocovariance function of general MAs

Let ψ be a Lebesgue square-integrable function and Z a centered process with stationary and orthogonal increments, and assume for simplicity that $Z_0 = 0$ a.s. and $\mathrm{V}_Z(t) = t$. Let $X = \psi * Z = (\int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}Z_s)_{t\in\mathbb{R}}$ be a backward moving average and R_X its autocovariance function, i.e.

$$\mathrm{R}_X(t) = \mathrm{E}[X_t X_0] = \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s, \qquad t\in\mathbb{R},$$

and let $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$. The behavior of R_X at 0 or ∞ corresponds in large extent to the behavior of the kernel ψ at 0 or ∞, respectively.

Indeed, we have the following result, in which k_α and j_α are constants given by

$$\begin{aligned} k_\alpha &= \Gamma(1+\alpha)\Gamma(-1-2\alpha)\Gamma(-\alpha)^{-1}, && \alpha\in(-1,-1/2),\\ j_\alpha &= (2\alpha+1)\sin(\pi(\alpha+1/2))\Gamma(2\alpha+1)\Gamma(\alpha+1)^{-2}, && \alpha\in(-1/2,1/2). \end{aligned}$$

Proposition 4.6. *Let the setting be as described above.*

(i) For $t\to\infty$ and $\alpha\in(-1,-\frac{1}{2})$, $\psi(t)\sim ct^\alpha$ implies $\mathrm{R}_X(t)\sim(c^2k_\alpha)t^{2\alpha+1}$ provided $|\psi(t)|\le c_1t^\alpha$ for all $t>0$ and some $c_1>0$.

(ii) For $t\to\infty$ and $\alpha\in(-\infty,-1)$, $\psi(t)\sim ct^\alpha$ implies $\mathrm{R}_X(t)/t^\alpha\to c\int_0^\infty\psi(s)\,\mathrm{d}s$, and hence $\mathrm{R}_X(t)\sim(c\int_0^\infty\psi(s)\,\mathrm{d}s)t^\alpha$ provided $\int_0^\infty\psi(s)\,\mathrm{d}s\neq 0$.

(iii) For $t\to 0$ and $\alpha\in(-\frac{1}{2},\frac{1}{2})$, $\psi(t)\sim ct^\alpha$ implies $\bar{\mathrm{R}}_X(t)\sim(c^2j_\alpha/2)|t|^{2\alpha+1}$ provided ψ is absolutely continuous on $(0,\infty)$ with density ψ' satisfying $|\psi'(t)|\le c_2t^{\alpha-1}$ for all $t>0$ and some $c_2>0$.

Recall that a function $f\colon\mathbb{R}\to\mathbb{R}$ is said to be absolutely continuous on $(0,\infty)$ if there exists a locally integrable function f' such that for all $0<u<t$

$$f(t)-f(u) = \int_u^t f'(s)\,\mathrm{d}s.$$

Proof. (i): Let $\alpha\in(-1,-\frac{1}{2})$ and assume that $\psi(t)\sim ct^\alpha$ as $t\to\infty$ and $|\psi(t)|\le c_1t^\alpha$ for $t>0$, then

$$\begin{aligned} \mathrm{R}_X(t) &= \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s = t\int_0^\infty \psi(t(s+1))\psi(ts)\,\mathrm{d}s\\ &= t^{2\alpha+1}\int_0^\infty \frac{\psi(t(1+s))\psi(ts)}{(t(1+s))^\alpha(ts)^\alpha}(1+s)^\alpha s^\alpha\,\mathrm{d}s\\ &\sim t^{2\alpha+1}c^2\int_0^\infty(1+s)^\alpha s^\alpha\,\mathrm{d}s \qquad \text{as } t\to\infty. \end{aligned} \tag{4.5}$$

Since

$$\int_0^\infty (1+s)^\alpha s^\alpha\,\mathrm{d}s = \frac{\Gamma(1+\alpha)\Gamma(-1-2\alpha)}{\Gamma(-\alpha)} = k_\alpha,$$

(4.5) shows that $\mathrm{R}_X(t)\sim(c^2k_\alpha)t^{2\alpha+1}$ for $t\to\infty$.

(ii): Let $\alpha \in (-\infty, -1)$ and assume that $\psi(t) \sim ct^\alpha$ for $t \to \infty$. Note that $\psi \in L^1(\mathbb{R}_+, \lambda)$ and for some $K > 0$ we have for all $t \geq K$ and $s > 0$ that $|\psi(t+s)|/t^\alpha \leq 2|c|(t+s)^\alpha/t^\alpha \leq 2|c|$. Hence by applying Lebesgue's dominated convergence theorem we obtain,

$$\mathrm{R}_X(t) = t^\alpha \int_0^\infty \left(\frac{\psi(t+s)}{t^\alpha} \psi(s) \right) \mathrm{d}s \sim t^\alpha c \int_0^\infty \psi(s) \, \mathrm{d}s \qquad \text{for } t \to \infty.$$

(iii): By letting

$$f_t(s) := \frac{\psi(t(s+1)) - \psi(ts)}{t^\alpha} \qquad t > 0, \, s \in \mathbb{R},$$

we have

$$\mathrm{E}[(X_t - X_0)^2] = t \int [(\psi(t(s+1)) - \psi(ts)]^2 \, \mathrm{d}s = t^{2\alpha+1} \int |f_t(s)|^2 \, \mathrm{d}s. \tag{4.6}$$

As $t \to 0$, we find

$$f_t(s) = \frac{\psi(t(s+1))}{(t(s+1))^\alpha}(s+1)^\alpha - \frac{\psi(ts)}{(ts)^\alpha} s^\alpha \to c((s+1)_+^\alpha - s_+^\alpha).$$

Choose $\delta > 0$ such that $|\psi(x)| \leq 2x^\alpha$ for $x \in (0, \delta)$. By our assumptions we have for all $s \geq \delta$ that

$$\begin{aligned}
|f_t(s)| &= t^{-\alpha} \left| \int_{ts}^{t(1+s)} \psi'(u) \, \mathrm{d}u \right| \leq t^{-\alpha+1} \sup_{u \in [st, t(s+1)]} |\psi'(u)| \\
&\leq c_2 t^{-\alpha+1} \sup_{u \in [st, t(s+1)]} |u|^{\alpha-1} = c_2 t^{-\alpha+1} |ts|^{\alpha-1} = c_2 s^{\alpha-1},
\end{aligned}$$

and for $s \in [-1, \delta)$, $|f_t(s)| \leq 2c[(1+s)^\alpha + s_+^\alpha]$. This shows that there exists a function $g \in L^2(\mathbb{R}_+, \lambda)$ such that $|f_t| \leq g$ for all $t > 0$, and thus, by Lebesgue's dominated converging theorem, we have

$$\int |f_t(s)|^2 \, \mathrm{d}s \xrightarrow[t \to 0]{} c^2 \int \left((s+1)_+^\alpha - s_+^\alpha \right)^2 \, \mathrm{d}s = c^2 j_\alpha. \tag{4.7}$$

Together with (4.6), (4.7) shows that $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2) t^{2\alpha+1}$ for $t \to 0$. □

Remark 4.7. It would be of interest to obtain a general result covering Proposition 4.6(ii) in the case $\int_0^\infty \psi(s) \, \mathrm{d}s = 0$. Recall that ψ_f, given by (4.4), often satisfies that $\int_0^\infty \psi_f(s) \, \mathrm{d}s = 0$, according to Lemma 4.5.

Example 4.8. Consider the case where $\psi(t) = t^\alpha e^{-\lambda t}$ for $\alpha \in (-\frac{1}{2}, \infty)$ and $\lambda > 0$. For $t \to 0$, $\psi(t) \sim t^\alpha$, and hence $\bar{\mathrm{R}}_X(t) \sim (j_\alpha/2) t^{2\alpha+1}$ for $t \to 0$ and $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, by Proposition 4.6(iii) (compare with Barndorff-Nielsen et al. (2009)).

Note that if $X = \psi * Z$ is a moving average, as above, then by Proposition 4.6(i) and for $t \to \infty$, $\mathrm{R}_X(t) \sim c_1 t^{-\alpha}$ with $\alpha \in (0,1)$, provided that $\psi(t) \sim c_2 t^{-(\alpha+1)/2}$ and $|\psi(t)| \leq c_3 t^{-(\alpha+1)/2}$. This shows that X has long-range dependence of order α.

Let us conclude this subsection with a short discussion of when a MA $X = \psi * Z$ is a semimartingale. It is often very important that the process of interest is a semimartingale, especially in finance, where the semimartingale property the asset price is equivalent to that the capital process depends continuously on the chosen strategy, see e.g. Section 8.1.1 in Cont and Tankov (2004). In the case where Z is a Brownian motion, Theorem 6.5 in Knight (1992) shows that X is an $\mathcal{F}^Z$-semimartingale if and only if ψ is absolutely continuous on $[0,\infty)$ with a square-integrable density. (Here $\mathcal{F}_t^Z := \sigma(Z_s : s \in (-\infty, t])$). For a further study to the semimartingale property of pseudo moving averages and more general processes see Basse (2008, 2009a,b) in the Gaussian case, and Basse and Pedersen (2009) for the infinitely divisible case.

4.3.2 QOU processes with PMA noise

Let us return to the case of a QOU process driven by a PMA, so let Z be a centered Lévy process, $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function which is 0 on $(-\infty, 0)$ and satisfies that $f(t-\cdot) - f(-\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and let N be given by

$$N_t = \int_{\mathbb{R}} [f(t-s) - f(-s)] \, \mathrm{d}Z_s, \qquad t \in \mathbb{R}. \tag{4.8}$$

First we will consider the relationship between the behavior of the kernel of the noise N and that of the kernel ψ_f of the corresponding moving average X.

Proposition 4.9. *Let N be given by* (4.8)*, and X be a QOU process driven by N with parameter $\lambda > 0$.*

(i) Let $\alpha \in (-1, -\frac{1}{2})$ and assume that for some $\beta \geq 0$ and $c \neq 0$, $f \in C^1((\beta,\infty);\mathbb{R})$ with $f'(t) \sim ct^\alpha$ for $t \to \infty$. Then, for $t \to \infty$, we have $\mathrm{R}_X(t) \sim (\frac{c^2 k_\alpha}{\lambda^2}) t^{2\alpha+1}$, provided $|f(t)| \leq rt^\alpha$ for all $t > 0$ and some $r > 0$.

(ii) Let $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $f(t) \sim ct^\alpha$ for $t \to 0$. Then, for $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)|t|^{2\alpha+1}$, provided there exists a $\beta \geq 0$ such that $f \in C^2((\beta,\infty);\mathbb{R})$ with $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$, and that f is absolutely continuous on $(0,\infty)$ with density f' satisfying $\sup_{t\in(0,t_o)}|f'(t)|t^{1-\alpha} < \infty$ for all $t_0 > 0$.

Proof. (i): By partial integration, we have for $t \geq \beta$,

$$\psi_f(t) = e^{-\lambda t}\left(e^{\lambda a} f(a) - \lambda \int_{-\infty}^{a} e^{\lambda s} f(s) \, \mathrm{d}s\right) + e^{-\lambda t} \int_a^t e^{\lambda s} f'(s) \, \mathrm{d}s, \tag{4.9}$$

showing that $\psi_f(t) \sim (\frac{c}{\lambda}) t^\alpha$ for $t \to \infty$. Choose $k > 0$ such that $|\psi_f(t)| \leq (2c/\lambda)t^\alpha$ for all $t \geq k$. By (4.4) we have that $\sup_{t\in[0,k]}|\psi_f(t)t^{-\alpha}| < \infty$ since $\sup_{t\in[0,k]}|f(t)t^{-\alpha}| < \infty$, and hence there exists a constant $c_1 > 0$ such that $|\psi_f(t)| \leq c_1 t^\alpha$ for all $t > 0$. Therefore, (i) follows by Proposition 4.6(i).

(ii): Since $f \in C^2((\beta,\infty);\mathbb{R})$, it follows by (4.9) and partial integration that for $t > \beta$ and $t \to \infty$,

$$\begin{aligned}\psi_f'(t) &= f'(t) - \lambda\psi_f(t) = f'(t) - \lambda e^{-\lambda t} \int_\beta^t e^{\lambda s} f'(s) \, \mathrm{d}s + O(e^{-\lambda t}) \\ &= e^{-\lambda t} \int_\beta^t e^{\lambda s} f''(s) \, \mathrm{d}s + O(e^{-\lambda t}) = O(t^{\alpha-1}),\end{aligned}$$

where we in the last equality have used that $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$. Using that $|\psi_f'(t)| \le |f'(t)| + \lambda|\psi_f(t)|$ and $\sup_{t\in(0,t_0)}|f'(t)t^{1-\alpha}| < \infty$ for all $t_0 > 0$, it follows that there exists a $c_2 > 0$ such that $|\psi_f'(t)| \le c_1 t^{\alpha-1}$ for all $t > 0$. Moreover, for $t \to 0$, we have that $\psi_f(t) \sim ct^\alpha$. Hence, (ii) follows by Proposition 4.6(iii). □

Now consider the following set-up: Let $Z = (Z_t)_{t\in\mathbb{R}}$ be a centered and square-integrable Lévy process, and for $H \in (0,1)$, $r_0 \neq 0$, $\delta \ge 0$, let

$$f(t) = r_0(\delta \vee t)^{H-1/2}, \qquad \text{and} \qquad N_t^{H,\delta} = \int_{\mathbb{R}} [f(t-s) - f(-s)] \,\mathrm{d}Z_s. \quad (4.10)$$

Note that when $\delta = 0$ and Z is a Brownian motion then $N^{H,\delta}$ is a constant times the fBm of index H, and when $\delta > 0$ then $N^{H,\delta}$ is a semimartingale. We have the following corollary to Proposition 4.9:

Corollary 4.10. *Let $N^{H,\delta}$ be given by* (4.10), *and let $X^{H,\delta}$ be a QOU process driven by $N^{H,\delta}$ with parameter $\lambda > 0$. Then, for $H \in (\frac{1}{2}, 1)$ and $t \to \infty$,*

$$\mathrm{R}_{X^{H,\delta}}(t) \sim (r_0^2 k_{H-3/2}(H-1/2)/\lambda^2)t^{2H-2}, \qquad \delta \ge 0,$$

and for $H \in (0,1)$ and $t \to 0$,

$$\bar{\mathrm{R}}_{X^{H,\delta}}(t) \sim \begin{cases} (r_0^2\delta^{2-1}/2)|t|, & \delta > 0, \\ (r_0^2 j_{H-1/2}/2)|t|^{2H}, & \delta = 0. \end{cases} \quad (4.11)$$

Proof. For $H \in (\frac{1}{2}, 1)$, let $\beta = \delta$. Then, $f \in C^1((\beta,\infty);\mathbb{R})$ and for $t > \beta$, $f'(t) = ct^\alpha$ where $\alpha = H - 3/2 \in (-1, -\frac{1}{2})$ and $c = r(H - 1/2)$. Moreover, $|f(t)| \le r\delta t^\alpha$. Thus, Proposition 4.9(i) shows that $\mathrm{R}_{X^{H,\delta}}(t) \sim (c^2 k_\alpha/\lambda^2)t^{2\alpha+1} = (r^2(H-1/2)^2 k_{H-3/2}/\lambda^2)t^{2H-2}$. To show (4.11) assume that $H \in (0,1)$. For $t \to 0$, we have $f(t) \sim ct^\alpha$, where $c = r_0$ and $\alpha = H - 1/2 \in (-\frac{1}{2}, \frac{1}{2})$ when $\delta = 0$, and $c = r_0\delta^{H-1/2}$ and $\alpha = 0$ when $\delta > 0$. For $\beta = \delta$, $f \in C^2((\beta,\infty);\mathbb{R})$ with $f''(t) = r_0(H-1/2)(H-3/2)t^{H-5/2}$, showing that $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$ (both for $\delta > 0$ and $\delta = 0$). Moreover, f is absolutely continuous on $(0,\infty)$ with density $f'(t) = r_0(H-1/2)t^{H-3/2}1_{[\delta,\infty)}(t)$. This shows that $\sup_{t\in(0,t_0)}|f'(t)t^{1-\alpha}| < \infty$ for all $t_0 > 0$ (both for $\delta > 0$ and $\delta = 0$). Hence (4.11) follows by Proposition 4.9(ii). □

4.4 Stability of the autocovariance function

Let N be a PMA of the form (4.1), where Z is a centered square-integrable Lévy process and $f(t) = c_H t_+^{H-1/2}$ where $H \in (0,1)$. (Recall that if Z is a Brownian motion, then N is a fBm of index H). Let X be a QOU process driven by N with parameter $\lambda > 0$, and recall that by Proposition 4.2, X is a MA of the form

$$X_t = \int_{-\infty}^t \psi_H(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where

$$\psi_H(t) = c_H\left(t^{H-2/2} - \lambda e^{-\lambda t}\int_0^t e^{\lambda u}u^{H-1/2}\,\mathrm{d}u\right), \qquad t \ge 0.$$

Below we will discus some stability properties for the autocovariance function under minor modification of the kernel function.

For all bounded measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ with compact support let $X_t^f = \int_{-\infty}^t (\psi_H(t-s) - f(t-s))\,\mathrm{d}Z_s$. We will think of X^f as a MA where we have made a minor change of X's kernel. Note that if we let $Y_t^f = X_t - X_t^f = \int_{-\infty}^t f(t-s)\,\mathrm{d}Z_s$, then the autocovariance function $\mathrm{R}_{Y^f}(t)$, of Y^f, is zero whenever t is large enough, due to the fact that f has compact support.

Corollary 4.11. *We have the following two situations, in which $c_1, c_2, c_3 \neq 0$ are non-zero constants.*

(i) For $H \in (0, \frac{1}{2})$ and $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$, we have for $t \to \infty$,

$$\mathrm{R}_{X^f}(t) \sim c_2 \mathrm{R}_X(t) t^{1/2-H} \sim c_1 t^{H-3/2}.$$

(ii) For $H \in (\frac{1}{2}, 1)$, we have for $t \to \infty$,

$$\mathrm{R}_{X^f}(t) \sim \mathrm{R}_X(t) \sim c_3 t^{2H-2}.$$

Thus for $H \in (0, \frac{1}{2})$, the above shows that the behavior of the autocovariance function at infinity is changed dramatically by making a minor change of the kernel. In particular, if f is a positive function, not the zero function, then $\mathrm{R}_{X^f}(t)$ behaves as $t^{1/2-H}\mathrm{R}_X(t)$ at infinity. On the other hand, when $H \in (\frac{1}{2}, 1)$ the behavior of the autocovariance function at infinity doesn't change if we make a minor change to the kernel. That is, in this case the autocovariance functions has a stability property, contrary to the case where $H \in (0, \frac{1}{2})$.

Remark 4.12. Note that the dramatic effect appearing from Corollary 4.11(i) is associated to the fact that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$, as shown in Lemma 4.5.

Proof of Corollary 4.11. By Corollary 4.3 we have for $t \to \infty$ that $\psi_H(t) \sim ct^\alpha$ where $c = c_H(H-1/2)/\lambda$ and $\alpha = H - 3/2$. To show (i) assume that $H \in (0, \frac{1}{2})$, and hence $\alpha \in (-\infty, -1)$. According to Lemma 4.5 we have that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$ and hence $\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s \neq 0$ since $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$ by assumption. From Proposition 4.6(ii) and for $t \to \infty$ we have that $\mathrm{R}_{X^f}(t)(t) \sim c_1 t^{2\alpha+1} = c_1 t^{H-3/2}$, where $c_1 = c\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s$. On the other hand, by Corollary 2.9 we have that $\mathrm{R}_X(t) \sim (H(H-1/2)/\lambda^2) t^{2H-2}$ for $t \to \infty$, and hence we have shown (i) with $c_2 = c_1\lambda^2/(H(H-1/2)$. For $H \in (\frac{1}{2}, 1)$ we have that $\alpha \in (-1, -\frac{1}{2})$, and hence (ii) follows by Proposition 4.6(i). □

5 Conclusion

In recent applications of stochastics, particularly in finance and in turbulence, modifications of classic noise processes by time change or by volatility modifications are of central importance, see for instance Barndorff-Nielsen and Shephard (2010) and

Barndorff-Nielsen and Shiryaev (2010) and references given there. Prominent examples of such processes are $\mathrm{d}N_t = \sigma_t \,\mathrm{d}B_t$ where B is Brownian motion and σ is a predictable stationary process (cf. Barndorff-Nielsen and Shephard (2001)), and $N_t = L_{T_t}$, where L is a Lévy process and T is a time change process with stationary increments (cf. Carr et al. (2003)). The theory discussed in the present paper applies to processes of this type (cf. Corollary 2.6). In the applications mentioned the processes are mostly semimartingales. However there is a growing interest in non-semimartingale processes, see Barndorff-Nielsen and Schmiegel (2009), Barndorff-Nielsen et al. (2009, 2010), and the results above covers also such cases. An example in point is $N_t = \int_{\mathcal{X}} B_t^{(x)} \, m(\mathrm{d}x)$ where the processes $B^{(x)}$ are Brownian motions in different filtrations and m is a measure on some space $\mathcal{X}$.

Moreover, extensions of the theory to wider settings would be of interest, for instance to generalized Langevin equations

$$X_t = X_0 - \lambda \int_0^t (X * k)(s) \,\mathrm{d}s + N_t$$

where k is a deterministic function and $(X * k)(s) = \int_{-\infty}^s X_u k(s-u) \,\mathrm{d}u$ denotes the convolution between k and X, as occurring in statistical mechanics and biophysics, see Kou (2008) and references given there. We hope to discuss this in future work.

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