THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Failure Recovery via RESTART: Wallclock Models

Søren Asmussen and Anders Rønn-Nielsen

10

Research Report

No. 04 I March 2010

Failure Recovery via RESTART: Wallclock Models

Søren Asmussen Anders Rønn-Nielsen

Dept. of Mathematical Sciences, Aarhus University,

Ny Munkegade, DK-8000 Aarhus C, Denmark;

{asmus,anders}@imf.au.dk

Abstract

A task such as the execution of a computer program or the transfer of a file on a communications link may fail and then needs to be restarted. Let the ideal task time be a constant ℓ and the actual task time X, a random variable. Tail asymptotics for $\mathbb{P}(X > x)$ is given under three different models: **1**: a time-dependent failure rate $\mu(t)$ and a constant work rate $r(t) \equiv 1$; **2**: Poisson failures and a time-dependent deterministic work rate $r(t)$; **3**: as **2**, but $r(t)$ is random and a function of a finite Markov process. Also results close to being necessary and sufficient are presented for X to be finite a.s. The results complement those of Asmussen, Fiorini, Lipsky, Rolski & Sheahan [*Math. Oper. Res.* **33**, 932–944, 2008] who took $r(t) \equiv 1$ and assumed the failure rate to be a function of the time elapsed since the last restart rather than wallclock time

Keywords change of measure, computer reliability, fluid model, gaps, inhomogeneous Poisson process, Markov-modulation, Markov renewal theorem, tail asymptotics, time transformation

1 Introduction and Statement of Results

Tasks such as the execution of a computer program or the transfer of a file on a communications link may fail. There is a considerable literature on protocols for handling such failures. We mention in particular RESUME where the task is resumed after repair, REPLACE where the task is abandoned and a new one taken from the pile of waiting tasks, RESTART where the task needs to be restarted from scratch, and CHECKPOINTING where the task contains checkpoints such that performed work is saved at checkpoint times and that upon a failure, the task only needs to be restarted from the last checkpoint.

The protocols RESUME and REPLACE are fairly easy to analyze, see e.g. Kulkarni *et al.* [15], [16] and Bobbio & Trivedi [8]. In contrast, RESTART (Castillo [9], Chimento & Trivedi [10]) resisted analysis for a long time until the recent work of Sheahan *et al.* [18] and Asmussen *et al.* [5] (see also Jelenkovic & Tan [13, 14] for in part parallel work). Recent results for CHECKPOINTING as well as references to earlier work can be found in Asmussen & Lipsky [6].

The model of Asmussen *et al.* [5] assumes that failures occur at a time after each restart with the same distribution G for each restart (a particular important case is of course the exponential distribution). However, it is easy to imagine situations where the model behaviour is determined by the time of the day (the clock on the wall) rather than the time elapsed since the last restart. Think, e.g., of a time-varying load in the system which may influence the failure rate and/or the speed at which the task is performed. For example, the load could be identified with the number of busy tellers in a call centre or the number of users in a LAN (local area network) currently using the central server. The purpose of the present paper is to provide some first insight in the behaviour of such models.

We denote by X the total task time, including failures (a precise definition is given below). One of our goals is to describe the asymptotics of the tail $\mathbb{P}(X > x)$ as $x \to \infty$. For simple restart with constant task time and Poisson failures, this is easy via a renewal argument. In fact, the details as given in [5] lead to:

Proposition 1.1. *Consider simple RESTART with ideal task time ℓ and Poisson(μ^*) failures. Let $\gamma_0 = \gamma_0(\ell, \mu^*) > 0$ denote the root of*

$$1 \;=\; \int_0^\ell \mu^* \mathrm{e}^{(\gamma_0 - \mu^*)y}\, \mathrm{d}y\,. \tag{1}$$

Then $\mathbb{P}(X > x) \sim c_0 \mathrm{e}^{-\gamma_0 x}$ as $x \to \infty$ for some $0 < c_0 < \infty$

Here and in the following $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$. Similarly, we will write $f(x) \approx_{\log} g(x)$ if $\log f(x) \sim \log g(x)$. This is the logarithmic asymptotics familiar from large deviations theory (though we will not use results or tools from that area!). It summarizes the main asymptotical features, but does not allow to capture constants like c_0, prefactors of smaller magnitude etc.

It should be noted that c_0 is explicit given γ, but the value needs not concern us here.

The emphasis in [5] is on the more difficult case of a random rather than a constant ideal task time. However as a first attempt, we shall in the present paper throughout assume a constant ideal task time of length ℓ. We will consider three models:

Model 1 Failures at time t after the start of the task occur at deterministic rate $\mu(t)$ and the system works with a constant rate $r(t) \equiv 1$.

Model 2 Failures occur according to a Poisson(μ^*) process with constant rate μ^*. At time t after the start of the task, the system works on the task at rate $r(t)$.

Model 3 As Model 2, but the rate function $r(t)$ is given as $r(t) = r_{V(t)}$ where $\{V(t)\}_{t\geq 0}$ is an ergodic Markov process with $p < \infty$ states and $r_1, \ldots, r_p$ are constants with $r_i > 0$ for at least one i.

Models 1, 2 are self-explanatory. Model 3 could for example describe a LAN with p users, where $V(t)$ is the number of users currently using the central server and $r_0 = 0$, $r_i = r_1/i$ for $i > 1$.

Models 1 and 2 exhibit a feature not found in simple RESTART: it is possible that $\mathbb{P}(X = \infty) > 0$. This would occur in Model 1 if $\mu(t) \to \infty$ fast enough, and in Model 2 if $r(t) \to 0$ fast enough. Our first main result gives the critical rates:

Theorem 1.1. *(1) Consider Model 1. If* $\limsup_{t\to\infty} \mu(t)/\log t < 1/\ell$*, then* $X < \infty$ *a.s., whereas* $\mathbb{P}(X = \infty) > 0$ *if* $\liminf_{t\to\infty} \mu(t)/\log t > 1/\ell$*.*
(2) Consider Model 2 and assume that $\int_0^\infty r(s)\,\mathrm{d}s = \infty$ *and* $R(t) = \int_0^t r(s)\mathrm{d}s < \infty$ *for all* $t \geq 0$*. If* $\liminf_{t\to\infty} r(R(t)) \log t/\mu^* > \ell$*, then* $X < \infty$ *a.s., whereas* $\mathbb{P}(X = \infty) > 0$ *if* $\limsup_{t\to\infty} r(R(t)) \log t/\mu^* < \ell$*.*

The result shows that in Model 1 only a very modest rate of increase to ∞ of $\mu(t)$ may cause the task never to terminate, and that the same is the case for Model 2 with only a very modest rate of decrease to 0 of $r(t)$. In view of this, it seems reasonable to concentrate on decreasing $\mu(t)$ in Model 1 and increasing $r(t)$ in Model 2. The simplest case is of course the power case, and our second main result gives the asymptotics of $\mathbb{P}(X > x)$ in this case:

Theorem 1.2. *(1) Consider Model 1 and assume that* $\mu(t)$ *is strictly positive with* $\mu(t) \sim at^{-\beta}$ *with* $0 < \beta < 1$*. Then*

$$\mathbb{P}(X > x) \approx_{\log} \mathrm{e}^{-c_1 x \log x} = x^{-c_1 x},$$

where $c_1 = (1 - \beta)/\ell$*.*
(2) Consider Model 2 and assume that $r(t) \sim at^\eta$ *with* $\eta > 0$*. Then*

$$\mathbb{P}(X > x) \approx_{\log} \mathrm{e}^{-c_2 x^{\eta+1} \log x} = x^{-c_2 x^{\eta+1}},$$

where $c_2 = a\eta/(\eta + 1)\ell$*.*

Note that $\beta = 0$ in (1) or $\eta = 0$ in (2) corresponds to the standard RESTART setting, which is why we exclude these cases. Note also that in both Model 1 and Model 2 the decay rate is faster than any exponential. In Model 1 this is intuitive by comparing with Proposition 1.1 since $\gamma \to \infty$ as $\mu \to 0$ with ℓ fixed. This is also the intuitive explanation in Model 2, but to see this, one needs an intermediate step of time reversal given below.

For Model 3 it is trivial that $X < \infty$ a.s. because there is an infinity of sojourn periods in the state with $r_i > 0$ and the probability of task completion in such a period is > 0. For the asymptotics, we need properties of the fluid model

$$F(t) = \int_0^t r_{V(s)}\,\mathrm{d}s.$$

More precisely:

Theorem 1.3. *In Model 3, let* $\kappa(s)$ *denote the largest real value of the* $p \times p$ *matrix* $\boldsymbol{K}[s]$ *with* ij*th element*

$$\int_0^\infty \mu^* \mathrm{e}^{(s-\mu^*)t} \mathbb{P}_i\Big(F(t) < \ell, V(t) = j\Big)\,\mathrm{d}t.$$

Then $\kappa(s)$ *increases monotonically from* $\kappa(0) < 1$ *to* ∞ *in the interval* $s \in [0, \infty)$*. If* γ_3 *denotes the unique value with* $\kappa(\gamma_3) = 1$*, then* $\mathbb{P}_i(X > x) \sim d_i \mathrm{e}^{-\gamma_3 x}$ *for suitable constants* $d_1, \ldots, d_p$*.*

Here $d_1, \dots, d_p$ are again explicit, see Section 4, and as usual, $\mathbb{P}_i$ refers to the case $V(0) = i$.

The outline of proofs is that first Model 1 is considered (Section 2). The results for Model 2 then follow by exploiting the time-transformation connection between homogeneous and inhomogeneous Poisson processes (Section 3). Theorem 1.3 for Model 3 is an easy consequence of the Markov renewal theorem, once it has been recognized how to write up an appropriate Markov renewal equation.

Finally, Section 4 also contains a numerical example.

Notation For the Poisson process with constant rate μ^*, we write $S_1^*, S_2^*, \dots$ for the event times and $U_n^* = S_n^* - S_{n-1}^*$ for the interevent times ($S_0^* = 0$ is not considered an event time). Similarly, the notation $S_1, S_2, \dots$ and $U_n = S_n - S_{n-1}$ is used for the inhomogeneous Poisson process of failures in Model 1, and $S_1', S_2', \dots$ and $U_n' = S_n' - S_{n-1}'$ for a certain auxiliary inhomogeneous Poisson process with rate function $\mu'(s)$ in Section 2. The corresponding counting processes are denoted by $N^*(t), N(t), N'(t)$.

2 Proofs: Model 1

Let $N(t)$ denote the number of failures before t. Then the counting process $\{N(t)\}_{t\ge 0}$ given by $N(t) = \sup\{n : S_n < t\}$ is a time-inhomogeneous Poisson process with rate function $\{\mu(t)\}_{t\ge 0}$.

Define the stopping time $\tau = \inf\{n \in \mathbb{N} \,|\, U_n > \ell\}$. Then the total task time is the r.v. $X = S_{\tau-1} + \ell$ if $\tau < \infty$ and $X = \infty$ otherwise. Note that the problem of computing the r.v. $S_{\tau-1}$ may be seen as an inhomogeneous version of the classical problem of first gaps in homogeneous Poisson processes 378, 468–469

In the proof of Theorem 1.1(1) and in the following, define the integrated intensity as $M(t) = \int_0^t \mu(s)\,\mathrm{d}s$. It is then standard that $\{N(t)\}_{t\ge 0}$ can be represented by taking the event times as $S_n = M^{-1}\big(S_n^*\big)$.

Proof of Theorem 1.1. Let ℓ' be fixed, let $\mu'(s) = (\log s)^+/\ell'$ and define X', M', U_n', S_n' etc. the obvious way (the ideal task time remains ℓ, not ℓ'!). Then for $s > 1$,

$$
\begin{aligned}
M'(s) &= s \log s/\ell' + \mathrm{O}(1)\,, \\
M'(s+\ell) &= (s+\ell)\Big[\log s + \mathrm{O}(1/s)\Big]/\ell' \;+\; \mathrm{O}(1) \;=\; (s+\ell)\log s/\ell' \;+\; \mathrm{O}(1)\,,
\end{aligned}
$$

and hence

$$
\int_1^\infty \mu'(s) \exp\{-M'(s+\ell) + M'(s)\}\,\mathrm{d}s \tag{2}
$$

$$
= \int_1^\infty \mathrm{O}(1) \log s \cdot s^{-\ell/\ell'}\,\mathrm{d}s \begin{cases} < \infty & \text{if } \ell' < \ell \\ = \infty & \text{if } \ell' > \ell \end{cases} . \tag{3}
$$

For the intuition, note that (2) equals $\mathbb{E}\sum_1^\infty\{n : U_n' > \ell\}$, the expected number of interevent intervals for the N' process that potentially could complete the task.

Assume first $\ell' < \ell$ and let $A'(s)$ be event that $U_n' \le \ell$ for all n with $S_{n-1}' \le s$. Clearly, $\mathbb{P}\big(A'(s)\big) > 0$. Defining $K'(s)$ as the number of n with $S_{n-1}' > s$, $U_n' > \ell$

and letting $\mathcal{F}'(s) = \sigma\Big(N'(v) : v \le s\Big)$, we have

$$\mathbb{E}\Big[K'(s) \,\Big|\, \mathcal{F}'(s)\Big] \;\le\; \int_s^\infty \mu'(v) \exp\{-M'(v+\ell) + M'(v)\}\,\mathrm{d}v\,.$$

By equation (3), we can choose s so large that this integral is (say) $< 1/2$ and get $\mathbb{P}\Big(K'(s) \ge 1 \,|\, \mathcal{F}'(s)\Big) \le 1/2$ such that

$$\begin{aligned}\mathbb{P}(X' = \infty) &= \mathbb{P}\Big(A'(s) \cap \{K'(s) = 0\}\Big)\\ &= \mathbb{E}\Big[I\Big(A'(s)\Big) \cdot \mathbb{P}\Big(K'(s) = 0\Big) \,\Big|\, \mathcal{F}'(s)\Big] \ge \mathbb{P}\Big(A'(s)\Big)/2 > 0\,.\end{aligned}$$

Let next $\ell' > \ell$. The above estimates for M' imply that ${M'}^{-1}(s) = s\ell'/\log s(1+\mathrm{o}(1))$ as $s \to \infty$, and hence that

$$S'_{n-1} \;=\; \frac{S^*_{n-1}\ell'}{\log S^*_{n-1}}\Big(1 + \mathrm{o}(1)\Big) \;=\; \frac{n\ell'}{\log n}\Big(1 + \mathrm{o}(1)\Big) \;\text{ a.s.}$$

Thus

$$\begin{aligned}\sum_{n=1}^\infty \mathbb{P}\Big(U'_n > \ell \,\Big|\, \mathcal{F}'(S'_{n-1})\Big) &\;=\; \sum_{n=1}^\infty \exp\{-M'(S'_{n-1} + \ell) + M'(S'_{n-1})\}\\ &= \sum_{n=1}^\infty \exp\{-\ell \log S'_{n-1}/\ell' + \mathrm{O}(1)\} \;=\; \infty \text{ a.s.}\end{aligned}$$

The conditional Borel-Cantelli lemma therefore implies that $U'_n > \ell$ for infinitely many n.

Now consider a general $\mu(s)$. If $\limsup_{s\to\infty} \mu(s)/\log s < 1/\ell$, then for some s_0 and some $\ell' > \ell$ we have $\mu(s) < \mu'(s)$ for all $s > s_0$. Then, realizing N' on (s_0, ∞) as the independent sum of N and an inhomogeneous Poisson process with rate $\mu'(s) - \mu(s)$, we may assume

$$\{S_{n-1} : S_{n-1} > s_0\} \;\subseteq \{S'_{n-1} : S'_{n-1} > s_0\}\,.$$

Since $U'_n > \ell$ for infinitely many n with $S'_{n-1} > s_0$, this implies $U_n > \ell$ for infinitely many n with $S_{n-1} > s_0$ and $X < \infty$. Similarly, if $\liminf_{s\to\infty} \mu(s)/\log s > 1/\ell$, then for some s_0 and some $\ell' > \ell$ we have $\mu(s) > \mu'(s)$ for all $s > s_0$, and $U_n > \ell$ for some n with $S_{n-1} > s_0$ implies $S'_n > \ell$ for some n with $S'_{n-1} > s_0$. Therefore the event that $U_n > \ell$ for some n with $S_{n-1} > s_0$ cannot have probability one, which as above implies $\mathbb{P}(X = \infty) > 0$. □

We next consider the proof of Theorem 1.2 (1), describing the tail of X in the most standard case, a Weibull type rate function $\mu(t) \sim at^{-\beta}$ with $0 < \beta < 1$. Note that $\beta = 0$ corresponds to the simple RESTART setting with Poisson failures with $\mu^* = a$. $\beta < 0$ is excluded because then $\mathbb{P}(X = \infty) > 0$, and $\beta > 1$ is excluded because then $M(\infty) < \infty$, a case that appears somewhat pathological and that we do not study.

Before turning to the setup of Theorem 1.2 (1) we shall prove some less clear results for a general $\mu(t)$. Assume that $\mu(t)$ is decreasing with limit 0.

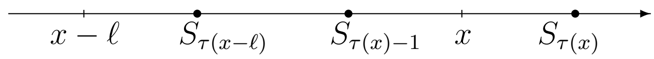

Figure 1: It holds that $x - S_{\tau(x)-1} \leq \ell$ when $U_{\tau(x-\ell)} \leq \ell$.

The probability $\mathbb{P}(X > x)$ can be written as $\mathbb{P}(X > x) = \mathbb{P}\big(B(x)\big)$ where

$$B(x) = \Big\{U_1 \leq \ell, \ldots, U_{\tau(x)-1} \leq \ell, x - S_{\tau(x)-1} \leq \ell\Big\}, \quad \tau(x) \;=\; \inf\{n : S_n > x\}\,.$$

Obviously, we must have $B(x) \subseteq C(x - \ell)$ where

$$C(x) = \{U_1 \leq \ell, \ldots, U_{\tau(x)} \leq \ell\}\,.$$

But in fact $x \geq S_{\tau(x-\ell)} \geq x - \ell$ implies that $U_{\tau(x-\ell)+1} \leq \ell, \ldots, U_{\tau(x)-1} \leq \ell$ and $x - S_{\tau(x)-1} \leq \ell$ (see Figure 1). That is,

$$B(x) = C(x - \ell)\,, \tag{4}$$

so deriving the asymptotics for $\mathbb{P}\big(C(x - t)\big)$ will solve the problem.

Choose $\gamma = \gamma(\ell)$ such that

$$1 = \int_0^\ell \mathrm{e}^{\gamma x}\,\mathrm{d}x = \frac{1}{\gamma}\Big[\mathrm{e}^{\gamma\ell} - 1\Big]\,. \tag{5}$$

Let $\mathbb{Q}$ be the probability measure where $(U_n)_{n\in\mathbb{N}}$ are i.i.d. each with density $u \to \mathrm{e}^{\gamma u}$ on $(0, \ell)$. Note that $\mathbb{Q}\big(C(x)\big) = 1$ since $\mathbb{Q}(U_1 \leq \ell) = 1$.

Proposition 2.1. *For any $\mu(t)$ it holds that*

$$\mathbb{P}\big(C(x)\big) \;=\; \mathbb{E}_{\mathbb{Q}}\Big[\Big(\prod_{k=1}^{\tau(x)} \mu(S_n)\Big) \exp\Big(-M(S_{\tau(x)}) - \gamma S_{\tau(x)}\Big)\Big]\,.$$

Proof. Define $(\mathcal{F}_n)_{n\in\mathbb{N}}$ as the natural filtration for $(U_n)_{n\in\mathbb{N}}$:

$$\mathcal{F}_n = \sigma(U_1, \ldots, U_n) \qquad (n \in \mathbb{N})\,.$$

Note that $\tau(x)$ is a stopping time with $\mathbb{P}(\tau(x) < \infty) = 1$ (such that $C(x)$ is well-defined) and that $C(x) \in \mathcal{F}_{\tau(x)}$.

Given S_{n-1} , the conditional distribution of U_n is determined by the density

$$f_{|S_{n-1}}(s) = \mu(s)\exp\Big(-M(s) + M(S_{n-1}))\Big) \qquad (s > S_{n-1})$$

(on $\{S_{n-1} < \infty\}$ in the case of $M(\infty) < \infty$). If $M(\infty) < \infty$, this distribution is defective with

$$\mathbb{P}_{|S_{n-1}}(U_n = \infty) = \exp\Big(-\int_{S_{n-1}}^\infty \mu(u)\,\mathrm{d}u\Big) = \exp\Big(-M(\infty) + M(S_{n-1}))\Big)\,.$$

Therefore the joint density of $(U_1,\dots,U_n)$ (w.r.t. the Lebesgue measure on $(0,\infty)^n$) is

$$
\begin{aligned}
g_n(u_1,\dots,u_n) = \prod_{k=1}^{n} g_{|s_{k-1}}(u_k) \ &= \ \prod_{k=1}^{n} \mu(s_k)\exp\Big(-M(s_k)+M(s_{k-1})\Big) \\
&= \Big(\prod_{k=1}^{n}\mu(s_k)\Big)\exp\{-M(s_n)\}
\end{aligned}
$$

for $(u_1,\dots,u_n)\in(0;\infty)^n$. Here the notation $s_k = u_1+\dots+u_k$ has been used.

Under $\mathbb{Q}$ the vector $(U_1,\dots,U_n)$ has density

$$
h_n(u_1,\dots,u_n) = \mathrm{e}^{\gamma s_n}\,.
$$

Define

$$
F_n = \{U_1\le \ell,\dots,U_n\le \ell\}\,.
$$

Then $F_n\in\mathcal{F}_n$. If $D_n\subseteq F_n$ and $D_n\in\mathcal{F}_n$ (that is $D_n=\{(U_1,\dots,U_n)\in C_n\}$, where $C_n\subseteq[0,\ell]^n$ is Borel measurable), we have

$$
\mathbb{P}(D_n) = \mathbb{E}_{\mathbb{Q}}\Big[\frac{g_n(U_1,\dots,U_n)}{h_n(U_1,\dots,U_n)};D_n\Big]\,.
$$

Thus by a standard extension to stopping times (e.g. [4] pp. 131–132)

$$
\mathbb{P}\big(C(x)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\frac{g_{\tau(x)}(U_1,\dots,U_{\tau(x)})}{h_{\tau(x)}(U_1,\dots,U_{\tau(x)})}\Big]\,,
$$

where we have used that $\mathbb{Q}\big(C(x)\big)=1$ and $\mathbb{P}(\tau(x)<\infty)=\mathbb{Q}(\tau(x)<\infty)=1$. When the expressions for g_n and h_n are inserted, this becomes the requested result. $\square$

Proposition 2.2. *If $\mu(t)$ is decreasing with limit 0 then*
(i)

$$
\mathbb{P}\big(C(x)\big) \ \le\ c_3\Big(\prod_{k=0}^{[x/\ell]}\mu(x-k\ell)\Big)\exp\Big(-M(x)-\gamma x\Big)
$$

as $x\to\infty$, for some constant c_3.
(ii)

$$
\begin{aligned}
\mathbb{P}\big(C(x)\big) \ \ge\ & c_4\exp\Big(\frac{1}{\ell}\log\Big(\mu(x+\ell)\Big)(1+\mathrm{o}(1))(x+\ell)\Big) \\
&\times\ \exp\Big(-M(x+\ell)-(1-1/\ell)\gamma x\Big)
\end{aligned}
$$

as $x\to\infty$, for some constant c_4.

Proof. For (i), recall that $x<S_{\tau(x)}\le x+\ell$. If either $\gamma>0$ or $\gamma<0$, the expression for $\mathbb{P}\big(C(x)\big)$ in Proposition 2.1 is bounded up by a constant times

$$
\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)}\mu(S_n)\Big]\exp\Big(-M(x)-\gamma x\Big) \tag{6}
$$

Left is exploring the behaviour of the expectation in (6). First rewrite it as

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)}\mu(S_k)\Big] = \mathbb{E}_{\mathbb{Q}}\Big[\Big(\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big)\Big(\prod_{k=\tau(x)-\lfloor x/\ell\rfloor}^{\tau(x)}\mu(S_k)\Big)\Big]$$
$$\leq \mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big]\Big(\prod_{k=0}^{\lfloor x/t\rfloor}\mu(x-k\ell)\Big).$$

In the inequality we have used that μ is decreasing and the fact that

$$S_{\tau(x)} > x,\ S_{\tau(x)-1} > x-\ell\ ,\ S_{\tau(x)-2} > x-2\ell\ ,\ldots,\ S_{\tau(x)-\lfloor x/\ell\rfloor} > x-\lfloor x/\ell\rfloor\ell\,.$$

Since $\lim_{t\to\infty}\mu(t) = 0$ the second factor above obviously decreases – very fast – to 0. We show that the first factor – the expectation – decreases to 0 and thereby is bounded such that we have the result from the theorem.

We have

$$\frac{\tau(x)}{x/\mathbb{E}_{\mathbb{Q}}[U_1]} \stackrel{x\to\infty}{\longrightarrow} 1 \quad \mathbb{Q}\text{–a.s.}$$

Since $\mathbb{E}_{\mathbb{Q}}[U_1] < \ell$ and therefore $x/\mathbb{E}_{\mathbb{Q}}[U_1] - \lfloor x/\ell\lfloor\to\infty$, this yields that

$$\tau(x) - \Big\lfloor\frac{x}{\ell}\Big\rfloor \stackrel{x\to\infty}{\longrightarrow} \infty \quad \mathbb{Q}\text{–a.s.}$$

Together with the fact that $S_n\to\infty$ $\mathbb{Q}$–a.s., this leads to

$$\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k) \stackrel{x\to\infty}{\longrightarrow} 0 \quad \mathbb{Q}\text{–a.s.}$$

because the factors in the product decrease to 0 and $\tau(x)-\lfloor x/\ell\rfloor-1\to\infty$. Now let $a>0$ be a constant such that $\mu(t)<1$ for $t>a$ and define the stopping time

$$\sigma = \inf\{n\in\mathbb{N}\,|\,S_n>a\}\,.$$

Then we have the following upper bound for the integrand in the expectation:

$$\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k) = \Big(\prod_{k=1}^{(\tau(x)-[x/\ell]-1)\wedge\sigma}\mu(S_k)\Big)\Big(\prod_{k=(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big)$$
$$\leq \prod_{k=1}^{(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}\lambda(S_k) \leq \prod_{k=1}^{(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}\mu(0) \leq \mu(0)^{\sigma}\,.$$

From Lemma 5.1 in the Appendix we have that this upper bound has finite expectation. Hence by dominated convergence we can conclude that

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big] \xrightarrow{x\to\infty} 0\,.$$

(ii): As in (i) we have

$$\mathbb{P}\big(C(x)\big) \;=\; \mathbb{E}_{\mathbb{Q}}\Big[\Big(\prod_{k=1}^{\tau(x)}\mu(S_n)\Big)\exp\big(-M(S_{\tau(x)})-\gamma S_{\tau(x)}\big)\Big]$$

and similarly the r.h.s. is bounded below by a constant times

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)}\mu(S_k)\Big]\exp\Big(-M(x+\ell)-\gamma x\Big)\,. \tag{7}$$

Recall that $S_k \le x+\ell$ on $\{\tau(x)\ge k\}$ so that a lower bound for (7) is

$$\mathbb{E}_{\mathbb{Q}}\Big[(\mu(x+\ell)^{\tau(x)}\Big]\exp\Big(-\Lambda(x+\ell)-\gamma x\Big)\,.$$

From Proposition 5.1 in the Appendix we have that this is bounded below by

$$\exp\Big(-\varphi^{-1}(\mu(x+\ell))(x+\ell)\Big)\exp\Big(-M(x+\ell)-\gamma x\Big)$$

with

$$\varphi(z) \;=\; \frac{z+\gamma-1}{\mathrm{e}^{\ell(z+\gamma-1)}-1}\,.$$

Combined with the result from Proposition 5.2 in the Appendix this gives

$$\mathbb{P}\Big(C(x)\Big) \;\ge\; \exp\Big(\frac{1}{\ell}\log(\mu(x+\ell))(1+\mathrm{o}(1))(x+\ell)\Big)\exp\Big(-M(x+\ell)-\gamma x\Big)$$

when $x\to\infty$ (remember that $\mu(x)\to 0$). $\square$

$\square$

As a result of Proposition 2.2 and (4), we immediately get

Corollary 2.1. *In the setup from above we have that*

(i) $\mathbb{P}\Big(A(x)\Big) \;\le\; c_5\Big(\prod_{k=1}^{\lfloor x/\ell\rfloor}\mu(x-(k+1)\ell)\Big)\exp\Big(-M(x-\ell)-\gamma x\Big)$ *as* $x\to\infty$, *for some constant* c_5.

(ii)

$$\mathbb{P}\Big(A(x)\Big) \;\ge\; c_6\exp\Big(\frac{1}{\ell}\log\Big(\mu(x)\Big)\Big(1+\mathrm{o}(1)\Big)x\Big)\exp\Big(-M(x)-(1-1/\ell)\gamma x\Big)$$

as $x\to\infty$, *for some constant* c_6.

In the case where $M(\infty)<\infty$ the result becomes simpler:

Corollary 2.2. *If furthermore* $M(\infty)<\infty$ *it holds that*

(i) $\mathbb{P}(X>x) \;\le\; c_7\Big(\prod_{k=1}^{\lfloor x/\ell\rfloor}\mu(x-(k+1)\ell)\Big)\exp\Big(-\gamma x\Big)$ *as* $x\to\infty$, *for some constant* c_7.

(ii)

$$\mathbb{P}(X>x) \;\ge\; c_8\exp\Big(\frac{1}{\ell}\log\Big(\mu(x)\Big)(1+o(1))x\Big)\exp\Big(-(1-1/\ell)\gamma x\Big)$$

as $x\to\infty$, *for some constant* c_8.

Proof of Theorem 1.1(i). If the intensity process is strictly positive and satisfies $\mu(s) \sim as^{-\beta}$ with $0 < \beta < 1$, then $\underline{\mu}$, $\overline{\mu}$ exists on the form $cs^{-\beta}$ with $\underline{\mu} \leq \mu(s) \leq \overline{\mu}(s)$. With $\underline{M}$, $\overline{M}$ the corresponding integrated intensity processes we have e.g. $\underline{M}(s) = cs^{1-\beta}/(1-\beta)$. Let furthermore $\underline{X}$, $\overline{X}$ denote total task times corresponding to $\underline{\mu}$, $\overline{\mu}$ respectively. Then $\mathbb{P}(\underline{X} > x) \leq \mathbb{P}(X > x) \leq \mathbb{P}(\overline{X} > x)$ and

$$
\begin{aligned}
&\prod_{k=1}^{\lfloor x/\ell \rfloor} \overline{\mu}(x - k\ell) \\
&\quad = \prod_{k=1}^{\lfloor x/\ell \rfloor} \frac{a}{(b + x - k\ell)^{1-\beta}} \\
&\quad = \prod_{k=1}^{\lfloor x/\ell \rfloor} \frac{a}{\ell^{1-\beta}(\frac{b+x}{\ell} - k)^{1-\beta}} \\
&\quad \leq C \frac{1}{\left(\frac{b+x}{\ell} - \left\lfloor \frac{x}{\ell} \right\rfloor\right)^{1-\beta}} \prod_{k=1}^{\lfloor x/\ell \rfloor - 1} \frac{1}{(\left\lfloor \frac{b+x}{\ell} \right\rfloor - k)^{1-\beta}} \\
&\quad = C \frac{1}{\left(\frac{b+x}{\ell} - \left\lfloor \frac{x}{\ell} \right\rfloor\right)^{1-\beta}} \left(\frac{\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - \left\lfloor \frac{x}{\ell} \right\rfloor\right)!}{\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)!} \right)^{1-\beta} \\
&\quad \leq C \frac{1}{\left(\frac{b+x}{\ell} - \left\lfloor \frac{x}{\ell} \right\rfloor\right)^{1-\beta}} \left(\frac{\left(\left\lfloor \frac{b}{\ell} \right\rfloor + 1\right)!}{\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)!} \right)^{1-\beta} \\
&\quad \sim \tilde{C} \frac{1}{\left(x\sqrt{\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)} \left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)^{\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1} \exp\left(- \left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)\right)\right)^{1-\beta}} .
\end{aligned}
$$

From Corollary 2.1 (i) we have that

$$
\begin{aligned}
\mathbb{P}(X > x) \leq C &\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1 \right)^{-(1-\beta)\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)} \\
&\times \exp\Big(- \frac{a}{\beta}(b+x)^{\beta} + (\frac{1-\beta}{\ell} - \gamma)x \Big) x^{-\frac{3(1-\beta)}{2}}
\end{aligned}
$$

when $x \to \infty$ for some constant C. The expression on the r.h.s. above is

$$
\begin{aligned}
&\approx_{\log} \left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1 \right)^{-(1-\beta)\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)} \\
&\approx_{\log} x^{\frac{1-\beta}{\ell} x} .
\end{aligned}
$$

From (ii) we get

$$
\begin{aligned}
\mathbb{P}(X > x) \geq \tilde{C} &\exp\Big(- \frac{1}{\ell}(1-\beta) \log\big(b + x\big)(1 + o(1))x \Big) \\
&\times \exp\Big(- \frac{a}{\beta}(b+x)^{\beta} - (1 - 1/\ell)\gamma x \Big) .
\end{aligned}
$$

Here the first factor decreases faster than the second so that this

$$\begin{aligned}
&\approx_{\log} \exp\Big(-\frac{1}{\ell}(1-\beta)\log\Big(b+x\Big)(1+o(1))x\Big) \\
&\approx_{\log} \exp\Big(-\frac{1}{\ell}(1-\beta)\log\Big(b+x\Big)x\Big) \\
&\approx_{\log} x^{-\frac{1-\beta}{\ell}x}\,.
\end{aligned}$$

All together we have shown that

$$\mathbb{P}(X > x) \approx_{\log} x^{\frac{1-\beta}{\ell}x}$$

when $x \to \infty$. $\square$

3 Proofs: Model 2

Recall that in Model 2 the failure times $(N_t^*)_{t\geq 0}$ form a homogeneous Poisson process with intensity parameter μ^*, event times $(S_n^*)_{n\in\mathbb{N}}$, and interevent times $(U_n^*)_{n\in\mathbb{N}}$. Again, ℓ is the ideal task time, and it is assumed that at time t the system works on the task at rate $r(t)$, where r is a nonnegative measurable function.

Define the continuous and increasing function R as

$$R(t) \;=\; \int_0^t r(s)\,\mathrm{d}s\,.$$

It is obvious that $\mathbb{P}(X=\infty) > 0$ if $R(\infty) < \infty$, so assume that $R(\infty) = \infty$. Also assume that $R(t) < \infty$ for all $t \geq 0$.

A straightforward calculation shows that the inverse R^{-1} of R is a continuous and increasing function and given by

$$R^{-1}(y) \;=\; \int_0^y \frac{1}{r\big(R(s)\big)}\,\mathrm{d}s\,.$$

Since $R(t)$ is the amount of work that has been spent on the task up to time t provided the task has not been completed, the total task time in absence of failures is given by $R(X) = \ell$, i.e. $X = R^{-1}(\ell)$. More generally, if the task is not completed at the time S_{n-1}^* of the $(n-1)$th failure, then the task is still uncompleted at the time S_n^* of the nth failure if and only if $R^{-1}(S_n^*) - R^{-1}(S_{n-1}^*) < \ell$ [these observations are close to some standard facts in storage processes, see [3] p. 381].

It follows that the total task time X can be calculated as follows. First define the time ω as

$$\omega = \inf\Big\{n\in\mathbb{N}\;\Big|\;\int_{S_{n-1}^*}^{S_n^*} r(t)\,\mathrm{d}t > \ell\Big\}\,,$$

and let ℓ^* satisfy

$$\int_{S_{\omega-1}^*}^{\ell^*} r(t)\,\mathrm{d}t = \ell\,.$$

Then the total task time X is

$$X = S_{\omega-1}^* + \ell^*\,,$$

if $\omega < \infty$ and $X = \infty$ when $\omega = \infty$.

Proof of Theorem 1.1(ii). From the definition of N we can construct another point process: Let S'_n, $n \in \mathbb{N}$ be defined by $S'_n = R(S^*_n)$ for all $n \geq 0$. Since $S'_n = R^{-1}(S^*_n)$ it is well–known that $(S_n)_{n\in\mathbb{N}}$ are the event times of an inhomogeneous Poisson process with rate function $\mu(t) = \mu^*/r\big(R(t)\big)$.

It is directly seen that also

$$\omega \;=\; \inf\{n \in \mathbb{N} \,|\, S'_n > \ell\}\,.$$

Applying this yields the following definition of the total task time X' corresponding to S'_n, $n \in \mathbb{N}$

$$X' = \begin{cases} S'_{\omega-1} + \ell & ,\ \omega < \infty \\ \infty & ,\ \omega = \infty \end{cases}$$

Especially we have $\{X = \infty\} = \{X' = \infty\}$ and hence the theorem follows from Theorem 1.1(i). □

Define

$$f(x) = ax^\eta, \qquad F(x) = \int_0^x f(y)\,\mathrm{d}y = \frac{a}{\eta+1}x^{\eta+1}\,.$$

Lemma 3.1. *If* $r(x) \sim f(x)$ *then*

$$R(x) \sim F(x) = \frac{a}{\eta+1}x^{\eta+1} \quad \textit{and} \quad r(R(x)) \sim f(F(x)) = a\left(\frac{a}{\eta+1}\right)^\eta x^{\eta(\eta+1)}\,.$$

Proof. Given $\epsilon > 0$, there exits a x_0 exists such that

$$(1-\epsilon)f(x) \leq r(x) \leq (1+\epsilon)f(x) \quad \text{for } x > x_0\,.$$

Hence

$$(1-\epsilon)\int_{x_0}^x f(y)\,\mathrm{d}y \leq \int_{x_0}^x r(y)\,\mathrm{d}y \leq (1+\epsilon)\int_{x_0}^x f(y)\,\mathrm{d}y \quad \text{for } x > x_0$$

Using $\int_0^x f(x)\,\mathrm{d}x \to \infty$ it is seen that choosing $x > x_0$ large enough gives

$$\frac{\int_0^x r(y)\,\mathrm{d}y}{\int_0^x f(y)\,\mathrm{d}y} = \frac{\int_0^{x_0} r(y)\,\mathrm{d}y + \int_{x_0}^x r(y)\,\mathrm{d}y}{\int_0^{x_0} f(y)\,\mathrm{d}y + \int_{x_0}^x f(y)\,\mathrm{d}y} \in (1-2\epsilon, 1+2\epsilon)\,.$$

For the second result write

$$\frac{r(R(x))}{f(F(x))} = \frac{r(R(x))}{f(R(x))}\frac{f(R(x))}{f(F(x))}\,,$$

where the first factor obviously has limit 1. For the second factor $x_0 > 0$ can be found given ϵ such that for $x > x_0$

$$(1-\epsilon)F(x) < R(x) < (1+\epsilon)F(x)$$

and hence

$$f((1-\epsilon)F(x)) < f(R(x)) < f((1+\epsilon)F(x))$$

for $x > x_0$. Furthermore

$$f((1-\epsilon)F(x)) < f(F(x)) < f((1+\epsilon)F(x))$$

so it is obtained that

$$\frac{f((1-\epsilon)F(x))}{f((1+\epsilon)F(x))} < \frac{f(R(x))}{f(F(x))} < \frac{f((1+\epsilon)F(x))}{f((1-\epsilon)F(x))}.$$

Since

$$\frac{f((1+\epsilon)F(x))}{f((1-\epsilon)F(x))} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\eta}$$

has limit 1 as $\epsilon \to 0$, the proof is complete. □

□

Proof of Theorem 1.2(i). Note that $\mathbb{P}(X = \infty) = 0$. With X' defined as in the proof of Theorem 1.1 it holds on $\{X < \infty\}$ that

$$\begin{aligned}
\{X > x\} &= \{R(X) > R(x)\} \\
&= \left\{ \int_0^{S'_{\omega-1}+\ell^*} r(t)\,\mathrm{d}t > R(x) \right\} \\
&= \{R(S'_{\omega-1}) + \ell > R(x)\} \\
&= \{X' > R(x)\}\,.
\end{aligned} \tag{8}$$

Recall that X' is the total task time for a nonhomogeneous Poisson process with intensity process $(\mu(t))_{t\geq 0}$ where $\mu(t) = \frac{\mu^*}{r(R(t))}$. From Lemma 3.1 we have that

$$r(R(t)) \sim f(F(t)) = a\left(\frac{a}{\eta+1}\right)^{\eta} t^{\eta(\eta+1)}\,,$$

and hence $(\mu(t))$ has a form that suits the theorem for Model 1. Since also $R(t) \sim F(t)$ applying the result for Model 1 to the relation (8) yields

$$P(X > x) = P(X' > R(x)) \approx_{\log} R(x)^{-\frac{\eta(\eta+1)}{\ell}R(x)} \approx_{\log} F(x)^{-\frac{\eta(\eta+1)}{\ell}F(x)}\,.$$

□

4 Proofs: Model 3

The renewal argument in [5] leading to Proposition 1.1 for simple RESTART uses a geometric sum representation of $D = X - \ell$. It is instructive for the following to give a direct variant in the present setup, where the failure times occur according to a Poisson process. Define $Z(x) = \mathbb{P}(D > x)$ and let $z(x)$, $Z_0(x)$ be the contributions to $Z(x)$ from the events $U > x$ that the first failure time exceeds x, resp. $U \leq x$. A failure at time $t \leq x$ will contribute to $Z_0(x)$ if and only if $t \leq \ell$, which readily leads to

$$Z_0(x) \;=\; \int_0^x Z(x-t)\mu\mathrm{e}^{-\mu t} I(t \leq \ell)\,\mathrm{d}t\,.$$

Similarly but easier, $z(x) = \int_x^\infty \mu e^{-\mu t} I(t \le \ell)\,dt$, and altogether,

$$Z(x) \;=\; z(x) + Z_0(x) \;=\; z(x) + \int_0^x Z(x-t)g(t)\,dt\,,$$

where $g(t)$ is the defective density $\mu e^{-\mu t} I(t \le \ell)$. The rest is then standard renewal theory (e.g. [3] V.7).

Now consider Model 3 and write again $D = X - \ell$. Define $Z_i(x) = \mathbb{P}_i(D > x)$. We then get the following Markov renewal equation:

Proposition 4.1.

$$Z_i(x) \;=\; z_i(x) \;+\; \sum_{j=1}^p \int_0^x F_{ij}(dt) Z_j(x-t)\,dt\,, \tag{9}$$

where F_{ij} *has density* $\mu e^{-\mu t}\mathbb{P}_i\big(F(t) \le \ell, V(t) = j\big)$ *and*

$$z_i(x) \;=\; \int_x^\infty \mu e^{-\mu t}\mathbb{P}_i\big(F(t) \le \ell\big)\,.$$

Proof. We condition again on the time $U = t$ of the first failure. Then for $D > 0$ it is necessary that $F(t) \le \ell$, and therefore $z_i(x)$ is the contribution to $Z_i(x)$ from the event $U > x$. Similarly, conditioning in addition on $V(t)$ shows that the second term in (9) is the contribution from the event $U \le x$. □

The proof of Theorem 1.3 is now a straightforward adaptation of the defective Markov key renewal theorem, [3] pp. 209–210. To give the value of D_i is also straightforward from the expressions there, but the formulas are tedious and therefore omitted.

For computational purposes, one therefore needs to evaluate $\mathbb{P}(F(t) \le \ell, V(t) = j)$. The four most common approaches are:

a) to let $g(t, f; ij) = (d/df)\mathbb{P}\big(F(t) \le f, V(t) = j\big)$ and derive a set of PDE's for the $g(t, f; ij)$;

b) the transform inversion method of Ahn & Ramaswami [1];

c) the series expansion of Sericola [17];

d) simulation of $\mathbb{P}_i\big(F(t) \le \ell, V(t) = j\big)$.

Example 4.1. Consider a LAN with N users. Each sends a task of an exponential(ν) duration to the central unit at rate λ (no more tasks are sent before completion), the central unit works at rate 1 and uses standard processor sharing (works simultaneous on all tasks at the same rate). Thus, it seems reasonable to take $V(t) \in \{0, \ldots, N\}$ as the number of tasks currently with the server, let

$$q_{i(i+1)} = (N - i)\lambda\,, \quad q_{i(i-1)} = \nu$$

and all other off-diagonal q_{ij} equal to zero, and take $r_i = 1/i$ for $i > 0$, $r_0 = 0$. The model for $V(t)$ is an example of the so–called *Palm's Machine Repair Problem* (described in [3] III.3), with only a single repairman.

With $\pi_j = \lim_{t\to\infty} \mathbb{P}_i\Big(V(t) = j\Big)$, the average service rate is $r^* = \sum_1^p \pi_i/i$ where π is the stationary distribution ov V. If failures occur at rate μ and a user sends a task of length ℓ to the central unit, a reasonable question is then how the exponential decay rate $\gamma(\ell)$ of this total task duration compares to that γ^* of simple RESTART with service rate r^* (that is, ideal task duration ℓ/r^*). To illustrate this, we took $N = 10$, $\lambda = 1$, $\ell = 1$ and considered 3×3 combinations of ν, μ: ν chosen such that $\mathbb{E}_\pi V(0) = 2\,,5\,,8$ (low, moderate and heavy load) and $\mu = 1/5\,,1\,,5$ (low, moderate and high failure rate). We used method d) and obtained the following table over γ and γ^* (the vaules of γ^* are in $(\cdot)$):

$\mathbb{E}_\pi V(0)$ μ	2	5	8
1/5	0.683 (0.744)	0.259 (0.304)	0.079 (0.079)
1	0.134 (0.121)	0.040 (0.021)	0.011 (0.004)
5	0.144 (0.030)	0.212 (0.050)	0.235 (0.094)

It is seen that in some cases, not at least when μ is large, there are major differences between γ and γ^*. Also note that even a rather small difference will have a substantial influence on the decay of $\mathbb{P}_i(X > x)$. □

5 Appendix

The following result is well–known and can e.g. be seen as a immediate consequence of Lemma 1 p. 144 in [7].

Lemma 5.1. *Assume that $(U_n)_{n\in\mathbb{N}}$ are iid variables with $U_n > 0$. Let $(S_n)_{n\in\mathbb{N}}$ be the corresponding random walk, that is $S_n = \sum_{k=1}^n U_k$ for $n \in \mathbb{N}$. Define*

$$\tau = \inf\{n \in \mathbb{N} \,|\, S_n > a\}$$

for some $a > 0$. Then

$$\mathbb{E}[t^\tau] < \infty \qquad \text{for all } t > 0\,.$$

Proposition 5.1. *Assume that $(U_n)_{n\in\mathbb{N}}$ are iid variables each with density $t \mapsto \mathrm{e}^{\gamma t}$ on $[0,t]$. Let $(S_n)_{n\in\mathbb{N}}$ be the corresponding random walk, that is $S_n = \sum_{k=1}^n U_k$ for $n \in \mathbb{N}$. Define*

$$\tau(x) = \inf\{n \in \mathbb{N} \,|\, S_n > x\} = \inf\{n \in \mathbb{N} \,|\, S_n - x > 0\}$$

for some $x > 0$. Then

$$\exp\Big(-\varphi^{-1}(z)(x+t)\Big) \le \mathbb{E}[z^{\tau(x)}] \le \exp\Big(-\varphi^{-1}(z)x\Big)$$

for all $0 < z < 1$, where

$$\varphi(\theta) = \frac{\theta+\gamma}{\mathrm{e}^{t(\theta+\gamma)} - 1}\,.$$

Proof. Because the U_k–variables are bounded we have for all $\theta > 0$ that

$$h(\theta) = \mathbb{E}\left[e^{\theta U_1}\right] < \infty\,.$$

Consequently

$$M_n(\theta) = \frac{e^{\theta S_n}}{h(\theta)^n} \qquad (n \in \mathbb{N})$$

is a martingale with mean 1. Define

$$\tau(x) = \inf\{n \in \mathbb{N} \mid S_n > x\}\,.$$

Then by optional stopping we have

$$1 = \mathbb{E}\left[\frac{e^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}; \tau(x) \le n\right] + \mathbb{E}\left[\frac{e^{\theta S_n}}{h(\theta)^n}; \tau(x) > n\right]\,.$$

Let $n \to \infty$ and note that $e^{\theta S_n} \le e^{\theta x}$ on $\{\tau(x) > n\}$. By dominated convergence, we have

$$1 = \mathbb{E}\left[\frac{e^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}; \tau(x) < \infty\right] = \mathbb{E}\left[\frac{e^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}\right]\,.$$

Since $x < S_{\tau(x)} \le x + t$ this yields

$$\mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] e^{\theta x} < 1 \le \mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] e^{\theta(x+t)}$$

and thereby

$$e^{-\theta(x+t)} \le \mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] < e^{-\theta x}\,.$$

Now consider the function $\theta \mapsto h(\theta) = \mathbb{E}[\exp(\theta U_1)]$. It is strictly increasing with $h(0) = 1$ and $\lim_{\theta\to\infty} h(\theta) = \infty$. Hence $\varphi(\theta) = 1/h(\theta)$ is strictly decreasing so that the inverse φ^{-1} is well–defined (on $]0, 1]$). Furthermore

$$\begin{aligned} h(\theta) &= \mathbb{E}\left[e^{\theta U_1}\right] \\ &= \int_0^t e^{\theta y} e^{\gamma y}\, dy \\ &= \frac{1}{\theta + \gamma}\left[e^{(\theta+\gamma)t} - 1\right]\,. \end{aligned}$$

This concludes the proof of the proposition. □

Proposition 5.2. *For the function $z \mapsto \varphi^{-1}(z)$ studied in Proposition 5.1 it holds that*

$$\varphi^{-1}(z) = -\frac{1}{t}\log(z)(1 + o(1))\,,$$

as $z \downarrow 0$.

Proof. We have that $\theta(z) = \varphi^{-1}(z)$ can be found as the solution w.r.t. θ of the equation

$$z = \frac{\theta + \gamma}{\mathrm{e}^{t(\theta+\gamma)} - 1}$$

which can be rewritten as

$$\theta + \gamma = z\left(\mathrm{e}^{t(\theta+\gamma)} - 1\right) . \tag{10}$$

Now let $\delta > 0$ and define $\theta_\delta(z)$ by

$$\theta_\delta(z) = -\frac{\delta}{t}\log(z) - \gamma .$$

With $\theta = \theta_\delta$ the r.h.s. of (10) becomes $z^{1-\delta} - z$ and the l.h.s. is of order $\log z$ when $z \downarrow 0$. If $\delta > 1$ the r.h.s. increases faster than the l.h.s. as $z \downarrow 0$. With z small enough we thereby have

$$\theta_\delta(z) + \gamma \le z\left(\mathrm{e}^{t(\theta_\delta(z)+\gamma)} - 1\right) .$$

Note that the r.h.s. in (10) is an increasing and convex function of θ while the l.h.s. is affine. From that we can deduce that $\theta(z) < \theta_\delta(z)$. Similarly in the $\delta \le 1$ case we can see that $\theta(z) > \theta_\delta(z)$. Hence

$$\theta(z) = -\frac{1}{t}\log(z)(1 + o(1))$$

as wanted. □

Acknowledgements

We are grateful to Lester Lipsky for inventing the term “Wallclock models”.

References

[1] S. Ahn & V. Ramaswami (2005) Efficient algorithms for transient analysis of stochastic fluid flow models. *J. Appl. Probab.* **42**, 531–549.

[2] L.N. Andersen & S. Asmussen (2008) Parallel computing, failure recovery and extreme values. *J. Statist. Th. Pract.* **2**, 279–292.

[3] S. Asmussen (2003) *Applied Probability and Queues* (2nd ed.). Springer-Verlag.

[4] S. Asmussen & P. W. Glynn (2007) *Stochastic Simulation: Algorithms and Analysis.* Springer-Verlag.

[5] S. Asmussen, P. Fiorini, L. Lipsky, T. Rolski & R. Sheahan (2008) On the distribution of total task times for tasks that must restart from the beginning if failure occurs. *Math. Oper. Res.* **33**, 932–944.

[6] S. Asmussen & L. Lipsky (2009) Failure recovery via RESTART: limit theorems for checkpointing. *Working paper.*

[7] K.B. Athreya & P.E. Ney (1972) Branching Processes. Springer, Berlin.

[8] A. Bobbio & K. Trivedi (1990) Computation of the distribution of the completion time when the work requirement is a PH random variable. *Stochastic Models* **6**, 133–150.

[9] X. Castillo & D.P. Siewiorek (1980) A performance-reliability model for computing systems. *Proc FTCS-10, Silver Spring, MD, IEEE Computer Soc.*, 187–192.

[10] P.F. Chimento, Jr. & K.S. Trivedi (1993) The completion time of programs on processors subject to failure and repair. *IEEE Trans. on Computers* **42**(1).

[11] R. DePrisco, A. Mayer, & M. Yung (1994) Time-optimal message-efficient work performance in the presence of faults. *Proc. 13th ACM PODC*, 161–172.

[12] W. Feller (1971) *An Introduction to Probability Theory and Its Applications* **II** (2nd ed.). Wiley.

[13] P. Jelenković & J. Tan (2007) Can retransmissions of superexponential documents cause subexponential delays? In *Proceedings of IEEE INFOCMO'07*, pp. 892–900, Anchorage.

[14] P. Jelenković & J. Tan (2007) Characterizing heavy-tailed distributions induced by retransmissions. *Submitted*.

[15] V. Kulkarni, V. Nicola, & K. Trivedi (1986) On modeling the performance and reliability of multimode systems. *The Journal of Systems and Software* **6**, 175–183.

[16] V. Kulkarni, V. Nicola, & K. Trivedi (1987) The completion time of a job on a multimode system. *Adv. Appl. Probab.* **19**, 932–954.

[17] B. Sericola (1998) Transient analysis of stochastic fluid models. *Performance Evaluation* **32**, 245–263.

[18] R. Sheahan, L. Lipsky, P. Fiorini & S. Asmussen (2006) On the completion time distribution for tasks that must restart from the beginning if failure occurs. *ACM SIGMETRICS Performance Evaluation Review* **34**, 24–26.