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Introduction into integral geometry and stereology

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Abstract

This text is the extended version of two talks held at the Summer Academy "Stochastic Geometry, Spatial Statistics and Random Fields" in the Soellerhaus, Germany, in September 2009. It forms (with slight modifications) a chapter of the Springer lecture notes "Lectures on Stochastic Geometry, Spatial Statistics and Random Fields" and is a self-containing introduction into integral geometry and its applications in stereology.

The most important integral geometric tools for stereological applications are kinematic formulas and results of Blaschke-Petkantschin type. Therefore, Crofton's formula and the principal kinematic formula for polyconvex sets are stated and shown using Hadwiger's characterization of the intrinsic volumes. Then, the linear Blaschke-Petkantschin formula is proved together with certain variants for flats containing a given direction (vertical flats) or contained in an isotropic subspace. The proofs are exclusively based on invariance arguments and an axiomatic description of the intrinsic volumes.

These tools are then applied in model-based stereology leading to unbiased estimators of specific intrinsic volumes of stationary random sets from observations in a compact window or a lower dimensional flat. Also, Miles-formula for stationary and isotropic Boolean models with convex particles are derived. In design-based stereology, Crofton's formula leads to an unbiased estimator of intrinsic volumes from isotropic uniform random flats. To estimate the Euler characteristic, which cannot be estimated using Crofton's formula, the disector design is presented. Finally we discuss design-unbiased estimation of intrinsic volumes from vertical and from isotropic sections.

1 Integral geometric foundations of stereology

In the early 60ies stereology was a collection of mathematical methods to extract spatial information of a material of interest from sections. Modern stereology may be considered as "sampling inference for geometrical objects" [5, 9] thus emphasizing the two main columns stereology rests upon: sampling theory and geometry. In this first Section on integral geometry, we will discuss two of the most important geometric concepts for stereology: kinematic integral formulas and results of Blaschke-Petkantschin type. The proofs are exclusively based on invariance arguments and an

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axiomatic description of the intrinsic volumes. Section 2 is then devoted to Stereology and describes in detail, how geometric identities lead to unbiased estimation procedures. The influence from sampling theory will also be mentioned in that later section.

1.1 Intrinsic volumes and kinematic integral formula

We start which a deliberately vague question on how to sample a set $K \subset \mathbb{R}^d$. In order to avoid possibly costly measurements on the whole of K, we sample K with a "randomly moved" sampling window $M \subset \mathbb{R}^d$ and consider only the part of K that is inside the moved window. To fix ideas, we assume that K and M are elements of the family $\mathcal{K}$ of convex bodies (compact convex subsets) in $\mathbb{R}^d$, that the (orientation preserving) motion is the composition of a translation with a random vector $\xi \in \mathbb{R}^d$ and a random rotation $\rho \in SO_d$ (special orthogonal group). Assume further that $f : \mathcal{K} \to \mathbb{R}$ is a functional which gives, for each observation, the measured value (think of the volume). What is the expected value of $f(K \cap \rho(M + \xi))$?

To make this question meaningful, we have to specify the distributions of ρ and ξ. One natural condition would be that the distribution of $K \cap \rho(M+\xi)$ is independent of the location and orientation of M. In particular, this implies that ρ should be *right invariant*: $\rho \circ R$ and ρ have the same distribution for any deterministic $R \in SO_d$. The space SO_d, identified with the family of all orthonormal matrices in $\mathbb{R}^{d \times d}$ with determinant 1 and endowed with the induced topology, becomes a compact topological group. The theory of invariant measures implies that there is a unique right invariant probability measure on SO_d, which we denote by ν. This measure, also called *normalized Haar measure*, has even stronger invariance properties: it is inversion invariant in the sense that ρ and ρ^{-1} have the same distribution. Together with the right invariance, this implies that ν is also left invariant in the obvious sense. The measure μ is therefore the natural measure on SO_d, its role being comparable to the one of the Lebesgue measure ν_d on $\mathbb{R}^d$. We will therefore just write $dR = \nu(dR)$ when integrating with respect to this measure. The matrix corresponding to the random rotation ρ can be constructed explicitly by applying the Gram-Schmidt orthonormalization algorithm to a d-tuple $(\eta_1, \dots, \eta_d)$ formed by random i.i.d. uniform vectors in the unit sphere $\mathbb{S}^{d-1}$. Note that the vectors $\eta_1, \dots, \eta_d$ are almost surely linearly independent.

Similar considerations for the random translation vector ξ lead to the contradictory requirement that the distribution of ξ should be a multiple of the Lebesgue measure ν_d on $\mathbb{R}^d$. In contrast to SO_d, the group $\mathbb{R}^d$ is only locally compact but not compact. We therefore have to modify our original question. In view of applications we assume that the moved window hits a fixed reference set $A \in \mathcal{K}$, which contains K. Using the invariant measures defined above we then have

$$\mathbf{E} f(K \cap \rho(M + \xi)) = \frac{\int_{SO_d} \int_{\mathbb{R}^d} f(K \cap R(M + x))\, dx\, dR}{\int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}\{A \cap R(M + x) \neq \emptyset\}\, dx\, dR}, \tag{1}$$

where we assumed $f(\emptyset) = 0$. Enumerator and denominator of this expression are of the same form and we first consider the special case of the denominator with

$M = B_r(o)$, $r > 0$, and $A = K \in \mathcal{K}$:

$$\int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}\{K \cap R(B_r(o) + x) \neq \emptyset\}\, dx\, dR = \nu_d(K \oplus B_r(o)).$$

By a fundamental result in convex geometry, this volume is a polynomial of degree at most d in $r > 0$, usually written as

$$\nu_d(K \oplus B_r(o)) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K),$$

where κ_j denotes the volume of the j-dimensional unit ball. This result is the *Steiner formula*. It defines important functionals, the *intrinsic volumes* $V_0, \dots, V_d$. They include the *volume* $V_d(K) = \nu_d(K)$, the *surface area* $2V_{d-1}(K)$ of the boundary of K (when $\operatorname{int} K \neq \emptyset$) and the trivial functional $V_0(K) = \mathbf{1}\{K \neq \emptyset\}$, also called *Euler(-Poincaré) characteristic* $\chi(K)$. The Steiner formula implies

$$\int_{SO_d} \int_{\mathbb{R}^d} V_0(K \cap R(B_r(o) + x))\, dx\, dR = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K).$$

Already in this special case with M being a ball, the intrinsic volumes play an essential role to express kinematic integrals explicitly. We will soon see that this even holds true when V_0 is replaced by a function $f : \mathcal{K} \to \mathbb{R}$ satisfying some natural properties. To do so, we clarify basic properties of V_j first.

It is easily seen from the Steiner formula that $V_j : \mathcal{K} \to \mathbb{R}$ is invariant under rigid motions and is homogeneous of degree j. Here, we call a function $f : \mathcal{K} \to \mathbb{R}$

- *invariant under rigid motions* if $f(R(K+x)) = f(K)$ for all $K \in \mathcal{K}$, $R \in SO_d$, and $x \in \mathbb{R}^d$, and
- *homogeneous of degree j* if $f(\alpha K) = \alpha^j f(K)$ for all $K \in \mathcal{K}$, $\alpha \geq 0$.

Using convexity properties, V_j can be shown to be additive and monotone (with respect to set inclusion). Here $f : \mathcal{K} \to \mathbb{R}$ is

- *additive* if $f(\emptyset) = 0$ and

  $$f(K \cup M) = f(K) + f(M) - f(K \cap M)$$

  for $K, M \in \mathcal{K}$ with $K \cup M \in \mathcal{K}$ (implying $K \cap M \neq \emptyset$).
- *monotone* if $f(K) \leq f(M)$ for all $K, M \in \mathcal{K}$, $K \subset M$.

Already a selection of these properties is sufficient to characterize intrinsic volumes axiomatically. This is the content of Hadwiger's famous characterization theorem.

Theorem 1.1 (Hadwiger). *Suppose $f : \mathcal{K} \to \mathbb{R}$ is additive, motion invariant and monotone. Then there exist $c_0, \dots, c_d \geq 0$ with*

$$f = \sum_{j=0}^{d} c_j V_j.$$

This shows that the intrinsic volumes are essentially the only functionals that share some natural properties with the volume. We will use this result without proof. It implies in particular, that under the named assumptions on f, we only have to consider the enumerator of (1) for $f = V_j$. In view of Hadwiger's characterization the following result gives a complete answer to our original question for a large class of measurement functions f.

Theorem 1.2 (Principal kinematic formula)**.** *Let $j \in \{0, \dots, d\}$ and $K, M \in \mathcal{K}$. Then*

$$\int_{SO_d} \int_{\mathbb{R}^d} V_j(K \cap R(M + x))\, dx\, dR = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(K) V_{d-k+j}(M),$$

where the constants are given by

$$c_{s_1,\dots,s_k}^{r_1,\dots,r_k} = \prod_{i=1}^{k} \frac{r_i! \kappa_{r_i}}{s_i! \kappa_{s_i}}. \tag{2}$$

In certain cases the formula remains valid even when the rotation integral is omitted. This is trivially true for $M = B_r(o)$, but also for $j = d$ and $j = d - 1$.

Proof. We denote the left hand side of the principal kinematic formula by $f(K, M)$. The functional $f(K, M)$ is symmetric in K and M due to the invariance properties of ν_d, ν and V_j. The homogeneity of V_j and a substitution yield

$$f(\alpha K, \alpha M) = \alpha^{d+j} f(K, M), \quad \alpha > 0.$$

As $f(K, \cdot)$ is additive, motion invariant and monotone, Hadwiger's characterization theorem implies the existence of constants $c_0(K), \dots, c_d(K) \geq 0$ (depending on K) with $f(K, \cdot) = \sum_{k=0}^{d} c_k(K) V_k$. Hence, for $\alpha > 0$,

$$\begin{aligned}
\sum_{k=0}^{d} c_k(K) V_k(M) \alpha^k &= \sum_{k=0}^{d} c_k(K) V_k(\alpha M) = f(K, \alpha M) \\
&= \alpha^{d+j} f(\frac{1}{\alpha} K, M) = \alpha^{d+j} f(M, \frac{1}{\alpha} K) \\
&= \sum_{k=0}^{d} c_k(M) V_k(K) \alpha^{d-k+j}, \quad \alpha > 0.
\end{aligned}$$

Comparison of the coefficients of these polynomials yields $c_k(M) = 0$ for $k < j$ and that $c_k(M)$ is proportional to $V_{d-k+j}(M)$ for $k \geq j$. This gives the principal kinematic formula with unknown constants. The constants are then determined by appropriate choices for K and M, for which the integrals can be calculated explicitly; see also the comment after Theorem 1.4. □

This solves our original question for $f = V_j$. Formula (1) now gives

$$\mathbf{E} V_j(K \cap \vartheta(M + \xi)) = \frac{\sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(K) V_{d-k+j}(M)}{\sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(A) V_{d-k}(M)},$$

if (ϑ, ξ) has its natural distribution on $\{(R,x) : R(M+x) \cap A \neq \emptyset\}$. As invariant integrations like in Theorem 1.2 do always lead to functionals in the linear span of $V_0, \dots, V_d$, an iterated version for $k+1$ convex bodies can be shown by induction.

Theorem 1.3 (Iterated principal kinematic formula). *Let $j \in \{0, \dots, d\}$, $k \geq 1$, and $K_0, \dots, K_k \in \mathcal{K}$. Then*

$$\int_{SO_d} \int_{\mathbb{R}^d} \cdots \int_{SO_d} \int_{\mathbb{R}^d} V_j(K_0 \cap R_1(K_1 + x_1) \cap \cdots \cap R_k(K_k + x_k))\, dx_1\, dR_1 \cdots dx_k\, dR_k$$
$$= \sum_{\substack{m_0,\dots,m_k=j \\ m_0+\cdots+m_k=kd+j}}^{d} c_{j,d,\dots,d}^{d,m_0,\dots,m_k} V_{m_0}(K_0) \cdots V_{m_k}(K_k)$$

with constants given by (2).

For $j = d$ and $j = d-1$ the rotation integrals on the left hand side can be omitted.

Theorem 1.2 has a counterpart where M is replaced by an affine subspace. For $k \in \{0, \dots, d\}$ let $\mathcal{L}_k^d$ be the *Grassmannian* of all k-dimensional linear subspaces of $\mathbb{R}^d$. The image measure of ν on SO_d under $R \mapsto RL_0$, $L_0 \in \mathcal{L}_k^d$ fixed, is a rotation invariant probability measure on $\mathcal{L}_k^d$, and integration with respect to it is denoted by dL. It is the only rotation invariant distribution on $\mathcal{L}_k^d$. Similarly, let $\mathcal{E}_k^d$ be the space of all (affine) k-dimensional flats in $\mathbb{R}^d$. The elements $E \in \mathcal{E}_k^d$ are called *k-flats* and can be parametrized in the form $E = R(L_0 + x)$ with $R \in SO_d$, $x \in L_0^\perp$, and a fixed space $L_0 \in \mathcal{L}_k^d$. The function $(R,x) \mapsto R(L_0 + x)$ on $SO_d \times L_0^\perp$ maps the measure $\nu \otimes \nu_{d-k}$ to a motion invariant measure on $\mathcal{E}_k^d$. Integration with this measure is denoted by dE. Up to a factor, this is the only motion invariant measure on $\mathcal{E}_k^d$.

We will later need families of subspaces containing or contained in a given space $L \in \mathcal{L}_k^d$, $0 \leq k \leq d$:

$$\mathcal{L}_r^L = \begin{cases} \{M \in \mathcal{L}_r^d : M \subset L\}, & \text{if } 0 \leq r \leq k, \\ \{M \in \mathcal{L}_r^d : M \supset L\}, & \text{if } k < r \leq d \end{cases}$$

Again, there is a uniquely determined invariant probability measure on $\mathcal{L}_r^L$. We will write dM when integrating with respect to it; the domain of integration will always be clear from the context. For $r \leq k$ existence and uniqueness of this probability measure follow from identifying L with $\mathbb{R}^k$. For $r > k$, this measure is obtained as image of $\int_{\mathcal{L}_{r-k}^{L^\perp} \cap (\cdot)} dM$ under $M \mapsto M \oplus L$. An invariance argument also shows that

$$\int_{\mathcal{L}_k^d} \int_{\mathcal{L}_r^L} f(L,M)\, dM\, dL = \int_{\mathcal{L}_r^d} \int_{\mathcal{L}_k^M} f(L,M)\, dL\, dM \tag{3}$$

holds for any measurable $f : \{(L,M) \in \mathcal{L}_k^d \times \mathcal{L}_r^d : M \in \mathcal{L}_r^L\} \to [0, \infty)$. The corresponding family of incident flats will only be needed for $0 \leq r \leq k \leq d$ and is defined as the space

$$\mathcal{E}_r^E = \{F \in \mathcal{E}_r^d : F \subset E\}$$

of all r-flats contained in a fixed k-flat E. Integration with respect to the invariant measure on this space will again be denoted by dF and can be derived by identifying E with $\mathbb{R}^k$ as in the linear case.

We can now state the announced counterpart of Theorem 1.2 with M replaced by a k-flat.

Theorem 1.4 (Crofton formula)**.** *For $0 \leq j \leq k < d$ and $K \in \mathcal{K}$ we have*

$$\int_{\mathcal{E}_k^d} V_j(K \cap E)\, dE = c_{j,d}^{k,d-k+j} V_{d-k+j}(K)$$

with $c_{j,d}^{k,d-k+j}$ given by (2).

This follows (apart from the value of the constant) directly from Hadwiger's characterization theorem, as the left hand side is additive, motion invariant, monotone and homogeneous of degree $d-k+j$. The constant is derived by setting $K = B_1(o)$. That the same constants also appear in the principal kinematic formula is not coincidental, but a consequence of a deeper connection between the principal kinematic formula and Crofton integrals; see also the 2nd item of the paragraph *Further Reading* at the end of this Section.

The results for V_j can be extended to *polyconvex sets*, i.e. sets in

$$\mathcal{R} = \{K \subset \mathbb{R}^d : \exists\, m \in \mathbb{N},\, K_1, \ldots, K_m \in \mathcal{K} \text{ with } K = \bigcup_{i=1}^m K_i\}.$$

In fact, additivity suggests how to define $V_j(K \cup M)$ for two convex bodies K, M, which not necessarily satisfy $K \cup M \in \mathcal{K}$. Induction then allows extension of V_j on $\mathcal{R}$. That such an extension is well-defined (it does not depend on the representation of $K \in \mathcal{R}$ as a union of convex bodies) follows from a result of Groemer [11]. We denote the extension of V_j on $\mathcal{R}$ again by V_j. Using induction on m, additivity implies the *inclusion-exclusion principle*

$$V_j(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^m (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} V_j(K_{i_1} \cap \cdots \cap K_{i_r})$$

for all $m \in \mathbb{N}$ and $K_1, \ldots, K_m \in \mathcal{R}$. This principle in particular implies that Theorems 1.2, 1.3, and 1.4 remain valid with the convexity assumption replaced by the assumption that all occuring sets are polyconvex.

Crofton's formula allows to derive mean values like in (1), where the moved convex body is replaced by a k-flat. A random k-flat E hitting a given reference space $A \in \mathcal{B}(\mathbb{R}^d)$ with the natural distribution

$$\mathbf{P}(E \in \cdot) = \frac{\int_{(\cdot)} \mathbf{1}\{E' \cap A \neq \emptyset\}\, dE'}{\int_{\mathcal{E}_k^d} \mathbf{1}\{E' \cap A \neq \emptyset\}\, dE'} \tag{4}$$

is called an *IUR* (isotropic uniform random) k-flat in A. For $K \in \mathcal{R}$ with $K \subset A$, Crofton's formula for an IUR k-flat in $A \in \mathcal{R}$ gives the mean value

$$\mathbf{E} V_j(K \cap E) = \frac{c_{j,d}^{k,d-k+j} V_{d-k+j}(K)}{c_{0,d}^{k,d-k} V_{d-k}(A)} \tag{5}$$

for $0 \le j \le k < d$. Hence, up to a known multiplicative constant depending on A, the random variable $V_j(K \cap E)$ is an unbiased estimator of $V_{d-k+j}(K)$. The relations (5) are sometimes called *fundamental stereological formulas*. We will discuss them and related stereological relations in more detail in Section 2.

It is not difficult to construct an IUR k-flat E in a compact set A. For $k = 0$ the flat E is a point, uniformly distributed in A. For $k > 0$ choose an $r > 0$ with $A \subset B_r(o)$ and a linear space $L_0 \in \mathcal{L}_k^d$. If $\rho \in SO_d$ is a random rotation with distribution ν and η is independent of ρ with the uniform distribution on $B_r(o) \cap L_0^\perp$, then $E = \rho(L_0 + \eta)$ is an IUR k-flat in $B_r(o)$. Conditioning on the event $E \cap A \neq \emptyset$ yields an IUR k-flat in A. The construction of an IUR k-flat can be simplified when $k = 1$ (IUR line) or $k = d-1$ (IUR hyperplane). To obtain an IUR line in $B_r(o)$ one can choose a uniform vector $\eta \in \mathbb{S}^{d-1}$ and, given η, a uniform point $\xi \in B_r(o) \cap \eta^\perp$. The line E parallel to η passing through ξ then is an IUR line in $B_r(o)$. In a similar way, an IUR hyperplane can be constructed by representing it by one of its normals and its closest point to o. For $d = 2$ and $d = 3$, which are the most important cases in applications, the construction of the random rotation ρ can thus be avoided.

It should be noted that an IUR k-flat in $B_1(o)$ cannot be obtained by choosing $k + 1$ i.i.d. uniform points in $B_1(o)$ and considering their affine hull H. Although H has almost surely dimension k, its distribution is not coinciding with the natural distribution of E in (4). The k-flat H is called *point weighted* k-flat, and its (non-constant) density with respect to $\int_{(\cdot)} \mathbf{1}\{E \cap B_1(o) \neq \emptyset\} dE$ can be calculated explicitly using the affine Blaschke-Petkantschin formula. As formulas of Blaschke-Petkantschin type play an important role in stereology, we discuss them in detail in the next section.

1.2 Blaschke-Petkantschin formulas

Suppose we have to integrate a function of q-tuples $(x_1, \ldots, x_q)$ of points in $\mathbb{R}^d$ with respect to the product measure ν_d^q. In several applications computations can be simplified by first integrating over all q-tuples of points in a q-dimensional linear subspace L (with respect to ν_q^q) and subsequently integrating over all linear subspaces with respect to the Haar measure $\int_{\mathcal{L}_q^d \cap (\cdot)} dL$. The case $q = 1$, $d = 2$ corresponds to the well-known integration in the plane using polar coordinates. The Jacobian appearing in the general transformation formula turns out to be a power of

$$\nabla_q(x_1, \ldots, x_q) = \nu_q([0, x_1] \oplus \cdots \oplus [0, x_q]),$$

where $[0, x_1] \oplus \cdots \oplus [0, x_q]$ is the parallelepiped spanned by the vectors $x_1, \ldots, x_q$. To simplify notation, we will just write dx for integration with respect to Lebesgue measure in $\mathbb{R}^k$, as the appropriate dimension k can be read off from the domain of integration under the integral sign.

Theorem 1.5 (Linear Blaschke-Petkantschin formula)**.** *Let $q \in \{1, \ldots, d\}$ and $f : (\mathbb{R}^d)^q \to [0, \infty)$ be measurable. Then*

$$\int_{(\mathbb{R}^d)^q} f(x) dx = b_{dq} \int_{\mathcal{L}_q^d} \int_{L^q} f(x)\, \nabla_q^{d-q}(x)\, dx\, dL,$$

with
$$b_{dq} = \frac{\omega_{d-q+1} \cdots \omega_d}{\omega_1 \cdots \omega_q},$$
where $\omega_j = j\kappa_j$ denotes the surface area of the unit ball in $\mathbb{R}^j$.

For the proof, which is by induction on q, we use a generalization of the polar coordinate formula.

Lemma 1.6. *Let $r \in \{0, \ldots, d-1\}$, $L_0 \in \mathcal{L}_r^d$ be fixed and $f : \mathbb{R}^d \to [0, \infty)$ be measurable. Then*
$$\int_{\mathbb{R}^d} f(x)\, dx = \frac{\omega_{d-r}}{2} \int_{\mathcal{L}_{r+1}^{L_0}} \int_M f(x) d(x, L_0)^{d-r-1}\, dx\, dM,$$
where $d(x, L_0)$ is the distance of x to L_0.

Proof. Let $L_0(u) = \{L_0 + \alpha u : \alpha \geq 0\}$ be the positive hull of L_0 and u. Then Fubini's theorem and spherical coordinates (in $L_o^\perp$) yield
$$\begin{aligned}
\int_{\mathbb{R}^d} f(z)\, dz &= \int_{L_0} \int_{L_0^\perp} f(x+y)\, dy\, dx \\
&= \int_{L_0} \int_0^\infty \int_{\mathbb{S}^{d-1} \cap L_0^\perp} f(x + \alpha u) \alpha^{d-r-1}\, du\, d\alpha\, dx \\
&= \int_{\mathbb{S}^{d-1} \cap L_0^\perp} \int_{L_0(u)} f(x) d(x, L_0)^{d-r-1}\, dx\, du \\
&= \frac{\omega_{d-r}}{2} \int_{\mathcal{L}_{r+1}^{L_0}} \int_M f(x) d(x, L_0)^{d-r-1}\, dx\, dM.
\end{aligned}$$
This concludes the proof of Lemma 1.6. □

Proof of Theorem 1.5. For $q = 1$ the assertion reduces to Lemma 1.6 with $r = 0$. We assume now that the assertion is true for some $q \in \mathbb{N}$ and all dimensions d, and use the fact that
$$\nabla_{q+1}(x_1, \ldots, x_{q+1}) = \nabla_q(x_1, \ldots, x_q) d(x_{q+1}, L), \tag{6}$$
if $x_1, \ldots, x_q \in L$, $L \in \mathcal{L}_q^d$. Fubini's theorem, the induction hypothesis and Lemma 1.6 give
$$\begin{aligned}
I &:= \int_{(\mathbb{R}^d)^{q+1}} f(z)\, dz = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^q} f(x, y)\, dx\, dy \\
&= b_{dq} \int_{\mathbb{R}^d} \int_{\mathcal{L}_q^d} \int_{L^q} f(x, y) \nabla_q^{d-q}(x)\, dx\, dL\, dy \\
&= b_{dq} \int_{\mathcal{L}_q^d} \int_{L^q} \int_{\mathbb{R}^d} f(x, y)\, dy \nabla_q^{d-q}(x)\, dx\, dL \\
&= b_{dq} \frac{\omega_{d-q}}{2} \int_{\mathcal{L}_q^d} \int_{L^q} \int_{\mathcal{L}_{q+1}^L} \int_M f(x, y) d(y, L)^{d-q-1}\, dy\, dM \nabla_q^{d-q}(x)\, dx\, dL \\
&= b_{dq} \frac{\omega_{d-q}}{2} \int_{\mathcal{L}_{q+1}^d} \int_M \int_{\mathcal{L}_q^M} \int_{L^q} f(x, y) d(y, L)^{d-q-1} \nabla_q^{d-q}(x)\, dx\, dL\, dy\, dM,
\end{aligned}$$

where the integrals over q and $(q+1)$-dimensional subspaces may be interchanged due to (3). From (6) and an application of the induction hypothesis for a q-fold integral over the $(q+1)$-dimensional space M with function $f(\cdot,y)\nabla_{q+1}(\cdot,y)^{d-q-1}$, we get

$$\begin{aligned}
I &= b_{dq}\frac{\omega_{d-q}}{2}\int_{\mathcal{L}^d_{q+1}}\int_M\int_{\mathcal{L}^M_q}\int_{L^q} f(x,y)\nabla_{q+1}(x,y)^{d-q-1}\nabla_q(x)\,dx\,dL\,dy\,dM\\
&= \frac{b_{dq}\omega_{d-q}}{2b_{(q+1)q}}\int_{\mathcal{L}^d_{q+1}}\int_M\int_{M^q} f(x,y)\nabla_{q+1}(x,y)^{d-q-1}\,dx\,dy\,dM\\
&= b_{d(q+1)}\int_{\mathcal{L}^d_{q+1}}\int_{M^{q+1}} f(z)\nabla_{q+1}(z)^{d-q-1}\,dz\,dM.
\end{aligned}$$

This concludes the proof. □

Amazingly, this shows that the linear Blaschke-Petkantschin formula follows by a relatively simple induction on q and a suitable use of spherical coordinates in subspaces of $\mathbb{R}^d$.

There are many formulas of Blaschke-Petkantschin type in the literature. Following [22] we can describe their common feature: Instead of integrating q-tuples of geometric objects (usually points or flats) directly, a 'pivot' is associated to this tuple (usually span or intersection) and integration of the q-tuple is first restricted to one pivot, followed by an integration over all possible pivots. For integrations the natural measures are used, and a Jacobian comes in. In Theorem 1.5, the pivot is the linear space (almost everywhere) spanned by the q points $x_1,\dots,x_q$. As an affine subspace of dimension q is spanned by $q+1$ affine independent points, a similar formula for affine q-flats is to be expected. Such a formula actually holds and is called *affine Blaschke-Petkantschin formula*. Although its Jacobian is different from the Jacobian in the linear case, the affine Blaschke-Petkantschin formula can be directly derived from Theorem 1.5. We refer to [22, Theorem 7.2.7] for details and give instead an example of another Blaschke-Petkantschin formula, where there is only one initial geometric element, namely an affine k-flat. The pivot is a linear space of dimension $r>k$ containing it.

Theorem 1.7. *Let $1\le k<r\le d-1$ and let $f:\mathcal{E}^d_k\to[0,\infty)$ be measurable. Then*

$$\int_{\mathcal{E}^d_k} f(E)\,dE = \frac{\omega_{d-k}}{\omega_{r-k}}\int_{\mathcal{L}^d_r}\int_{\mathcal{E}^L_k} f(E)d(o,E)^{d-r}\,dE\,dL.$$

Proof. If $L\in\mathcal{L}^d_k$ is fixed, the restriction of the measure

$$\int_{\mathcal{L}^L_r}\int_{\mathbb{S}^{d-1}\cap L^\perp\cap M}\mathbf{1}\{u\in(\cdot)\}\,du\,dM$$

on $\mathbb{S}^{d-1}\cap L^\perp$ is invariant with respect to all rotations of $L^\perp$ (leaving L fixed), and must thus be a multiple of $\int_{\mathbb{S}^{d-1}\cap L^\perp}\mathbf{1}\{u\in(\cdot)\}\,du$. The factor is $\omega_{r-k}/\omega_{d-k}$.

Hence, integrating

$$\int_0^\infty f(\alpha(\cdot)+L)\alpha^{d-k-1}\,d\alpha$$

with respect to this measure, and using spherical coordinates in $L^\perp$ gives

$$\frac{\omega_{r-k}}{\omega_{d-k}}\int_{L^\perp} f(x+L)\,dx = \int_{\mathcal{L}_r^L}\int_{\mathbb{S}^{d-1}\cap L^\perp\cap M}\int_0^\infty f(\alpha u+L)\alpha^{d-k-1}\,d\alpha\,du\,dM.$$

A back transformation of spherical coordinates appearing on the right in the $(r-k)$-dimensional space $L^\perp \cap M$ yields

$$\frac{\omega_{r-k}}{\omega_{d-k}}\int_{L^\perp} f(x+L)\,dx = \int_{\mathcal{L}_r^L}\int_{L^\perp\cap M} f(x+L)\,\|x\|^{d-r}\,dx\,dM.$$

Integration with respect to $L \in \mathcal{L}_k^d$ leads to

$$\begin{aligned}\int_{\mathcal{E}_k^d} f(E)\,dE &= \frac{\omega_{d-k}}{\omega_{r-k}}\int_{\mathcal{L}_k^d}\int_{\mathcal{L}_r^L}\int_{L^\perp\cap M} f(x+L)\,\|x\|^{d-r}\,dx\,dM\,dL\\ &= \frac{\omega_{d-k}}{\omega_{r-k}}\int_{\mathcal{L}_r^d}\int_{\mathcal{L}_k^M}\int_{L^\perp\cap M} f(x+L)d(x+L,0)^{d-r}\,dx\,dL\,dM\\ &= \frac{\omega_{d-k}}{\omega_{r-k}}\int_{\mathcal{L}_r^d}\int_{\mathcal{E}_k^M} f(E)d(o,E)^{d-r}\,dE\,dM,\end{aligned}$$

where (3) was used. This completes the proof. □

We also notice an example of a Blaschke-Petkantschin formula, where the pivot is spanned by an initial geometric element and a fixed subspace. We only consider initial geometric elements and fixed subspaces of dimension one here, although versions for higher dimensional planes (and q-fold integrals, $q > 1$) exist. The Jacobian appearing in the following relation is a power of the *generalized determinant* $[L, L_0]$ of two subspaces L and L_0. In the special case we consider here, L and L_0 are lines and $[L, L_0]$ is just the sine of the angle between them.

Lemma 1.8. *Let $L_0 \in \mathcal{L}_1^d$ be a fixed line. Then*

$$\int_{\mathcal{L}_1^d} f(L)\,dL = \frac{\omega_2\omega_{d-1}}{\omega_1\omega_d}\int_{\mathcal{L}_2^{L_0}}\int_{\mathcal{L}_1^M} f(L)[L,L_0]^{d-2}\,dL\,dM$$

holds for any measurable $f : \mathcal{L}_1^d \to [0,\infty)$.

Proof. An invariance argument implies

$$\int_{\mathbb{R}^d} f(\mathrm{span}\{x\})\mathbf{1}\{\|x\|\le 1\}\,dx = \kappa_d\int_{\mathcal{L}_1^d} f(L)\,dL,$$

where $\mathrm{span}\{x\}$ is the line containing x and o. Using this and Lemma 1.6 twice, first with $r = 1$ in $\mathbb{R}^d$ and then with $r = 0$ in M, we get

$$\begin{aligned}\int_{\mathcal{L}_1^d} f(L)\,dL &= \kappa_d^{-1}\int_{\mathbb{R}^d} f(\mathrm{span}\{x\})\mathbf{1}\{\|x\|\le 1\}\,dx\\ &= \frac{\omega_{d-1}}{2\kappa_d}\int_{\mathcal{L}_2^{L_0}}\int_M f(\mathrm{span}\{x\})\mathbf{1}\{\|x\|\le 1\}d(x,L_0)^{d-2}\,dx\,dM\\ &= \frac{\omega_2\omega_{d-1}}{4\kappa_d}\int_{\mathcal{L}_2^{L_0}}\int_{\mathcal{L}_1^M} f(L)\int_L \mathbf{1}\{\|x\|\le 1\}d(x,L_0)^{d-2}\,\|x\|\,dx\,dL\,dM.\end{aligned}$$

The innermost integral is

$$[L, L_0]^{d-2} \int_{L \cap B_1(o)} \|x\|^{d-1} \, dx = \frac{2}{d} [L, L_0]^{d-2},$$

and the claim follows. $\square$

An affine version of Lemma 1.8 is obtained by replacing $f(L)$ by $\int_{L^\perp} f(x+L)\,dx$, where now f is a nonnegative measurable function on $\mathcal{E}_1^d$. Lemma 1.8 and Fubini's theorem then imply

$$\int_{\mathcal{E}_1^d} f(E)\,dE = \frac{\omega_2 \omega_{d-1}}{\omega_1 \omega_d} \int_{\mathcal{L}_2^{L_0}} \int_{M^\perp} \int_{\mathcal{E}_1^{M+x}} f(E)[E, L_0]^{d-2}\,dE\,dx\,dM, \tag{7}$$

where $[E, L_0] := [L, L_0]$, if $L \in \mathcal{L}_1^d$ is parallel to $E \in \mathcal{E}_1^d$.

Further reading

1. Hadwiger's characterization theorems (with either a monotonicity or a continuity assumption on f) can be found in the monograph [13]. A simplified proof (for the characterization based on continuity) can be found in [14], see also [15] or [3].

2. Hadwiger [13] showed a general kinematic formula, where the intrinsic volume V_j in the principal kinematic formula is replaced by an additive continuous functional on $\mathcal{K}$ and the right hand side involves Crofton-type integrals with f as functional. In particular, this shows that the constants in the principal kinematic formula are the same as in corresponding Crofton formulas, facilitating their calculation.

3. There are numerous generalizations of the principal kinematic formula and the Crofton formula. Local versions exist, where the intrinsic volumes are replaced by support measures (generalized curvature measures). When the averaging with respect to rotations is omitted, one obtains translative integral formulas; see [21]. For instance, the principal kinematic formula in its translative form still allows on the right hand side for a sum of $d - j + 1$ summands distinguishable by their homogeneity properties, but these summands depend on the relative position of K and M. Iterated versions of the principal translative formula exist, but in contrast to Theorem 1.3 new functionals appear when the number of convex bodies is increased; see [27], where a translative formula of Crofton-type and for half-spaces is derived as well. Integral geometric formulas for convex cylinders can be seen as joint generalizations of the principal kinematic formula and Crofton's formula. Details can be found in [22].

   We discussed integral geometric formulas for polyconvex sets. However, they are valid for considerably larger set classes. Already Federer [10] showed that the principal kinematic formula holds for sets of positive reach. Zähle [29] and Rother & Zähle [19] extended kinematic integral formulas to even larger set classes containing the class of so-called U_{PR}-sets. A set is an element of U_{PR} if it can be written as locally finite union of sets of positive reach such that any finite nonempty intersection of them has again positive reach. The mentioned results even hold locally, that is, for curvature measures.

4. The idea to base proofs of Blaschke-Petkantschin formulas on invariance arguments is due to Miles [16]. We followed mainly the presentation of his and Petkantschin's [18] results in [22]. Santaló's monograph [20] is a general reference for Blaschke-Petkantschin formulas. His proofs use differential forms.

2 Stereology

The purpose of this lecture is to give an introduction into stereology with a special emphasis on the usefulness of integral geometric tools. *Stereology* (gr.: "stereos" means solid) is a sub-area of stochastic geometry and spatial statistics dealing with the estimation of geometric characteristics (like volume, area, perimeter or particle number) of a structure from samples. Typically samples are sections with or projections onto flats, intersections with full-dimensional test sets or combinations of those.

2.1 Motivation

Unlike tomography, stereology is not aiming for a full-dimensional reconstruction of the geometry of the structure, but rather trying to assess certain key properties. This is what makes stereology extremely efficient and explains its widespread use in many applied sciences. As estimation is based on samples of the structure, one has to assure that these samples are in a certain sense representative for the structure as a whole – at least concerning the geometric characteristics of interest. Stereologists therefore assume that the structure is "statistically homogeneous", a property that only was vaguely defined in the early literature. The former East German stochastics school of J. Mecke, D. Stoyan and collaborators (see [24] and the references therein) made this concept rigorous by considering the structure $Z \subset \mathbb{R}^d$ as a random closed set which is stationary (i.e. the distribution of $Z+x$ is independent of $x \in \mathbb{R}^d$). Often it was also assumed that Z is also isotropic (the distribution of RZ is independent of $R \in SO_d$). As (weak) model assumptions on Z are needed, this approach is called the *model-based approach* . The stationarity assumption is appropriate in many applications in geology, metallurgy and materials science. It is, however, often hard to check in other disciplines and certainly inappropriate in anatomy and soil science. In these cases the *design-based approach* has to be used, where the structure of interest is considered deterministic, and the selection of the sample is done in a controlled randomized way.

The Australian statisticians R. E. Miles and P. J. Davy [17], [9], [8] made this rigorous by pointing out the strong analogy between stereology and sample surveys. Sample surveys (think of opinion polls) infer properties of the whole population (e.g. the total number of citizens voting for the democratic party) from a randomized sample of the population. In a simplified stereological situation, where a feature of interest K is contained in a reference space A, the space A corresponds to the

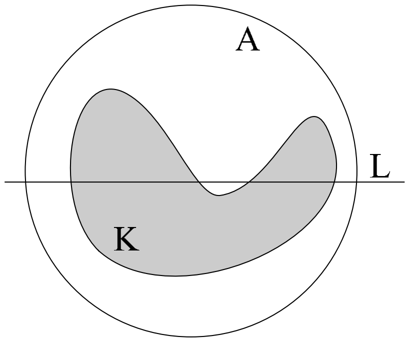

Figure 1: General setting in stereology

total population, the intersection with a set L corresponds to a sample, and K corresponds to the subpopulation of interest to us.

This analogy is more than a formal one and allows among other things to transfer variance reducing methods like systematic random sampling, unequal probability sampling and stratification to stereology. We return to design-based stereology in a later section, and start with model-based methods.

2.2 Model-based stereology

In model-based stereology we assume that the structure of interest $Z \subset \mathbb{R}^d$ is a stationary random closed set (see e.g. [22]). We want to use integral geometric formulas from Section 1.1, which we only have shown for polyconvex sets. The assumption that Z is stationary and polyconvex is not suitable, as a stationary set $Z \neq \emptyset$ is known to be almost surely unbounded. Instead we assume that Z is almost surely locally polyconvex, i.e. $Z \cap K$ is polyconvex for all $K \in \mathcal{K}$, almost surely. We denote by $N(Z \cap K)$ the minimal number of convex bodies that is needed to represent $Z \cap K$ as their union and assume the integrability condition

$$\mathbf{E}2^{N(Z\cap[0,1]^d)} < \infty. \tag{8}$$

Following [22] we call a random set $Z \subset \mathbb{R}^d$ a *standard random set* if

(i) Z is stationary,
(ii) Z is a.s. locally polyconvex, and
(iii) Z satisfies (8).

The class of standard random sets forms the most basic family of random sets which is flexible enough to model real-world structures reasonably. To define *mean intrinsic volumes per unit volume*, one might consider $\mathbf{E}V_j(Z \cap W)/\nu_d(W)$ for an observation window $W \in \mathcal{K}$ with $\nu_d(W) > 0$. But this definition is inapt, as can already be seen in the special case $j = d-1$ corresponding to surface area estimation: in addition to the surface area of ∂Z in W also the surface area of ∂W in Z is contributing, leading to an overestimation of the mean surface area per unit volume. In order to eliminate such edge effects one defines

$$\overline{V}_j(Z) = \lim_{r\to\infty} \frac{\mathbf{E}V_j(Z \cap rW)}{\nu_d(rW)},$$

where $W \in \mathcal{K}$, $\nu_d(W) > 0$ as before. If Z is a standard random set then $\overline{V}_j(Z)$ exists and is independent of W. It is called the *j-th specific intrinsic volume of Z* . If, in addition, Z is isotropic, the principal kinematic formula holds for $\overline{V}_j(Z)$.

Theorem 2.1. *Let Z be an isotropic standard random set, and $j \in \{0, \ldots, d\}$. Then*

$$\mathbf{E}V_j(Z \cap W) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \overline{V}_k(Z) V_{d-k+j}(W), \quad W \in \mathcal{K}.$$

If W is a ball, $j = d$ or $j = d-1$, the isotropy assumption can be dropped.

Proof. Let Z be defined on the abstract probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Fix $W \in \mathcal{K}$ and $r > 0$. It can be shown that

$$\begin{aligned} f : \mathbb{R}^d \times \Omega &\to \mathbb{R} \\ (x, \omega) &\mapsto V_j((Z(\omega) \cap W) \cap (B_r(o) + x)) \end{aligned}$$

is measurable, and that the integrability condition (8) implies integrability of f with respect to $\nu_d \otimes \mathbf{P}$. This will allow us to use Fubini's theorem later in the proof. The motion invariance of V_j and stationarity and isotropy of Z imply

$$\begin{aligned} &\mathbf{E}V_j((Z \cap W) \cap (B_r(o) + x)) \\ &\quad = \mathbf{E}V_j((RZ + x) \cap W \cap (RB_r(o) + x)) \\ &\quad = \mathbf{E}V_j((Z \cap B_r(o)) \cap R^{-1}(W - x)) \end{aligned}$$

for $x \in \mathbb{R}^d$, $R \in SO_d$. Fubini's theorem and the invariance properties of ν_d and ν imply

$$\begin{aligned} &\mathbf{E} \int_{SO_d} \int_{\mathbb{R}^d} V_j((Z \cap W) \cap R(B_r(o) + x))\, dx\, dR \\ &\quad = \mathbf{E} \int_{SO_d} \int_{\mathbb{R}^d} V_j((Z \cap B_r(o)) \cap R(W + x))\, dx\, dR, \end{aligned}$$

so the sets W and $B_r(o)$ can be interchanged. The principal kinematic formula, applied on both sides, yields

$$\sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \mathbf{E}V_k(Z \cap W) V_{d-k+j}(B_r(o)) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \mathbf{E}V_k(Z \cap B_r(o)) V_{d-k+j}(W).$$

Now we divide both sides by $\nu_d(B_r(o))$ and let r tend to infinity. As

$$\frac{V_{d-k+j}(B_r(o))}{\nu_d(B_r(o))} = r^{j-k} \frac{V_{d-k+j}(B_1(o))}{\kappa_d},$$

the claim follows. In the cases where the principal kinematic formula holds even without averaging over all rotations, isotropy is not needed in the above proof. □

Theorem 2.1 shows that

$$\mathbf{E} \begin{pmatrix} V_0(Z \cap W) \\ \vdots \\ V_d(Z \cap W) \end{pmatrix} = A \cdot \begin{pmatrix} \overline{V}_0(Z) \\ \vdots \\ \overline{V}_d(Z) \end{pmatrix}$$

with a triangular matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$, which is regular if $\nu_d(W) > 0$. Hence $A^{-1}\begin{pmatrix} V_0(Z\cap W) \\ \vdots \\ V_d(Z\cap W)\end{pmatrix}$ is an unbiased estimator of $\begin{pmatrix} \overline{V}_0(Z) \\ \vdots \\ \overline{V}_d(Z)\end{pmatrix}$ and can be determined from observations of Z in the full-dimensional window W alone.

If E is a k-flat, and Z is a standard random set in $\mathbb{R}^d$, then $Z \cap E$ is a standard random set in E (in particular, stationarity refers to invariance of $\mathbf{P}_{Z\cap E}$ under all translations in E). If Z is isotropic, then $Z \cap E$ is isotropic in E.

Theorem 2.2 (Crofton's formula for random sets)**.** *If Z is an isotropic standard random set and $E \in \mathcal{E}_k^d$ with $0 \le j \le k < d$, then*

$$\overline{V}_j(Z \cap E) = c_{j,d}^{k,d-k+j}\overline{V}_{d-k+j}(Z).$$

Theorem 2.2 follows readily from Theorem 2.1. Due to stationarity one may assume $o \in E$. Then set $W = B_r(o) \cap E$ in the principal kinematic formula for random sets, divide by $\nu_k(W)$ and let r tend to infinity.

The concept of standard random sets is not suited for simulation purposes, as it cannot be described by a finite number of parameters. To obtain more accessible random sets, germ-grain models are employed. If $\varphi = \{\xi_1, \xi_2, \dots\}$ is a stationary point process in $\mathbb{R}^d$ and $K_0, K_1, \dots$ are i.i.d. nonempty compact random sets, independent of φ, the random set

$$Z = \bigcup_{i=1}^{\infty}(\xi_i + K_i)$$

is called a stationary *germ-grain model*. The points of φ are considered as germs to which the grains K_i are attached. The set K_0 is called the *typical grain* and its distribution will be denoted by $\mathbf{Q}$. If K_0 is almost surely convex, Z is called a *germ-grain model with convex grains*. We will always assume convexity. To assure that Z can be written as a union of finitely many grains $x_i + K_i$ when considered in a bounded window, a condition on $\mathbf{Q}$ is required. We assume throughout

$$\overline{V}_j(K_0) = \mathbf{E}V_j(K_0) < \infty \ \text{ for all } j = 0, \dots, d.$$

This condition is equivalent to saying that the mean number of grains $x_i + K_i$ that hit any bounded window is finite.

We will consider stationary germ-grain models for which the underlying point process is a Poisson process. These are called *stationary Boolean models*, and are examples of standard random sets. The iterated principal kinematic formula implies a wonderful result for the specific intrinsic volumes of Boolean models.

Theorem 2.3. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex typical grain K_0, based on a stationary Poisson point process Π_λ with intensity λ. Then*

$$\overline{V}_d(Z) = 1 - e^{-\lambda\overline{V}_d(K_0)}.$$

and

$$\overline{V}_{d-1}(Z) = \lambda\overline{V}_{d-1}(K_0)e^{-\lambda\overline{V}_d(K_0)}.$$

If $j \in \{0, \dots, d-2\}$ and K_0 is isotropic we have

$$\overline{V}_j(Z) = \lambda e^{-\lambda \overline{V}_d(K_0)} \left[\overline{V}_j(K_0) - c_j^d \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \lambda^{s-1} \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}(K_0) \right].$$

The constants appearing in the previous theorem are again given by (2). Note that they are slightly different from the incorrect constants in [22, Theorem 9.1.4].

Sketch. To avoid technicalities we assume that K_0 is almost surely contained in a ball $B_\delta(o)$ for some fixed $\delta > 0$. Then $Z \cap W = \bigcup_{i=1}^{\infty} [(\xi_i + K_i) \cap W]$ only depends on the Poisson process $\Pi_\lambda = \{\xi_1, \xi_2, \dots\}$ in the bounded window $W^\delta = W \oplus B_\delta(o)$. The number of points of $\Pi_\lambda \cap W^\delta$ is Poisson distributed with parameter $\lambda V_d(W^\delta)$, and, given this number is n, the n points of $\Pi_\lambda \cap W^\delta$ are i.i.d. uniform in W^δ. If these points are denoted by $\xi_1, \dots, \xi_n$, the inclusion-exclusion principle gives

$$\begin{aligned}
&\mathbf{E}[V_j(Z \cap W) | \#(\Pi_\lambda \cap W^\delta) = n] \\
&= \mathbf{E}\left[V_j\Big(\bigcup_{i=1}^{n} [(\xi_i + K_i) \cap W] \Big) | \#(\Pi_\lambda \cap W^\delta) = n \right] \\
&= \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < \dots < i_r \le n} \frac{\Phi_{i_1,\dots,i_r}(j)}{V_d(W^\delta)^r}
\end{aligned}$$

with

$$\begin{aligned}
&\Phi_{i_1,\dots,i_r}(j) \\
&\quad = \mathbf{E}_{K_{i_1},\dots,K_{i_r}} \int_{W^\delta} \cdots \int_{W^\delta} V_j(W \cap (K_{i_1} + x_{i_1}) \cap \dots \cap (K_{i_r} + x_{i_r}))\, dx_{i_1} \cdots dx_{i_r} \\
&\quad = \mathbf{E}_{K_1,\dots,K_r} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} V_j(W \cap (K_1 + x_1) \cap \dots \cap (K_r + x_r))\, dx_1 \cdots dx_r.
\end{aligned}$$

Here we used that $K_1, K_2, \dots$ are i.i.d., and contained in $B_\delta(o)$. Hence

$$\begin{aligned}
\mathbf{E}V_j(Z \cap W) &= \sum_{n=1}^{\infty} \frac{(\lambda V_d(W^\delta))^n}{n!} e^{-\lambda V_d(W^\delta)} \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \frac{\Phi_{1,\dots,r}(j)}{V_d(W^\delta)^r} \\
&= e^{-\lambda V_d(W^\delta)} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r!} \Phi_{1,\dots,r}(j) V_d(W^\delta)^{-r} \sum_{n=r}^{\infty} \frac{(\lambda V_d(W^\delta))^n}{(n-r)!} \\
&= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r!} \lambda^r \Phi_{1,\dots,r}(j).
\end{aligned}$$

For $j = d$ and $j = d-1$, the iterated principal kinematic formula without the average over all rotations can be applied to simplify $\Phi_{1,\dots,r}(j)$. For the volume, we have

$$\Phi_{1,\dots,r}(d) = \mathbf{E}_{K_1,\dots,K_r} V_d(W) V_d(K_1) \cdots V_d(K_r) = V_d(W) (\overline{V}_d(K_0))^r,$$

and for half the surface area we get

$$\Phi_{1,\dots,r}(d-1) = V_{d-1}(W)(\overline{V}_d(K_0))^r + r V_d(W) \overline{V}_{d-1}(K_0) (\overline{V}_d(K_0))^{r-1}.$$

Thus

$$\mathbf{E}V_d(Z \cap W) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r!}(\lambda \overline{V}_d(K_0))^r V_d(W) = V_d(W)\big(1 - e^{-\lambda \overline{V}_d(K_0)}\big)$$

and

$$\begin{aligned}\mathbf{E}V_{d-1}(Z \cap W) = V_{d-1}(W)&\big(1 - e^{-\lambda \overline{V}_d(K_0)}\big) \\ &+ V_d(W)\lambda \overline{V}_{d-1}(K_0) e^{-\lambda \overline{V}_d(K_0)}.\end{aligned}$$

Replacing W by rW, dividing by $\nu_d(rW)$ and letting r tend to infinity yields the claim for $j = d$ and $j = d-1$. For $j < d-1$, isotropy of K_0 implies that

$$\begin{aligned}\Phi_{i_1,\dots,i_r}(j) = \mathbf{E}_{K_1,\dots,K_r} &\int_{SO_d}\int_{\mathbb{R}^d} \cdots \int_{SO_d}\int_{\mathbb{R}^d} \\ &V_j(W \cap R_1(K_1 + x_1) \cap \cdots \cap R_r(K_r + x_r))\, dx_1\, dR_1 \cdots dx_r dR_r.\end{aligned}$$

The claim then follows in a similar way as before by applying the iterated principal kinematic formula and sorting the resulting expressions according to their homogeneity. In the final result s is the number of terms with homogeneity smaller than d. This concludes the sketch of the proof. □

Specialized to two dimensions, the formulas in Theorem 2.3 read

$\overline{V}_2(Z) = 1 - e^{-\lambda \overline{V}_2(K_0)}$	(specific area)
$2\overline{V}_1(Z) = 2\lambda \overline{V}_1(K_0) \cdot e^{-\lambda \overline{V}_2(K_0)}$	(specific perimeter)
$\overline{V}_0(Z) = e^{-\lambda \overline{V}_2(K_0)} \big(\lambda - \frac{1}{\pi}(\lambda \overline{V}_1(K_0))^2\big)$	(specific Euler characteristic)

The last relation requires isotropy. If all the quantities on the left side are known, these relations can be used to determine the mean intrinsic volumes of K_0 and the intensity of Π_λ. Hence, measurement (estimation) of the specific intrinsic volumes allows to estimate $\overline{V}_2(K_0)$, $\overline{V}_1(K_0)$ and λ, which determine all the parameters of Z if $\mathbf{Q}$ is a suitable distribution with at most two real parameters.

2.3 Design-based stereology

We know turn to design-based stereology, where the structure of interest is assumed to be a deterministic set, and the sampling is randomized in a suitable way. We have already derived the set of fundamental stereological formulas (5) from Crofton's formula, where the set $K \in \mathcal{R}$ was sampled by IUR k-flats. Recall that if K is contained in the reference space A, and E is an IUR k-flat in A, then

$$\left[\frac{c_{0,d}^{k,d-k}}{c_{j,d}^{k,d-k+j}} V_{d-k}(A)\right] V_j(K \cap E) \tag{9}$$

is an unbiased estimator for $V_{d-k+j}(K)$ for $0 \le j \le k < d$. This shows that $V_m(K)$ can unbiasedly be estimated from k-dimensional sections if $m \ge d-k$. For $m < d-k$, unbiased estimation of $V_m(K)$ from IUR k-flat sections is impossible: If K is a set with relative interior points contained in an m-dimensional subspace,

$m < d - k$, then $V_m(K) > 0$, but $K \cap E = \emptyset$ almost surely. In particular, the Euler characteristic $V_0(K)$ cannot be estimated from IUR sections. Therefore, the *disector technique* has been suggested in [23]. The basic idea is to work with hyperplanes and to replace the section plane by a pair of parallel $(d-1)$-flats (E, E_ε) of distance $\varepsilon > 0$ apart. The flats must be randomized, but averaging with respect to rotations is not required, so it is enough to choose E as a FUR (fixed orientation uniform) k-flat in A with $k = d-1$. A *FUR* k-flat E in A is obtained by uniformly translating a fixed subspace $L_0 \in \mathcal{L}_k^d$ with a translation vector in $x \in L_0^\perp$ such that $E = L_0 + x$ hits A. In other words, E has distribution

$$\mathbf{P}(E \in \cdot) = c(A)^{-1} \int_{A|L_0^\perp} \mathbf{1}\{L_0 + x \in \cdot\} dx,$$

where $c(A) = \nu_{d-k}(A|L_0^\perp)$ is the projected volume of A on $L_0^\perp$. To describe the disector let E be a FUR $(d-1)$-flat in A, parallel to some deterministic $L_0 \in \mathcal{L}_{d-1}^d$, and let $E_\varepsilon = E + \varepsilon u$, where $u \in L_0^\perp$ is a unit vector.

To fix ideas let K be a union of m disjoint convex particles $K_1, \dots, K_m$. Let N_{E,E_ε} be the number of particles that hit E, but not E_ε. Then $V_0(K) = m$ is the number of particles and can be estimated unbiasedly by

$$\widehat{V}_0 = \frac{c(A)}{\varepsilon} N_{E,E_\varepsilon}$$

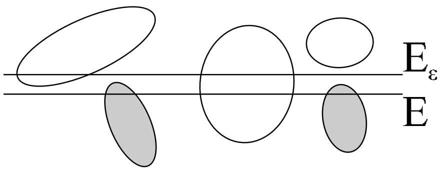

Figure 2: The disector technique: only the two shaded particles are counted.

if, almost surely, none of the particles is located between E and E_ε, that is, if the projected height of K_i on a line orthogonal to E is at least ε for all $i = 1, \dots, m$. If the approximate size of the particles is known, this can be achieved choosing ε small enough. The unbiasedness follows from

$$\varepsilon \mathbf{E} \widehat{V}_0 = \sum_{i=1}^m \int_{-\infty}^{\infty} \mathbf{1}\{(L + tu) \cap K_i \neq \emptyset\} \mathbf{1}\{(L + (t+\varepsilon)u) \cap K_i = \emptyset\}\, dt = m\varepsilon,$$

as the integrand is one exactly on an interval of length ε. In applications N_{E,E_ε} is often approximated by a comparison of $K \cap E$ and $K \cap E_\varepsilon$ using a priori information on the particles. However, strictly speaking, this estimator requires more information than just these intersections. To decide whether two profiles in E and E_ε originate from the same particle, the part of K between E and E_ε must be known. In a typical biological application, this is achieved using confocal microscopy. By continuously moving the focal plain from E_ε to E, one obtains N_{E,E_ε} by counting all particles that come into focus during this process. The method can be extended to sets K in more general set classes, but then, tangent points between the planes with normal u have to be counted according to whether they are convex, concave or of saddle type.

We return to the fundamental stereological formulas and discuss possible improvements. We restrict for illustration to perimeter estimation of $K \in \mathcal{R}$ from linear sections ($k = 1$) in the plane ($d = 2$). By (9) with $j = 0$ the random number

$$\hat{V}_1 = 2V_1(A)V_0(K \cap E) \tag{10}$$

is an unbiased estimator of the perimeter $2V_1(K)$ of $K \subset A \in \mathcal{R}$, if $E \in \mathcal{L}_1^2$ is IUR in A. To reduce the variance, one could repeat the measurements with n i.i.d. random lines E and consider the arithmetic mean of the corresponding estimates (10). However, the variance reduction is generally only of order $1/n$, as the estimates are uncorrelated. It may happen that some of the sampling lines are close to one another, and the corresponding intersection counts are therefore very similar and contain redundant information. In classical survey sampling one uses *systematic random sampling* in such situations: sampling from a linearly ordered population of units can generally be improved by choosing every m-th unit in both directions from a randomly selected starting unit, $m > 1$. This way, units that are close to one another (and tend to be similar) are not in the same sample, and samples become negatively correlated. This concept, transferred to the random translation of $E \subset \mathbb{R}^2$ leads to sampling with a IUR grid of lines of distance h apart:

$$G = \{\eta^\perp + (\xi + mh)\eta : m \in \mathbb{Z}\},$$

where η is uniform in $\mathbb{S}^1$, and ξ is independent of η and uniform in $[0, h]$. It is not difficult to show that

$$\mathbf{E}V_0(K \cap G) = \frac{1}{h} \int_{\mathcal{E}_1^2} V_0(K \cap E)\, dE = \frac{2}{\pi h} V_1(K), \qquad K \in \mathcal{R},$$

where we used Crofton's formula. Hence $\pi h V_0(K \cap G)$ is an unbiased estimator for the perimeter of K. This estimator is called *Steinhaus estimator* and does not involve any reference space A. Similar variance reduction procedures are possible in the case of sampling with k-flats in $\mathbb{R}^d$.

The assumption of IUR section planes is sometimes too strong: it is either impracticable or not desired to use fully randomized sections. For instance, when analyzing sections of the skin in biology it is natural to use sections parallel to a fixed axis, the normal of the skin surface. This way, different layers of tissue in the section can be distinguished more easily. The common axis is usually thought to be the vertical direction, and the samples are therefore called *vertical sections*. We restrict to planar vertical sections in three-dimensional space to avoid technicalities.

Let $L_0 \in \mathcal{L}_1^3$ be the vertical axis and A a Borel set in $\mathbb{R}^3$. A random 2-flat H in $\mathbb{R}^3$ is called a *VUR* (vertical uniform random) 2-flat in A if it has the natural distribution on

$$\{E \in \mathcal{E}_2^3 : E \cap A \neq \emptyset,\ E \text{ is parallel to } L_0\}.$$

Explicitly, $P(H \in \cdot)$ coincides up to a normalizing constant with

$$\int_{\mathcal{L}_2^{L_0}} \int_{A|L^\perp} \mathbf{1}\{L + x \in (\cdot)\}\, dx dL.$$

For $A \in \mathcal{K}$ the normalizing constant is $\pi/(2V_1(A|L_0^\perp))$. As vertical flats all contain the vertical axis, they are surely not IUR, so Crofton's formula cannot be applied directly. The key idea is to choose a random line E in H in such a way that E is IUR in $\mathbb{R}^3$ and apply Crofton's formula to E. Given H, this random line E will have a density with respect to the natural measure on $\mathcal{E}_1^H$, and this density can be determined using Blaschke-Petkantschin formulas.

Let $K \in \mathcal{R}$ be contained in the reference space $A \in \mathcal{K}$, and fix a vertical axis $L_0 \in \mathcal{L}_1^3$. From (7) with $d = 3$,

$$f(E) = V_0(K \cap E)\mathbf{1}\{E \cap A \neq \emptyset\},$$

and Crofton's formula

$$\int_{\mathcal{L}_2^{L_0}} \int_{A|L^\perp} \int_{\{E \in \mathcal{E}_1^{L+x} : E \cap A \neq \emptyset\}} V_0(K \cap E)[E, L_0]\, dE\, dx\, dL$$
$$= \frac{2}{\pi} \int_{\mathcal{E}_1^3} V_0(K \cap E)\, dE = \frac{1}{\pi} V_2(K).$$

Hence, if H is a VUR 2-flat in A with vertical axis L_0,

$$\mathbf{E} \int_{\{E \in \mathcal{E}_1^H : E \cap A \neq \emptyset\}} V_0(K \cap E)[E, L_0]\, dE = \frac{1}{2V_1(A|L_0^\perp)} V_2(K).$$

This can be interpreted as follows. Let H be a VUR 2-flat in A with vertical axis L_0, and let $E \in \mathcal{E}_1^H$ be a random 1-flat hitting A with density $[E, L_0]$ with respect to the invariant measure on $\mathcal{E}_1^H$. Then $4V_1(A|L_0^\perp)V_0(K \cap E)$ is an unbiased estimator of the surface area $2V_2(K)$ of K. In applications, one usually counts the number of intersections of E with the boundary of K. This number coincides almost surely with $2V_0(K \cap E)$. Instead of sine-weighted test lines in H a test curve, the *cycloid* is used. It incorporates the weighting, as its orientation distribution is proportional to the sine. Variance reduction can be achieved by systematic random sampling: instead of only counting intersections with one cycloid, intersections with a periodic grid of cycloids are determined.

The last stereological concept that we will discuss here is the so-called local design. It is again motivated by applications: When sampling a biological cell it is convenient to consider only sections of the cell with planes through a given reference point, which usually is the cell nucleus or the nucleolus. For a mathematical description we assume that the reference point is the origin. The branch of stereology dealing with inference on $K \in \mathcal{R}$ from sections $K \cap L$, $L \in \mathcal{L}_r^d$, $1 \le r \le d-1$, is called *local stereology*. Like in the case of vertical sections, Crofton's formula cannot be applied directly, but only after a sub-sampling in L with a suitably weighted affine plane. Theorem 1.7 and Crofton's formula imply for $0 \le j \le k < r \le d-1$

$$\int_{\mathcal{L}_r^d} \int_{\mathcal{E}_k^L} V_j(K \cap E) d(E, o)^{d-r}\, dE\, dL$$
$$= \frac{\omega_{r-k}}{\omega_{d-k}} \int_{\mathcal{E}_k^d} V_j(K \cap E)\, dE = \frac{\omega_{r-k}}{\omega_{d-k}} c_{j,d}^{k,d-k+j} V_{d-k+j}(K).$$

Stereologically this can be interpreted as follows: Let $K \in \mathcal{R}$ be contained in the reference space $B_o(s)$ with $s > 0$. Let $L \in \mathcal{L}_r^d$ be an isotropic random plane. Give L, let E be a random plane in L with density proportional to $\mathbf{1}\{E \cap B_o(s) \neq \emptyset\}\, d(E, o)^{d-r}$

with respect to the invariant measure on $\mathcal{E}_k^L$. Then $cV_j(K \cap E)$ is an unbiased estimator for $V_{d-k+j}(K)$, where the constant is given by

$$c = \binom{r}{k} \frac{\omega_{d-k}}{\omega_{r-k}} c_{0,r,d-k+j}^{j,r-k,d} \frac{\kappa_r}{\kappa_k} s^{r-k}.$$

Note that $(K \cap L) \cap E = K \cap E$, so the estimator depends on K only through $K \cap L$. The intrinsic volume $V_m(K)$ can be estimated from r-dimensional isotropic sections with the above formula only if $m > d - r$. That there cannot exist any unbiased estimation procedure for $m \leq d - r$ is clear: for an m-dimensional ball K contained in a m-dimensional linear subspace, we have $K \cap E = \{o\}$ almost surely, so the radius of K is almost surely invisible in the sections.

Further reading

1. Besides the monograph of Schneider and Weil [22] on stochastic geometry and integral geometry, the classical book of Stoyan, Kendall, and Mecke [24] is recommended as reference for the model-based approach. Concerning design-based stereology, Baddeley and Jensen's monograph [5] includes also recent developments.

2. We introduced the specific intrinsic volumes of a standard random set Z. It is shown in [28] that corresponding specific φ-values of Z exists whenever φ is an additive, translation invariant functional on $\mathcal{R}$ satisfying a certain boundedness condition. Specific intrinsic volumes can also be defined as Lebesgue densities of average curvature measures of Z; see e.g. Corollary 9.4.1 and the references in [22]. As intrinsic volumes are also called quermass integrals, one finds the notion *quermass densities* for the specific intrinsic volumes in the earlier literature.

3. Using translative integral formulas, Theorem 2.1 was generalized to curvature measures of standard random sets that are not necessarily isotropic in [25]. In [26] Theorem 2.3 is generalized to stationary Boolean models that are not necessarily isotropic. It is shown that at least for small dimensions ($d \leq 4$), the underlying intensity is still determined by the Boolean model, but an estimation procedure would require more than just the measurement of the specific intrinsic volumes.

4. Vertical section designs in general dimensions are developed in [2]. Test systems in the vertical plane and practical sampling procedures are explained in [4].

5. The monograph [7] is an excellent introduction to local stereology, focusing on formulas for k-dimensional Hausdorff measures instead of intrinsic volumes. Such relations are based on generalized Blaschke-Petkantschin formulas for Hausdorff measures. A local stereological formula for the intrinsic volumes, as presented here, is a relatively recent development taken from [12, 1] based on ideas in [6].

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