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Abstract

Stochastic integration on the predictable σ-field with respect to σ-finite L^0-valued measures, also known as formal semimartingales, is studied. In particular, the triplet of such measures is introduced and used to characterize the set of integrable processes. Special attention is given to Lévy processes indexed by the real line. Surprisingly, many of the basic properties break down in this situation compared to the usual $\mathbb{R}_+$ case.

Keywords: stochastic integration; semimartingales; Lévy processes; vector measures
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1 Introduction

Recently there has been growing interest in stochastic integrals of the form

$$\int_{-\infty}^{\infty} \phi_s \cdot m(\mathrm{d}s), \tag{1.1}$$

where ϕ is an $\mathbb{R}^n$-valued predictable processes indexed by $\mathbb{R}$ and m is an n-dimensional σ-finite L^0-valued measure on the predictable σ-field induced by a filtration $\mathscr{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$; or in the terminology of Schwartz (1981), m is a *formal semimartingale*. By the Bichteler-Dellacherie Theorem there is a one to one correspondence between semimartingales indexed by compact intervals and finite L^0-valued measures m on the predictable σ-field. However, a Lévy process indexed by $\mathbb{R}$ does not induce a finite L^0-valued measure, only a σ-finite one, and hence integrals with respect to $\mathbb{R}$-indexed Lévy processes can not be defined within the usual semimartingale framework.

The main purpose of this paper is to give applicable conditions for integrals of the form (1.1) to exist. Important examples include

$$(\alpha)\colon \int_{-\infty}^{\infty} (g(t-s)\sigma_s)\cdot \mathrm{d}Z_s \qquad \text{and} \qquad (\beta)\colon \int_{-\infty}^{\infty} f(s, Z_{s-})\cdot \mathrm{d}Z_s, \tag{1.2}$$

where Z is an $\mathbb{R}^n$-valued Lévy process indexed by $\mathbb{R}$ (i.e., has independent stationary increments), σ is an $\mathbb{R}^n$-valued predictable stationary process, g is an $\mathbb{R}^{n^2}$-valued deterministic function, $t \in \mathbb{R}$, and $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function. The setting (α) is in Barndorff-Nielsen and Schmiegel (2007, 2008, 2009) used for modeling an interesting new class of moving averages; here we recall e.g. from Doob (1990) that moving averages provide a large class of stationary processes. With $n = 2$ and $Z = (Z^1, Z^2)$ such that $\lim_{t\to\infty} Z_t^2 = \infty$ a.s., it is shown in Basse-O'Connor et al. (2010b) that the integral (β) with $f(s, x_1, x_2) = (e^{x_2} 1_{\{s\leq 0\}}, 0)$, exists if and only if there exists a stationary distribution to the generalized Ornstein-Uhlenbeck process driven by Z, and in this case the stationary solution X is of the form

$$X_t = e^{-Z_t^2} \int_{-\infty}^{t} e^{Z_{s-}^2} \, \mathrm{d}Z_s^1, \qquad t \in \mathbb{R}.$$

Integration of deterministic functions with respect to independently scattered random measures is characterized in Rajput and Rosiński (1989) and Marcus and Rosiński (2001). Moreover, when Z is a semimartingale and $t > 0$, Jacod and Shiryaev (2003) have characterized the set of predictable processes ϕ for which $\int_0^t \phi_s \cdot \mathrm{d}Z_s$ exists in terms of the triplet of Z. Cherny and Shiryaev (2005) have extended this characterization to include integrals of the form $\int_0^\infty \phi_s \cdot \mathrm{d}Z_s$. If in addition $Z = (Z_s)_{s\in[0,t]}$ is quasi-left continuous and $n = 1$, Kwapień and Woyczyński (1991), Theorem 6.1, have characterized the topology on the set of integrable functions $L(Z)$ in terms of the triplet of Z.

The above mentioned results are extended in Theorems 4.5–4.6 to the case of integrals of the form (1.1), i.e., to σ-finite L^0-valued measures. To obtain these results we show and apply a characterization of convergence in Emery's semimartingale topology, see Theorem 4.10. However, first the triplet of a σ-finite L^0-valued measure m is introduced in Theorem 4.2. Using these extensions we are able to give applicable conditions for integrals of the form (α)–(β) to exist. In (β) it is natural to consider the filtration $\mathcal{F}_t = \mathcal{F}_t^Z$, where $\mathcal{F}_t^Z = \sigma(Z_s : s \in (-\infty, t])$. Contrarily to $\mathbb{R}_+$, the following break down for Lévy processes Z indexed by $\mathbb{R}$:

(i) Z is *not* an $\mathcal{F}^Z$-Lévy process
(because $Z_t - Z_s$ is not independent of $\mathcal{F}_s^Z$ for all $-\infty < s < t < \infty$).

(ii) Even when Z is centered it is *not* a martingale in $\mathcal{F}^Z$ (or in any other filtration).

Despite of (i)–(ii), we show in Subsection 5.2 that a Lévy process Z induces a σ-finite L^0-valued measure in the filtration $\mathscr{F}^Z$. This result relies on an expansion of the filtration extending Jacod and Protter (1988), Theorems 2.6 and 2.9; see Appendix. In $\mathscr{F}^Z$ it does however not seem possible to calculate the triplet of Z explicitly. Therefore, in Theorem 5.3, we consider an expanded filtration in which Z still induces a σ-finite L^0-valued measure and in which we are able to calculate the triplet of Z explicitly; this gives in particular applicable conditions for integrals of the form (β) to exist. Theorem 5.5 concerns the import case of a square integrable Lévy process.

Before proceeding, we study in Section 2 integration theory for σ-finite measures m on a measurable space $(E, \mathscr{E})$ with values in a linear metric space F. In particular we see how m induces (in a canonical way) a set of integrable functions, called $L(m)$, by its semivariation. Vector valued integration theory is very well-developed; see e.g. Bichteler (1976, 1981); Bichteler and Jacod (1983); Curbera and Delgado (2007); Kwapień and Woyczyński (1992); Rolewicz (1985); Schwartz (1981); Turpin (1974, 1975) for nice treatments. However, only few of these references consider σ-finite rather than finite measures; one notable exception is Schwartz (1981), who calls a σ-finite measure a *formal measure*. Our approach differs slightly from that of Schwartz (1981), e.g., we start with σ-additive set functions m instead of integral mappings with certain continuity properties. In the subsequent sections focus is on the case $(E, \mathscr{E}) = (\mathbb{R} \times \Omega, \mathscr{P})$ and $F = L^0$.

2 Vector valued measures

Let $(E, \mathscr{E})$ denote a measurable space and for all $n \geq 1$ let $M(\mathscr{E}; \mathbb{R}^n)$ be the space of all $\mathbb{R}^n$-valued $\mathscr{E}$-measurable functions. Furthermore, $(F, \|\cdot\|)$ denotes an F-space in which unconditional convergence implies bounded multiplier convergence (see Rolewicz (1985)); i.e. for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $k \geq 1$ and $x_1, \dots, x_k \in F$

$$\Big\| \sum_{i=1}^{k} \epsilon_i x_i \Big\| < \delta \text{ for all } (\epsilon_i)_{i=1}^k \subseteq \{0,1\} \;\Rightarrow\; \Big\| \sum_{i=1}^{k} a_i x_i \Big\| < \epsilon \text{ for all } (a_i)_{i=1}^k \subseteq [-1,1]. \tag{2.1}$$

Here we follow Rolewicz (1985) and call $(F, \|\cdot\|)$ an F-space if and only if $d(x,y) := \|x - y\|$ defines a metric in which F is a linear complete metric space. We may and do always assume that $\|\cdot\|$ is increasing, that is, for all $a \in [-1,1]$, $\|ax\| \leq \|x\|$ (cf. Rolewicz (1985), Theorem I.2.2).

This covers in particular $F = L^0(\Omega, \mathcal{F}, \mathrm{P})$, the space of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, equipped with the F-norm $\| Z \|_0 := E[|Z| \wedge 1]$ cf. Ryll-Nardzewski and Woyczyński (1975), together with all locally convex F-spaces (e.g. Banach spaces) cf. Rolewicz (1985), Corollary III.6.6.

2.1 The one-dimensional case

We call a set function m defined on a subset of $\mathscr{E}$ an F-valued σ-finite measure on $(E, \mathscr{E})$ if there exists a sequence $(O_k)_{k\geq 1} \subseteq \mathscr{E}$ with $O_k \uparrow E$ such that, with $\mathscr{E}_m = \{A \in \mathscr{E} : A \subseteq O_k \text{ for some } k \geq 1\}$, the mapping $m \colon \mathscr{E}_m \to F$ satisfies $m(\emptyset) = 0$ and if $(A_i)_{i=1}^\infty \subseteq \mathscr{E}_m$ are disjoint sets with $\cup_{i=1}^\infty A_i \in \mathscr{E}_m$ then

$$m\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{i=1}^\infty m(A_i) \quad \text{in } F.$$

The sequence $(O_k)_{k\geq 1}$ is called m-feasible. Denote by $\mathscr{S}_m$ the vector space of all real-valued $\mathscr{E}_m$-simple functions, i.e., $\phi \in \mathscr{S}_m$ if and only if

$$\phi = \sum_{i=1}^{k} a_i 1_{A_i} \quad \text{for some } a_1, \dots, a_k \in \mathbb{R} \text{ and } A_1, \dots, A_k \in \mathscr{E}_m. \tag{2.2}$$

Clearly the A_i's may be assumed disjoint. Set for $\phi \in \mathscr{S}_m$ of the form (2.2) $m(\phi) = \sum_{i=1}^{k} a_i m(A_i)$. Then $m : \mathscr{S}_m \to F$ is linear and is called *the simple integral* with respect to m. We are looking for a linear extension $\int \cdot \, \mathrm{d}m$ of the simple integral defined on a vector space $\mathscr{D} \subseteq E^{\mathbb{R}}$ satisfying the DCT (Dominated Convergence Theorem), i.e. whenever $(\phi_k)_{k\geq 1}, \psi \subseteq \mathscr{D}$, with $|\phi_k| \leq \psi$ and $\phi_k \to \phi$ pointwise, we have $\phi \in \mathscr{D}$ and $\int \phi_k \, \mathrm{d}m \to \int \phi \, \mathrm{d}m$ in F. Note that the DCT ensures that every bounded $\mathscr{E}$-measurable function vanishing outside some O_k is in $\mathscr{D}$.

In order to state the main theorem we introduce the semivariation $\|\cdot\|_m$ of m, that is,

$$\|\phi\|_m = \sup_{\psi \in \mathscr{S}_m \colon |\psi| \leq |\phi|} \|m(\psi)\| \quad \text{for } \phi \in M(\mathscr{E}; \mathbb{R}).$$

It is readily seen that: (i) $|\phi_1| \leq |\phi_2|$ implies $\|\phi_1\|_m \leq \|\phi_2\|_m$; (ii) $\|\cdot\|_m$ is subadditive, i.e. $\|\phi_1 + \phi_2\|_m \leq \|\phi_1\|_m + \|\phi_2\|_m$; (iii) $\|m(\phi)\| \leq \|\phi\|_m$ for $\phi \in \mathscr{S}_m$.

Set furthermore

$$L^1(m) = \left\{\phi \in M(\mathscr{E}; \mathbb{R}) : \lim_{\lambda \to 0} \|\lambda \phi\|_m = 0\right\}$$

or equivalently

$$L^1(m) = \left\{\phi \in M(\mathscr{E}; \mathbb{R}) : \{m(\psi) : |\psi| \leq |\phi|, \, \psi \in \mathscr{S}_m\} \text{ is bounded in } F\right\}.$$

Note that $\|\cdot\|_m$, and hence $L^1(m)$, is invariant of the choice of $(O_k)_{k\geq 1}$. Thus, it is $L^1(m)$ rather than $(O_k)_{k\geq 1}$ that is important. Indeed, if $(\tilde{O}_k)_{k\geq 1} \subseteq \mathscr{E}$ with $\tilde{O}_k \uparrow E$ is another m-feasible sequence then the two semivariations agree; that is,

$$\|\phi\|_m = \sup_{\psi \in \tilde{\mathscr{S}}_m \colon |\psi| \leq |\phi|} \left\|m(\psi)\right\| \quad \text{for } \phi \in M(\mathscr{E}; \mathbb{R}).$$

Here $\tilde{\mathscr{S}}_m$ denotes the simple functions relative to $(\tilde{O}_k)_{k\geq 1}$. To see this, we may and do assume $\tilde{O}_k \subseteq O_k$ for all $k \geq 1$, implying that the left-hand side dominates the right-hand side. To get the other inequality observe that $A = \bigcup_{k=1}^{\infty} A \cap (\tilde{O}_k \setminus \tilde{O}_{k-1})$ for $A \in \mathscr{E}_m$. The σ-additivity of m on each O_k therefore ensures that for all $\psi \in \mathscr{S}_m$ there exists a sequence $(\psi_k)_{k\geq 1} \subseteq \tilde{\mathscr{S}}_m$ with $|\psi_k| \leq |\psi|$ for all k such that $m(\psi_k) \to m(\psi)$ in F. But this means that $\|m(\psi_k)\| \to \|m(\psi)\|$, proving the result.

Theorem 2.1. *Assume that m is locally bounded, i.e. the set $\{m(B) : B \in \mathcal{E} \cap O_k\}$ is bounded in F for all $k \geq 1$. Then $L^1(m)$ is a linear space and equipped with $\|\cdot\|_m$ it is an F-space containing $\mathscr{S}_m$ as a dense subspace. The simple integral extends to $L^1(m)$ by $\|\cdot\|_m$-continuity and the extension $\int \cdot \, \mathrm{d}m \colon L^1(m) \to F$ satisfies the DCT.*

More generally, if $(\phi_k)_{k\geq 1} \subseteq L^1(m)$, $\phi_k \to \phi$ pointwise and there exists $\psi \in L^1(m)$ such that $|\phi_k| \leq \psi$ for all $k \geq 1$, then $\phi_k \to \phi$ in $L^1(m)$.

Moreover, $\phi \in L^1(m)$ if and only if there exists $(\phi_k)_{k\geq 1} \subseteq \mathscr{S}_m$ such that $\phi_k \to \phi$ pointwise and for all $A \in \mathscr{E}$, $\lim_k \int_A \phi_k \,\mathrm{d}m$ exists in F.

Remark 2.2. In the important case $F = L^0(\Omega, \mathcal{F}, \mathrm{P})$ all σ-finite measures are locally bounded cf. Talagrand (1981).

Proof of Theorem 2.1. According to Bichteler (1976) the first part follows by verifying:

(E_1) The simple integral is continuous when $\mathscr{S}_m$ is given the Schwartz inductive topology coming from uniform convergence on each O_k, $k \geq 1$.

(E_2) $m(\phi_k) \to 0$ in F if $(\phi_k)_{k\geq 1} \subseteq \mathscr{S}_m$ and $\phi_k \downarrow 0$ pointwise.

(E_3) $m(\phi_k) \to 0$ in F for every positive disjoint sequence $(\phi_k)_{k\geq 1} \subseteq \mathscr{S}_m$ majorized by some $\psi \in \mathscr{S}_m$.

In the proof we follow Kwapień and Woyczyński (1992), Theorem 7.1.2, who consider the case $F = L^0$; see also Rolewicz (1985), Theorem III.6.2.

Let $(\phi_k)_{k\geq 1} \subseteq \mathscr{S}_m$ be given such that $\phi_k \to 0$ as defined in E_1, i.e. there is an $l \geq 1$ such that $\{\phi_k \neq 0\} \subseteq O_l$ for all k and $\sup_{x\in O_l}|\phi_k(x)| \to 0$. For a given $\epsilon > 0$ let $\delta > 0$ be chosen according to (2.1). Since m is locally bounded, for all $l \geq 1$ there exists $c > 0$ such that $\|cm(B)\| < \delta$ for all $B \in \mathcal{E} \cap O_l$. Fixing $k_0 \geq 1$ such that $|\phi_k| \leq c1_{O_l}$ for $k \geq k_0$ and writing for each k

$$\phi_k = \sum_{i=1}^{r_k} a_{i,k} 1_{A_{i,k}} \quad \text{where } (a_{i,k})_{1\leq i\leq r_k} \subseteq \mathbb{R} \text{ and } (A_{i,k})_{1\leq i\leq r_k} \subseteq \mathscr{E} \text{ are pairwise disjoint,} \tag{2.3}$$

we see from (2.1), with $cm(A_{1,k}), \dots, cm(A_{r_k,k})$ playing the role of the x's and $a_{i,k}/c$, $i = 1, \dots, r_k$, that of the a's, that for $k \geq k_0$

$$\|m(\phi_k)\| = \Big\| \sum_{i=1}^{r_k} \frac{a_{i,k}}{c}\, c\,m(A_{i,k}) \Big\| < \epsilon \;\text{ since }\; \Big\| \sum_{i=1}^{r_k} \epsilon_i c\, m(A_{i,k}) \Big\| = \Big\| c\,m\big(\bigcup_{i:\,\epsilon_i=1} A_{i,k}\big) \Big\| < \delta.$$

To prove (E_2) and (E_3) it suffices to show

$$(\phi_k)_{k\geq 1}, \psi \subseteq \mathscr{S}_m,\ |\phi_k| \leq \psi \leq 1 \text{ and } \phi_k \to 0 \text{ pointwise } \Rightarrow m(\phi_k) \to 0.$$

So let $(\phi_k)_{k\geq 1}$ and ψ in $\mathscr{S}_m$ be given according to the left-hand side and assume that ϕ_k is given by (2.3). Write, for $c > 0$, $\phi_{k1} = \phi_k 1_{\{|\phi_k|\leq c\}}$ and $\phi_{k2} = \phi_k 1_{\{|\phi_k|>c\}}$. As $\{\phi_k \neq 0\} \subseteq \{\psi \neq 0\} \subseteq O_l$ for all k and some $l \geq 1$ we may, using (E_1) for fixed $\epsilon > 0$, chose $c > 0$ such that $\|m(\phi_{k1})\| < \epsilon$ for all k. For all $(B_k)_{k\geq 1} \subseteq \mathcal{E}$, $\limsup_k (B_k \cap \{|\phi_k| > c\}) = \emptyset$. Thus, due to the σ-additivity on $\mathcal{E} \cap O_l$, $\lim_{k\to\infty} m(B_k \cap \{|\phi_k| > c\}) = 0$ in F, and so there exists $k_0 \geq 1$ such that $\sup_{B\in\mathcal{E}}\|m(B\cap\{|\phi_k|>c\}\|<\delta$ for $k\geq k_0$, where δ is chosen according to (2.1). For given $\epsilon_1,\dots,\epsilon_{k_n}\subseteq\{0,1\}$ we have as above

$$\Big\|\sum_{i=1}^{r_k}\epsilon_i m(A_{i,k}\cap\{|\phi_k|>c\})\Big\| = \Big\|m\big(\bigcup_{i:\,\epsilon_i=1} A_{i,k}\cap\{|\phi_k|>c\}\big)\Big\| < \delta$$

which by (2.1) gives

$$\|m(\phi_{k2})\| = \Big\|\sum_{i=1}^{r_k} a_{i,k} m(A_{i,k}\cap\{|\phi_k|>c\})\Big\| < \epsilon \quad \text{for } k\geq k_0.$$

The *only if*-part of the last statement follows since $\mathscr{S}_m$ is dense. To get the *if*-part set $\nu(A)=\lim_{k\to\infty}\int_A\phi_k\,\mathrm{d}m$ for $A\in\mathcal{E}$. By Turpin (1974), Theorem 7.1.5, ν is an F-valued σ-finite measure on $(E,\mathcal{E})$. According to Turpin (1974), $\phi 1_{O_k}\in L^1(m)$ for all k and $\nu(A)=\int_A\phi\,\mathrm{d}m$ for all $A\in\bigcup_{k\geq 1}(\mathcal{E}\cap O_k)$. The σ-additivity of ν and fact that $O_n^c\downarrow\emptyset$ gives

$$0=\lim_{k\to\infty}\sup\{\|\nu(H)\|: H\in\mathcal{E},\, H\subseteq O_k^c\}$$

and so, arguing as above, we get by (2.1)

$$\lim_{k\to\infty}\sup\{\|\nu(\psi)\|:\psi\in\mathscr{S}_m,\,|\psi|\leq 1_{O_k^c}\}=0,$$

implying $\|\phi-\phi 1_{O_k}\|_m\to 0$. This proves the remaining part of the theorem. □

When $1_E\in L^1(m)$ we call m a finite measure, and in this case m extends to a measure defined on the entire σ-field $\mathscr{E}$.

2.2 The multivariate case

In the following we define and study the integral with respect to $m=(m^i)_{i\leq n}$, where $m^1,\dots,m^n$ are F-valued σ-finite and locally bounded measures as defined in Subsection 2.1. Let $(O_k)_{k\geq 1}\subseteq\mathscr{E}$ with $O_k\uparrow E$ be a sequence which is m^i-feasible for all i. Our construction of the integral is motivated by Bichteler and Jacod (1983), Section 3.

Setting

$$L^1(m)=\big\{\phi=(\phi^i)_{i=1}^n\in M(\mathscr{E};\mathbb{R}^n):\phi^i\in L^1(m^i)\text{ for all } i=1,\dots,n\big\}$$

gives a linear space stable under multiplication with bounded elements in $M(\mathscr{E};\mathbb{R})$. Define for $\phi\in L^1(m)$ the integral of ϕ with respect to m as $\int\phi\cdot\mathrm{d}m=\sum_{i=1}^n\int\phi^i\,\mathrm{d}m^i$. Set for $\phi\in M(\mathscr{E};\mathbb{R}^n)$,

$$\|\phi\|_m=\sup\big\{\big\|\int(\phi\psi)\cdot\mathrm{d}m\big\|:\psi\in M(\mathscr{E};\mathbb{R}),\ |\psi|\leq 1,\ \psi\phi\in L^1(m)\big\}, \tag{2.4}$$

and let

$$L(m)=\big\{\phi\in M(\mathscr{E};\mathbb{R}^n):\lim_{\lambda\to 0}\|\lambda\phi\|_m=0\big\}. \tag{2.5}$$

Fix $\phi \in M(\mathscr{E};\mathbb{R}^n)$ and set $\overline{O}_k = O_k \cap \{\|\phi\| \leq k\}$ for $k \geq 1$, and $\mathscr{E}_{\phi\bullet m} = \{A \in \mathscr{E} : A \subseteq \overline{O}_k \text{ for some } k \geq 1\}$. Then, $\phi \bullet m \colon \mathscr{E}_{\phi\bullet m} \to F$ defined as $\phi \bullet m(A) = \int (1_A\phi) \cdot \mathrm{d}m$ is an F-valued σ-finite and locally bounded measure on $(E,\mathscr{E})$. Note that $\phi \in L^1(m)$ implies $1_E \in L^1(\phi \bullet m)$ and hence, applying the DCT of $\phi \bullet m$ and the m^i's, we get in this case

$$\phi \bullet m(E) = \lim_{k\to\infty} \phi \bullet m(\overline{O}_k) = \lim_{k\to\infty} \int (1_{\overline{O}_k}\phi) \cdot \mathrm{d}m = \int \phi \cdot \mathrm{d}m,$$

which motivates the following definition:

Definition 2.3. $\phi \in M(\mathscr{E};\mathbb{R}^n)$ is said to be integrable with respect to m if $1_E \in L^1(\phi \bullet m)$, and in this case $\int \phi \cdot \mathrm{d}m := \phi \bullet m(E) \in F$ is the integral of ϕ with respect to m.

The following gives the basic properties (recall that an F^*-space is an F-space except that it is not necessarily complete).

Theorem 2.4. *$(L(m), \|\cdot\|_m)$ is an F^*-space, ϕ is integrable with respect to m if and only if $\phi \in L(m)$, and the mapping $L(m) \ni \phi \mapsto \int \phi \cdot \mathrm{d}m \in F$ is linear. Moreover, for $\phi \in M(\mathscr{E};\mathbb{R}^n)$ and $\psi \in M(\mathscr{E};\mathbb{R})$ we have $\psi \in L^1(\phi \bullet m)$ if and only if $\psi\phi \in L(m)$ and*

$$\int \psi \,\mathrm{d}(\phi \bullet m) = \int (\psi\phi) \cdot \mathrm{d}m \qquad \textit{for } \phi \in M(\mathscr{E};\mathbb{R}^n) \textit{ and } \psi \in L^1(\phi \bullet m).$$

Proof. For $\phi_1, \phi_2 \in M(\mathscr{E};\mathbb{R}^n)$ we can, using the DCT, show that $\|\phi_1 + \phi_2\|_m \leq \|\phi_1\|_m + \|\phi_2\|_m$ and hence it follows that $L(m)$ is an F^*-space. The last part of Theorem 2.4 follows once we have shown that $\|\psi\|_{\phi\bullet m} = \|\psi\phi\|_m$ for all $\phi \in M(\mathscr{E};\mathbb{R}^n)$ and all $\psi \in M(\mathscr{E};\mathbb{R})$. Let $\overline{\mathscr{S}}$ be the set of simple functions relative to the sequence $(\overline{O}_k)_{k\geq 1}$ introduced above. We have

$$\begin{aligned}\|\psi\|_{\phi\bullet m} &= \sup_{|\xi|\leq|\psi|,\xi\in\overline{\mathscr{S}}} \|(\phi \bullet m)(\xi)\| = \sup_{|\xi|\leq|\psi|,\xi\in\overline{\mathscr{S}}} \Big\| \int (\phi\xi) \cdot \mathrm{d}m \Big\| \\ &= \sup_{|\xi|\leq|\psi|,\phi\xi\in L^1(m)} \Big\| \int (\phi\xi) \cdot \mathrm{d}m \Big\| = \sup_{|\xi|\leq 1,(\xi\psi\phi)\in L^1(m)} \Big\| \int (\xi\psi\phi) \cdot \mathrm{d}m \Big\| = \|\psi\phi\|_m,\end{aligned}$$

where the third equality follows using the DCT of the m^i's.

Since $1 \in L^1(\phi \bullet m)$ if and only if $\phi \bullet m$ extends to a finite F-valued measure on $(E,\mathscr{E})$ the vector space property of $L(m)$ together with the linearity of the integral is clear. The proof can now be completed by applying the DCT of the one-dimensional σ-finite measure $\phi \bullet m$. □

3 Notation and basic definitions

Throughout the rest of the paper $(\Omega, \mathcal{F}, \mathrm{P})$ denotes a probability space and L^0 is $L^0(\Omega, \mathcal{F}, \mathrm{P})$. Recall that with $\|Z\|_0 = \mathrm{E}[|Z| \wedge 1]$ for $Z \in L^0$, $(L^0, \|\cdot\|_0)$ is an F-space.

Let $\mathscr{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ be a filtration, i.e., an increasing family of σ-fields satisfying the usual conditions of right-continuity and completeness. Set $\mathcal{F}_{-\infty} = \cap_{t\in\mathbb{R}}\mathcal{F}_t$ and $\mathcal{F}_\infty = \sigma(\cup_{t\in\mathbb{R}}\mathcal{F}_t)$. If $-\infty < a < b < \infty$, then as usual, an $\mathbb{R}^n$-valued process indexed by $[a,b]$ or $[a,\infty)$ is said to be a semimartingale with respect to $(\mathcal{F}_t)_{t\in[a,b]}$ respectively $(\mathcal{F}_t)_{t\in[a,\infty)}$ if it can be decomposed into the sum of an $\mathbb{R}^n$-valued càdlàg local martingale with respect to the same filtration and an $\mathbb{R}^n$-valued càdlàg adapted process of bounded variation on compacts; see Jacod and Shiryaev (2003) for the basic properties. An $\mathbb{R}^n$-valued process $X = (X_t)_{t\in\mathbb{R}}$ is a semimartingale with respect to $\mathscr{F}$ if $X_{-\infty} = \lim_{t\to-\infty} X_t$ exist a.s. and for all continuous and increasing functions $g\colon [0,\infty) \to [-\infty,\infty[$, $(X_{g(t)})_{t\in[0,\infty)}$ is an $(\mathcal{F}_{g(t)})_{t\in[0,\infty)}$-semimartingale. Finally, an $\mathbb{R}^n$-valued process $X = (X_t)_{t\in\mathbb{R}}$ is called a semimartingale up to infinity with respect to $\mathscr{F}$ if $X_{-\infty} = \lim_{t\to-\infty} X_t$ and $X_\infty = \lim_{t\to\infty} X_t$ exist a.s. and for all continuous and increasing functions $f\colon [0,1] \to [-\infty,\infty]$, $(X_{f(t)})_{t\in[0,1]}$ is an $(\mathcal{F}_{f(t)})_{t\in[0,1]}$-semimartingale. Unless there is a risk of confusion the filtration will typically not be mentioned explicitly.

Let $\mathcal{SM}$ be the space of all real-valued semimartingales up to infinity $X = (X_t)_{t\in\mathbb{R}}$ equipped with Emery's semimartingale topology

$$\|X\|_{\mathcal{SM}} = \sup_{\phi\in M(\mathscr{P};\mathbb{R}),|\phi|\leq 1} \Big\| \int_{-\infty}^{\infty} \phi_s \,\mathrm{d}X_s \Big\|_0, \qquad X \in \mathcal{SM}. \tag{3.1}$$

Recall that $(\mathcal{SM}, \|\cdot\|_{\mathcal{SM}})$ is an F-space (cf. Emery (1979) and Mémin (1980)).

An $\mathbb{R}^n$-valued process $X = (X_t)_{t\in\mathbb{R}}$ is called an *increment semimartingale* if for all $a \in \mathbb{R}$ the process $(X_t - X_a)_{t\geq a}$ is a semimartingale. Note that an increment semimartingale need not be adapted but all increments are adapted. The class of *increment local martingales*, defined in the obvious way, are studied in Basse-O'Connor et al. (2010a). In particular, Basse-O'Connor et al. (2010a), Remark 3.15, shows that if X is a continuous increment local martingale such that $X_{-\infty}$ exist a.s. then the process $X - X_{-\infty}$ is a continuous local martingale. There is no such result for continuous increment semimartingales. For example, if $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function for which $g_{-\infty} = \lim_{t\to-\infty} g_t$ exists and which is of bounded variation on compacts but of unbounded variation on $(-\infty, 0)$ then $(g_t)_{t\in\mathbb{R}}$ is an increment semimartingale but not a semimartingale.

Whenever $X = (X_t)_{t\in\mathbb{R}}$ is a semimartingale and τ is a truncation function on $\mathbb{R}^n$, i.e. $\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bounded function such that $\tau(x) = x$ in a neighborhood of zero, define $X(\tau)$ as

$$X_t(\tau) = X_t - \sum_{s\in(-\infty,t]} \big(\Delta X_s - \tau(\Delta X_s)\big), \qquad t \in \mathbb{R}.$$

Moreover, let μ^X denote the jump measure of X, that is,

$$\mu^X(A) = \sharp\big\{s \in \mathbb{R} : (s, \Delta X_s) \in A\big\}, \qquad A \in \mathscr{B}(\mathbb{R}\times\mathbb{R}^n_0),$$

where for a set D, $\sharp D$ denotes the number of elements in D, $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$ and $\Delta X_t = X_t - X_{t-}$ is the jump of X at t.

Let $(U,\mathscr{U})$ be a measurable space. Then $\mu=\mu(\omega;\mathrm{d}u)$ is said to be a random signed measure on U if for all $\omega\in\Omega$, $\mu(\omega;\cdot)$ is an $\mathbb{R}$-valued σ-finite measure on $(U,\mathscr{U})$ as defined in Subsection 2.1; i.e., $\mu(\omega;\emptyset)=0$ and there exist $(A_l)_{l\geq 1}\subseteq\mathscr{U}$ (depending in general on ω) such that $A_l\uparrow U$ and for all disjoint sets $(B_k)_{k\geq 1}\subseteq\mathscr{U}$ with $\cup_{k=1}^\infty B_k\subseteq A_l$ for some $l\geq 1$ we have

$$\mu\big(\omega;\bigcup_{k=1}^\infty B_k\big)=\sum_{k=1}^\infty \mu(\omega;B_k)\qquad\text{in }\mathbb{R}.$$

In case $A_l=U$ for all l, μ is called a finite random signed measure on U. If for all ω $\mu(\omega;\cdot)$ takes values in $[0,\infty)$, μ is called a random positive measure. In this case $\mu(\omega;\cdot)$ extends to a $[0,\infty]$-valued measure defined on the entire σ-field $\mathscr{U}$, and it is then a random measure in the sense of Jacod and Shiryaev (2003). Denote by $\mathrm{Var}(\mu;\cdot)$ the total variation of the random signed measure μ, that is, the positive random measure, finite on each A_l, given, for $A\in\mathscr{U}$ with $A\subseteq A_l$ for some l, by

$$\begin{aligned}&\mathrm{Var}(\mu;A)(\omega)\\&\quad=\sup\Big\{\sum_{i=1}^k|\mu(\omega;B_i)|: k\geq 1, (B_i)_{i=1}^k\subseteq\mathscr{U}\text{ are disjoint with }\cup_{i=1}^k B_i=A\Big\}.\end{aligned}$$

When $U=\mathbb{R}^n$ and $\mu(\omega;\{u\})=0$ for all $u\in U$ and $\omega\in\Omega$, we say that μ is continuous. When $\mu=(\mu^i)_{i\leq n}$ where each μ^i is a random signed measure on U we speak of μ as an n-dimensional random signed measure on U.

Let $\mathscr{P}$ denote the predictable σ-field on $\mathbb{R}\times\Omega$, i.e.,

$$\mathscr{P}=\sigma\Big((u,t]\times A: -\infty<u<t<\infty,\ A\in\mathcal{F}_u\Big),$$

and let $\tilde{\mathscr{P}}=\mathscr{P}\otimes\mathscr{U}$ and $\tilde{\Omega}=\mathbb{R}\times\Omega\times U$. A random measure μ on $\mathbb{R}\times U$ is said to be $\tilde{\mathscr{P}}$-σ-finite if there exists $(A_l)_{l\geq1}\subseteq\tilde{\mathscr{P}}$ such that $A_l\uparrow\tilde{\Omega}$ and such that $\mathrm{E}[\int 1_{A_l}(s,x)\,\mathrm{Var}(\mu;\mathrm{d}s\times\mathrm{d}x)]<\infty$ for all $l\geq 1$. For any $\tilde{\mathscr{P}}$-measurable functions $W=W(s,\omega,x)$ we will use the standard notation

$$W*\mu_t=\int_{-\infty}^t\int_U W(s,x)\,\mu(\mathrm{d}s\times\mathrm{d}x)\quad\text{and}\quad W*\mu=\int_{\mathbb{R}}\int_U W(s,x)\,\mu(\mathrm{d}s\times\mathrm{d}x),$$

whenever the integrals are well-defined. Furthermore, μ is called predictable if for all $\tilde{\mathscr{P}}$-measurable functions W, $(W*\mu_t)_{t\in\mathbb{R}}$ is predictable whenever it is well-defined. Optional random signed measures are defined similarly, see Jacod and Shiryaev (2003), Chapter II.1. Finally, a random signed measure μ on $\mathbb{R}$ is called $\mathscr{P}$-σ-finite if there is $(A_l)_{l\geq1}\subseteq\mathscr{P}$ such that $A_l\uparrow\mathbb{R}\times\Omega$ and $\mathrm{E}[\int 1_{A_l}(s)\,\mathrm{Var}(\mu;\mathrm{d}s)]<\infty$ for all $l\geq 1$.

4 Integration of predictable processes

In this section we introduce the stochastic integral on the predictable σ-field with respect to a σ-finite L^0-valued measure (cf. Section 2). In Schwartz (1981) such measures are called *formal semimartingales*; see also Emery (1982) and Bichteler and Jacod (1983). A key point in this paper is the introduction of the characteristic triplet of σ-finite measures; see Theorem 4.2. In the case of a finite measure (which corresponds to a semimartingale) this notion has been very successfully applied e.g. in Jacod and Shiryaev (2003). Thereafter, in Theorems 4.5–4.6, $L(m)$ is characterized in terms of the triplet of m, giving applicable conditions for a process to be integrable.

Consider the setting of Section 2 in the case where $(E, \mathscr{E}) = (\mathbb{R} \times \Omega, \mathscr{P})$; thus let $m = (m^i)_{i \leq n}$ denote an n-dimensional σ-finite L^0-valued measure on $(\mathbb{R} \times \Omega, \mathscr{P})$ and let $(O_k)_{k \geq 1} \subseteq \mathscr{P}$ satisfying $O_k \uparrow \mathbb{R} \times \Omega$ be m^i-feasible for all i; that is, m is defined on $\mathscr{P}_m = \{A \in \mathscr{P} : A \subseteq O_k \text{ for some } k \geq 1\}$. For all $t \in \mathbb{R}$ let $\Omega_t = (-\infty, t] \times \Omega$ and assume throughout that

(1) $\forall\, A \in \mathscr{P}_m,\ A \subseteq \Omega_t : m(A)$ is $\mathcal{F}_t$-measurable,

(2) $\forall\, A \in \mathscr{P}_m,\ u \in \mathbb{R},\ B \in \mathcal{F}_u : m\big(A \cap ((u, \infty) \times B)\big) = 1_B m\big(A \cap ((u, \infty) \times \Omega)\big)$.

Example 4.1. If m is a finite measure then the $\mathbb{R}^n$-valued process $X = (X_t)_{t \in \mathbb{R}}$ defined by $X_t = m(\Omega_t)$ can be chosen càdlàg and is then an $(\mathcal{F}_t)_{t \in \mathbb{R}}$-semimartingale up to infinity with $X_{-\infty} = 0$ by the Bichteler-Dellacherie Theorem (see Bichteler (1981) or Schwartz (1981)). Conversely, if we start out with an $\mathbb{R}^n$-valued $(\mathcal{F}_t)_{t \in \mathbb{R}}$-semimartingale X up to infinity and define m by $m(A) = \int_A 1 \,\mathrm{d}X$ for $A \in \mathscr{P}$ (where the integral is defined coordinatewise in the semimartingale sense) then m is a finite measure. When m is a finite measure we will often not distinguish between m and the corresponding semimartingale.

Another important example is when m is a Radon measure, by which we mean that m is a σ-finite L^0-valued measure with $O_k = [-k, k] \times \Omega$, $k \geq 1$. As a consequence of the Bichteler-Dellacherie Theorem it follows that m is a Radon measure if and only if there exists an $\mathbb{R}^n$-valued increment semimartingale $Z = (Z_t)_{t \in \mathbb{R}}$ with respect to $\mathscr{F}$ such that $m\big((u, t] \times B\big) = 1_B(Z_t - Z_u)$ for all $-\infty < u < t < \infty$ and $B \in \mathcal{F}_u$.

Note that if X is a semimartingale then $X_{-\infty}$ exists a.s.; however, in applications many processes of interest, such as Lévy processes, do not have a limit at $-\infty$, showing that it is not enough to consider finite measures.

Recall from Section 2 that an $\mathbb{R}^n$-valued predictable process $\phi = (\phi^i)_{i \leq n}$ induces a σ-finite L^0-valued measure on $(\mathbb{R} \times \Omega, \mathscr{P})$ defined as $\phi \bullet m(A) = \int_A \phi \cdot \mathrm{d}m$ for any predictable set A satisfying $A \subseteq \overline{O}_k = O_k \cap \{\|\phi\| \leq k\}$ for some k. The set of integrable predictable processes ϕ with respect to m is denoted by $L(m)$, and $L^1(m)$ is the subset of $L(m)$ for which $\int \phi \cdot \mathrm{d}m$ can be defined as $\int \phi \cdot \mathrm{d}m = \sum_i^n \int \phi^i \,\mathrm{d}m^i$. When m is induced by an increment semimartingale Z we often write $L(Z)$ instead of $L(m)$ and $\int \phi \cdot \mathrm{d}Z$ instead of $\int \phi \cdot \mathrm{d}m$. In the following we shall use both the characterization of $L(m)$ given in (2.5) as well the fact that $\phi \in L(m)$ if and only if $\phi \bullet m$ induces a finite one-dimensional L^0-valued measure. In this case $\int \phi \cdot \mathrm{d}m$ is defined as $\phi \bullet m(\mathbb{R} \times \Omega)$. According to (2.4) and Theorem 2.4 we have

$$\|\phi\|_m = \sup\{\|\int \psi \,\mathrm{d}(\phi \bullet m)\|_0 : \psi \in M(\mathscr{P}; \mathbb{R}), |\psi| \leq 1\}, \qquad \phi \in L(m),$$

which by (3.1) shows that $\|\phi\|_m = \|\phi \bullet m\|_{\mathcal{SM}}$. This implies that $(L(m), \|\cdot\|_m)$ is an F-space. To see this we only have to argue that $\|\cdot\|_m$ induces a complete metric, and by using Schwartz (1981), p. 427–428, we may and do assume that m is a finite measure, that is, corresponds to an $\mathbb{R}^n$-valued semimartingale up to infinity. Since $\|\phi\|_m = \|\phi \bullet m\|_{\mathcal{SM}}$ the metric induced by $\|\cdot\|_m$ coincides with the one imposed by Mémin (1980) so the result is given in his Théorème V.4.

For all $A \in \mathscr{P}$ and $\phi \in M(\mathscr{P};\mathbb{R}^n)$ satisfying $1_A\phi \in L(m)$ we use the notation $\int_A \phi \cdot \mathrm{d}m$ rather than $\int (1_A\phi) \cdot \mathrm{d}m$; if $A = (s,t] \times \Omega$ with $s < t$ we write $\int_{(s,t]}$ or $\int_s^t$ instead of $\int_A$ and similarly when $A = (s,\infty) \times \Omega$.

If $\phi 1_{\Omega_t^c} \in L(m)$ for all $t \in \mathbb{R}$ and $\lim_{t\to-\infty} \int_t^\infty \phi \cdot \mathrm{d}m$ exists a.s. the limit is called the *improper integral* of ϕ. Obviously, if $\phi \in L(m)$ then the improper integral exists but the converse is only true in special cases. For example, let $(\xi_t, \eta_t)_{t\in\mathbb{R}}$ be a bivariate Lévy process indexed by $\mathbb{R}$ with $(\xi_0, \eta_0) = (0,0)$ and let $t > 0$. Then the improper integral $\lim_{s\to-\infty} \int_s^t e^{\xi_u} \,\mathrm{d}\eta_u$ exists if and only if $(1_{\{u\le t\}} e^{\xi_u})_{u\in\mathbb{R}} \in L(\eta)$; see Basse-O'Connor et al. (2010b). The appropriate choice of filtration when working Lévy processes is discussed in Section 5. In another direction, if Z is a continuous increment local martingale and for all $s < t$, $\int_s^t \phi_u \cdot \mathrm{d}Z_u$ exists then the improper integral $\lim_{s\to-\infty} \int_s^t \phi_u \cdot \mathrm{d}Z_u$ exists if and only if $(1_{\{u\le t\}}\phi_u)_{u\in\mathbb{R}} \in L(Z)$; see Basse-O'Connor et al. (2010a).

4.1 The triplet of random measures

For all $k \geq 1$ let $X(k) = \{(X_t^i(k))_{i\le n} : t \in \mathbb{R}\}$ be given by $X_t(k) = m\big(O_k \cap \Omega_t\big)$. As mentioned in Example 4.1 we may and will assume that $X(k)$ is a càdlàg $\mathscr{F}$-semimartingale up to infinity for all $k \geq 1$. Note that $X(k)$ satisfies

$$X_t(k) = \int_{-\infty}^t 1_{O_k}(s) \,\mathrm{d}X_s(k+1), \qquad k \geq 1, t \in \mathbb{R}.$$

Thus, if there exists a $k \geq 1$ (depending on ω) such that $\Delta X_t^i(k)(\omega) \neq 0$ then $\Delta X_t^i(k)(\omega) = \Delta X_t^i(l)(\omega)$ for all $l > k$ and $i = 1, \ldots, n$. Define m's *jump process* $\Delta m = \{(\Delta m_t^i)_{i\le n} : t \in \mathbb{R}\}$ as

$$\Delta m_t^i = \begin{cases} \Delta X_t^i(k) & \text{whenever } \Delta X_t^i(k) \neq 0 \text{ for some } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and m's *jump measure* μ as

$$\mu(A) = \sharp\big\{t \in \mathbb{R} : (t, \Delta m_t) \in A\big\}, \qquad A \in \mathscr{B}(\mathbb{R} \times \mathbb{R}_0^n).$$

For every $A \in \mathscr{P}$ such that $1_A \in L^1(m^i)$ for $i = 1, \ldots, n$, define the semimartingale up to infinity $(m(A)_t)_{t\in\mathbb{R}}$ as $m(A)_t = (m^i(A)_t)_{i\le n} = m(A \cap \Omega_t)$. Call m a *continuous martingale measure* if for all such A the process $(m(A)_t)_{t\in\mathbb{R}}$ is an n-dimensional continuous local martingale. Note that since each $X(k)$ is a semimartingale, its jump measure is finite. Consequently μ, the jump measure of m, is a $\tilde{\mathscr{P}}$-σ-finite integer-valued random measure on $\mathbb{R} \times \mathbb{R}_0^n$ and hence by Jacod and Shiryaev (2003), Theorem II.1.8, has a predictable compensator ν. Thus, in particular ν is a $\tilde{\mathscr{P}}$-σ-finite measure on $\mathbb{R} \times \mathbb{R}_0^n$. Denote by $G_{\text{loc},\infty}(\mu)$ the set of $\mathbb{R}$-valued $\tilde{\mathscr{P}}$-measurable functions $W = W(s, \omega, x)$ for which $W * (\mu - \nu)$ exists up to infinity. That is, there is a localizing sequence $(\sigma_k)_{k \geq 1}$ with $P(\sigma_k = \infty) \to 1$ such that, with $\widetilde{W}$ defined as in Jacod and Shiryaev (2003), Definition II.1.27, we have $\mathrm{E}[(\sum_{s \leq \sigma_k} (\widetilde{W}_s)^2)^{1/2}] < \infty$. We use the notation $W * (\mu - \nu)$ for the integral over $\mathbb{R} \times \mathbb{R}_0^n$ of W with respect to $\mu - \nu$.

The next result concerns the canonical decomposition of m relative to a truncation function τ. Other decomposition results of L^p-valued measure are studied in Bichteler and Jacod (1983). As usual we generally suppress ω in random variables.

Theorem 4.2. *There exist a continuous martingale measure $m^c = (m^{c,i})_{i \leq n}$, a predictable compensator ν of μ and an n-dimensional $\mathscr{P}$-σ-finite random signed measure $B = (B^i)_{i \leq n}$ on $\mathbb{R}$ such that for all $A \in \mathscr{P}_m$, $1_A \in L^1(m^{c,i})$ for $i = 1, \dots, n$, and*

$$m(A) = m^c(A) + \big(1_A(s)\tau(x)\big) * (\mu - \nu) + \big(1_A(s)(x - \tau(x))\big) * \mu + \int_{\mathbb{R}} 1_A(s)\, B(\mathrm{d}s). \quad (4.1)$$

Furthermore, there exists an $n \times n$-dimensional predictable $\mathscr{P}$-σ-finite random signed measure $C = (C^{i,j})_{i,j \leq n}$ on $\mathbb{R}$ such that for $i, j = 1, \dots, n$ and $t \in \mathbb{R}$

$$\langle m^{c,i}(A)., m^{c,j}(A).\rangle_t = \int_{-\infty}^t 1_A(s)\, C^{i,j}(\mathrm{d}s), \qquad A \in \mathscr{P}_m$$

where $\langle \cdot, \cdot \rangle_t$ denotes the predictable quadratic variation. Moreover, (B, C, ν) is unique.

Finally, there are a $\mathscr{P}$-σ-finite predictable random positive measure $\lambda = \lambda(\omega; \mathrm{d}s)$ on $\mathbb{R}$, two predictable processes, $b = \{(b_t^i)_{i \leq n} : t \in \mathbb{R}\}$ with values in $\mathbb{R}^n$ and $c = \{(c_t^{i,j})_{i,j \leq n} : t \in \mathbb{R}\}$ taking values in the symmetric positive semidefinite $n \times n$-matrices, and a transition kernel $K = K(s, \omega; \mathrm{d}x)$ from $(\mathbb{R} \times \Omega, \mathscr{P})$ into $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ such that $K(s; \{0\}) = 0$ and $\int_{\mathbb{R}^n} \big(1 \wedge \|x\|^2\big)\, K(s; \mathrm{d}x) < \infty$ for all $s \in \mathbb{R}$ and

$$B(\mathrm{d}s) = b_s\, \lambda(\mathrm{d}s), \qquad C(\mathrm{d}s) = c_s\, \lambda(\mathrm{d}s), \qquad \nu(\mathrm{d}s \times \mathrm{d}x) = K(s; \mathrm{d}x)\lambda(\mathrm{d}s). \quad (4.2)$$

The triplet (B, C, ν) given in Theorem 4.2 is called the triplet of m and will play an important role in this paper. From the proof follows that if m is a finite measure then so is m^c. When $Z = (Z_t)_{t \in \mathbb{R}}$ is an increment semimartingale with associated random measure m, (B, C, ν) will also be called the triplet of Z.

Remark 4.3.

(i) Let us describe the right-hand side of (4.1) in more detail. The second term is defined coordinatewise and $W(s, x) = 1_A(s)\tau^i(x)$ is in $G_{\text{loc},\infty}(\mu)$ for all $i = 1, \dots, n$ (τ^i is the ith coordinate). The third term is defined coordinatewise as well and is in fact just a finite sum. Finally, $\int 1_A(s) \operatorname{Var}(B^i; \mathrm{d}s) < \infty$ a.s. for all $i = 1, \dots, n$.

(ii) In Section 2 we noted that $L(m)$ is invariant under the choice of $(O_k)_{k \geq 1}$. Similarly, it follows by uniqueness that the triplet is invariant under the choice of $(O_k)_{k \geq 1}$.

Proof of Theorem 4.2. For $k \geq 1$ let m_k be the finite measure defined by $m_k(A) = m(A \cap O_k)$ for $A \in \mathscr{P}$. In other words, m_k is the measure associated with $X(k)$ as described in Example 4.1. As shown in Jacod and Shiryaev (2003), Theorem II.2.34, there exists a unique triplet (B_k, C_k, ν_k) such that

$$X_t(k) = X_t^c(k) + \tau(x) * (\mu^{X(k)} - \nu_k)_t + (x - \tau(x)) * \mu_t^{X(k)} + B_k((-\infty, t]).$$

and so, denoting by m_k^c the n-dimensional finite measure corresponding to $X^c(k)$, we get, for $A \in \mathscr{P}$,

$$m_k(A) = m_k^c(A) + (1_A(s)\tau(x)) * (\mu_k - \nu_k) + (1_A(s)(x - \tau(x))) * \mu_k + \int_{\mathbb{R}} 1_A(s)\, B_k(\mathrm{d}s).$$

Since

$$m_k(A) = m_{k+1}(A) \quad \text{for } A \in \mathscr{D}_k := \{B \in \mathscr{P} : B \subseteq O_k\} \tag{4.3}$$

we have $m_k^c = m_{k+1}^c$ on $\mathscr{D}_k$ by uniqueness. Hence we can define a σ-finite measure m^c on $\mathscr{P}_m = \cup_{k=1}^{\infty} \mathscr{D}_k$ to be equal to m_k^c on each $\mathscr{D}_k$. Again by (4.3) it follows that

$$\begin{aligned} \mu_k(\mathrm{d}s \times \mathrm{d}x) &= 1_{O_k}(s)\mu_{k+1}(\mathrm{d}s \times \mathrm{d}x), \\ B_k(\mathrm{d}s) = 1_{O_k}(s)B_{k+1}(\mathrm{d}s), &\qquad C_k(\mathrm{d}s) = 1_{O_k}(s)C_{k+1}(\mathrm{d}s), \end{aligned}$$

and therefore we may define $B(\mathrm{d}s)$ to be equal to $B_k(\mathrm{d}s)$ on $\mathscr{D}_k$ and $C(\mathrm{d}s)$ to be equal to $C_k(\mathrm{d}s)$ on $\mathscr{D}_k$. By construction of (B, C, ν), (4.1) holds. Note also that μ by definition equals μ_k on $\mathscr{D}_k$ for all $k \geq 1$ and hence $\nu = \sup_{k \geq 1} \nu_k$ by monotone convergence. The uniqueness of (B, C, ν) follows by the uniqueness of (B_k, C_k, ν_k) for all $k \geq 1$.

Due to the fact that ν is a $\tilde{\mathscr{P}}$-σ-finite measure the existence of a $\mathscr{P}$-σ-finite measure $\tilde{\lambda}$ and a transition kernel $\tilde{K}$ such that $\nu(\mathrm{d}s \times \mathrm{d}x) = \tilde{K}(s; \mathrm{d}x)\, \tilde{\lambda}(\mathrm{d}s)$ follows by general disintegration theory, see e.g. Jacod and Shiryaev (2003), Chapter II, 1.2. The construction of b, c, K and λ satisfying (4.2) is now obvious. □

Throughout let τ_1 and τ_n denote truncation functions on respectively $\mathbb{R}$ and $\mathbb{R}^n$.

Proposition 4.4. *For all $\mathbb{R}^n$-valued predictable processes ϕ, the σ-finite measure $\phi \bullet m$ has jump measure $\mu_{\phi\bullet m}$ and continuous martingale measure $(\phi \bullet m)^c$ given by*

$$\mu_{\phi\bullet m} = \mu \circ \Big((s, x) \mapsto (s, \langle \phi_s, x \rangle)\Big)^{-1}, \qquad (\phi \bullet m)^c = \phi \bullet m^c,$$

and its triplet $(\tilde{B}, \tilde{C}, \tilde{\nu})$ is given by

$$\begin{aligned} \tilde{B}(\mathrm{d}s) &= \Big(\langle \phi_s, b_s \rangle + \int_{\mathbb{R}^n} \big[\tau_1(\langle \phi_s, x \rangle) - \langle \phi_s, \tau_n(x) \rangle\big]\, K(s, \mathrm{d}x)\Big)\, \lambda(\mathrm{d}s), \\ \tilde{C}(\mathrm{d}s) &= \langle \phi_s, c_s \phi_s \rangle\, \lambda(\mathrm{d}s), \qquad \tilde{\nu} = \nu \circ \Big((s, x) \mapsto (s, \langle \phi_s, x \rangle)\Big)^{-1}. \end{aligned}$$

Here for all $x, y \in \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x^i y^i$ denotes the usually inner product in $\mathbb{R}^n$. To prove Proposition 4.4 it is enough to consider each of the measures $m(\cdot \cap \overline{O}_k)$ for $k \geq 1$ where $\overline{O}_k$ is defined just below Example 4.1. However, $m(\cdot \cap \overline{O}_k)$ corresponds to an $\mathbb{R}^n$-valued semimartingale up to infinity and $\phi 1_{\overline{O}_k}$ is a bounded predictable process, so the results follows from Jacod and Shiryaev (2003), Chapter IX, Proposition 5.3.

Let $\phi \in L(m)$. Since in this case $\phi \bullet m$ is a finite measure it follows from Theorem 4.2 and Proposition 4.4 that $\phi \in L(m^c)$, $\tau_1\big(\langle \phi_s, x\rangle\big) \in G_{\mathrm{loc},\infty}(\mu)$, $\tilde{B}$ is a finite random signed measure on $\mathbb{R}$ and

$$\begin{aligned}\int \phi \cdot \mathrm{d}m & \\ &= \int \phi \cdot \mathrm{d}m^c + \tau_1\big(\langle \phi_s, x\rangle\big) * (\mu - \nu) + \big(\langle \phi_s, x\rangle - \tau_1(\langle \phi_s, x\rangle)\big) * \mu + \tilde{B}((-\infty, \infty)).\end{aligned} \tag{4.4}$$

4.2 A characterization of $L(m)$

In Section 2 we characterized $L(m)$ by means of $\|\cdot\|_m$ which, however, is rarely known explicitly. The next result characterizes $L(m)$ in terms of the triplet (B, C, ν) of m which is often known; see e.g. Section 5. We assume throughout this subsection that m has triplet of the form (4.2) with respect to $\tau = \tau_n$.

Theorem 4.5. *For all $\mathbb{R}^n$-valued predictable processes ϕ we have $\phi \in L(m)$ if and only if the following (a)–(c) are satisfied almost surely:*

$$(a)\quad \int_{\mathbb{R}} \Big| \langle \phi_s, b_s\rangle + \int_{\mathbb{R}^n} \big[\tau_1(\langle \phi_s, x\rangle) - \langle \phi_s, \tau_n(x)\rangle\big]\, K(s; \mathrm{d}x) \Big|\, \lambda(\mathrm{d}s) < \infty, \tag{4.5}$$

$$(b)\quad \int_{\mathbb{R}} \langle \phi_s, c_s \phi_s\rangle\, \lambda(\mathrm{d}s) < \infty, \qquad (c)\quad \int_{\mathbb{R}}\int_{\mathbb{R}^n} \big(1 \wedge |\langle \phi_s, x\rangle|^2\big)\, K(s; \mathrm{d}x)\, \lambda(\mathrm{d}s) < \infty. \tag{4.6}$$

Note that when m is a continuous measure (i.e., $\mu = 0$), (4.5)–(4.6) reduce to

$$\int_{\mathbb{R}} \Big(|\langle \phi_s, b_s\rangle| + \langle \phi_s, c_s \phi_s\rangle \Big)\, \lambda(\mathrm{d}s) < \infty \qquad \text{a.s.}$$

Set

$$\begin{aligned}
U(s, x) &= \Big| \langle x, b_s\rangle + \int_{\mathbb{R}^n} \big[\tau_1(\langle x, y\rangle) - \langle x, \tau_n(y)\rangle\big]\, K(s; \mathrm{d}y) \Big|, && s \in \mathbb{R}, x \in \mathbb{R}^n, \\
\Phi(s, x) &= \langle x, c_s x\rangle + U(s, x) + \int_{\mathbb{R}^n} (1 \wedge |\langle x, y\rangle|^2)\, K(s; \mathrm{d}y), && s \in \mathbb{R}, x \in \mathbb{R}^n, \\
L^{\Phi,0} &= \big\{\phi \in M(\mathscr{P}; \mathbb{R}^n) : \int_{\mathbb{R}} \Phi(s, \phi_s)\, \lambda(\mathrm{d}s) < \infty \ \text{ a.s.}\big\}, && \\
\Psi_{\Phi,0}(\phi) &= \mathrm{E}\big[\big| \int_{\mathbb{R}} \Phi(s, \phi_s)\, \lambda(\mathrm{d}s)\big| \wedge 1\big], && \phi \in L^{\Phi,0},
\end{aligned}$$

and $\tilde{U}(s,x)=\sup_{c\in[-1,1]}U(s,cx)$. Recall that $A\in\mathscr{P}_m$ is called a null-set for m if for all $B\in\mathscr{P}_m$ with $B\subseteq A$, $m(B)=0$. Moreover, a deterministic positive σ-finite measure κ on $\mathscr{P}$ with the same null-sets as m is called a control measure for m. The following result extends Theorem 4.5:

Theorem 4.6. *$L(m)=L^{\Phi,0}$ and for all $(\phi^k)_{k\geq 1}$ and ϕ in $L(m)$, $\phi^k\to\phi$ in $L(m)$ if and only if $\Psi_{\Phi,0}(\phi^k-\phi)\to 0$.*

Moreover $\lambda\otimes\mathrm{P}$, given by $(\lambda\otimes\mathrm{P})(A)=\mathrm{E}[\int_{\mathbb{R}}1_A(s)\,\lambda(\mathrm{d}s)]$, is a control measure for m whenever λ is chosen such that $\lambda\big(s\in\mathbb{R}:b_s=0,c_s=0,K(s;\mathbb{R}^n)=0\big)=0$ a.s.

When $n=1$ and m is finite and quasi-left continuous, Theorem 4.6 is obtained by Kwapień and Woyczyński (1991), Theorem 6.1, via decoupling techniques, whereas our approach is based on martingale theory.

Assume $n=1$ and set

$$\tilde{\Phi}(s,x)=c_sx^2+\tilde{U}(s,x)+\int_{\mathbb{R}}(1\wedge|xy|^2)\,K(s;\mathrm{d}y),$$

$$\Psi_{\tilde{\Phi},0}(\phi)=\mathrm{E}\big[\big|\int_{\mathbb{R}}\tilde{\Phi}(s,\phi_s)\,\lambda(\mathrm{d}s)\big|\wedge 1\big],\qquad \phi\in L^{\Phi,0}.$$

From Musielak-Orlicz theory (see Musielak (1983), Definition 7.2) it follows that $\Psi_{\tilde{\Phi},0}$ is a modular. Moreover, by arguing as in Rajput and Rosiński (1989) it can be shown that $\Psi_{\tilde{\Phi},0}$ satisfies the Δ_2-condition, i.e., there exists a constant $v>0$ such that $\Psi_{\tilde{\Phi},0}(2\phi)\leq v\Psi_{\tilde{\Phi},0}(\phi)$ for all $\phi\in L^{\Phi,0}$. Hence by Musielak (1983), Theorems 1.5 and 7.7, $L^{\Phi,0}$ is an F-space in the F-norm,

$$\|\phi\|_{\tilde{\Phi},0}=\inf\big\{t>0:\Psi_{\tilde{\Phi},0}(\phi/t)\leq t\big\},$$

and $\Psi_{\tilde{\Phi},0}(\phi^k)\to 0$ if and only if $\|\phi^k\|_{\tilde{\Phi},0}\to 0$. Therefore, by Lemma 4.9, we can restate Theorem 4.6 as follows:

Theorem 4.7. *For $n=1$, $L(m)$ and $L^{\Phi,0}$ are equivalent F-spaces, that is, the F-norms induce the same topology.*

4.3 Proofs of Theorems 4.5–4.6 and convergence in $\mathcal{SM}$

To prove Theorem 4.5 we need the following Lemmas 4.8–4.9.

Lemma 4.8. *Let μ be a $\tilde{\mathscr{P}}$-σ-finite optional random positive measure on $\mathbb{R}^n_0$ with predictable compensator ν and let $W=W(s,\omega,x)$ be a positive $\tilde{\mathscr{P}}$-measurable function on $\mathbb{R}\times\Omega\times\mathbb{R}^n_0$. Then $W*\nu<\infty$ a.s. implies $W*\mu<\infty$ a.s.*

Proof. For all $k\geq 1$ let $\sigma_k=\inf\{t\in\mathbb{R}:W*\nu_t>k\}$ which is a predictable stopping time since $(W*\nu_t)_{t\in\mathbb{R}}$ is a predictable process, and hence $(-\infty,\sigma_k)\in\mathscr{P}$ for all $k\geq 1$. Therefore,

$$\mathrm{E}[(W1_{(-\infty,\sigma_k)})*\mu]=\mathrm{E}[(W1_{(-\infty,\sigma_k)})*\nu]\leq k,$$

implying $W*\mu_{\sigma_k-}<\infty$ a.s. for all $k\geq 1$. Furthermore, since $\sigma_k=\infty$ for k sufficiently large we obtain $W*\mu<\infty$ a.s. □

The next lemma follows as Rajput and Rosiński (1989), Lemma 2.8.

Lemma 4.9. *There exists a constant $C_1 \in (0,\infty)$, only depending on τ_1, such that*

$$\tilde{U}(s,x) \le U(s,x) + C_1 \int_{\mathbb{R}^n} \big(1 \wedge |\langle x,y\rangle|^2\big)\, K(s;\mathrm{d}y), \qquad s \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Proof of Theorem 4.5. Let $\phi \in L(m)$. In this case $\phi \bullet m$ is a finite measure, and its triplet $(\tilde{B},\tilde{C},\tilde{\nu})$ is given in Proposition 4.4. From usual semimartingale theory we know that $\tilde{B}$ has finite total variation on $(-\infty,\infty)$ (implying (a)), $\tilde{C}$ has finite total variation on $(-\infty,\infty)$ a.s. (implying (b)) and finally $\int_{\mathbb{R}\times\mathbb{R}}(1\wedge|y|^2)\,\tilde{\nu}(\mathrm{d}s\times\mathrm{d}y) < \infty$ a.s. (implying (c)).

Conversely, assume that ϕ satisfies (4.5)–(4.6) and let us show that $\phi \in L(m)$, i.e., with

$$D = \{\psi \in M(\mathscr{P};\mathbb{R}) :\ |\psi| \le 1,\ \psi\phi \in L^1(m)\}, \tag{4.7}$$

we need by (2.4)–(2.5) to show that $\{\int(\psi\phi)\cdot\mathrm{d}m : \psi \in D\}$ is bounded in L^0. Let $\psi \in D$. By (a), (c) and Lemma 4.9 (with $C_1 > 0$ given there), we have

$$\begin{aligned}\int_{\mathbb{R}} U(s,\psi_s\phi_s)\,\lambda(\mathrm{d}s) &\le \int_{\mathbb{R}} \tilde{U}(s,\psi_s\phi_s)\,\lambda(\mathrm{d}s) \le \int_{\mathbb{R}} \tilde{U}(s,\phi_s)\,\lambda(\mathrm{d}s)\\ &\le C_1\Big[\int_{\mathbb{R}}\Big|\langle\phi_s,b_s\rangle + \int_{\mathbb{R}^n}\big[\tau_1(\langle\phi_s,x\rangle) - \langle\phi_s,\tau_n(x)\rangle\big]\,K(s;\mathrm{d}x)\Big|\,\lambda(\mathrm{d}s)\\ &\qquad + \int_{\mathbb{R}}\int_{\mathbb{R}^n}\big(1\wedge|\langle\phi_s,x\rangle|^2\big)\,K(s;\mathrm{d}x)\,\lambda(\mathrm{d}s)\Big] < \infty,\end{aligned}$$

which shows that

$$B(\psi) := \int_{\mathbb{R}}\Big(\langle\psi_s\phi_s,b_s\rangle + \int_{\mathbb{R}^n}\big[\tau_1(\langle\psi_s\phi_s,x\rangle) - \langle\psi_s\phi_s,\tau_n(x)\rangle\big]\,K(s;\mathrm{d}x)\Big)\,\lambda(\mathrm{d}s),$$

is well-defined and $\sup_{\psi\in D}|B(\psi)| < \infty$ a.s. In particular, $\{B(\psi) : \psi \in D\}$ is bounded in L^0.

For $\psi \in D$ we have by Lenglart's inequality (Lenglart (1977), Théorème I) that for all $\delta,\theta > 0$

$$\begin{aligned}\mathrm{P}\Big(\Big|\int(\psi\phi)\cdot\mathrm{d}m^c\Big| \ge \theta\Big) &\le \delta/\theta^2 + \mathrm{P}\Big(\int_{\mathbb{R}}\psi_s^2\langle\phi_s,c_s\phi_s\rangle\,\lambda(\mathrm{d}s) \ge \delta\Big)\\ &\le \delta/\theta^2 + \mathrm{P}\Big(\int_{\mathbb{R}}\langle\phi_s,c_s\phi_s\rangle\,\lambda(\mathrm{d}s) \ge \delta\Big),\end{aligned}$$

which by (b) shows that $\{\int(\phi\psi)\cdot\mathrm{d}m^c : \psi \in D\}$ is bounded in L^0.

Using (c) and the fact that $|\tau_1(x)| \le r(1\wedge|x|)$ for some $r > 0$, Lenglart's inequality shows

$$\begin{aligned}\mathrm{P}\Big(|\tau_1(\langle x,\psi_s\phi_s\rangle) * (\mu-\nu)| > \theta\Big) &\le \delta/\theta^2 + \mathrm{P}\Big(\tau_1(\langle x,\psi_s\phi_s\rangle)^2 * \nu > \delta\Big)\\ &\le \delta/\theta^2 + \mathrm{P}\Big(\big(1\wedge\langle x,\psi_s\phi_s\rangle^2\big) * \nu > \delta/r\Big) \le \delta/\theta^2 + \mathrm{P}\Big(\big(1\wedge\langle x,\phi_s\rangle^2\big) * \nu > \delta/r\Big)\end{aligned}$$

implying that $\{\tau_1(\langle x,\psi_s\phi_s\rangle)*(\mu-\nu):\psi\in D\}$ is bounded in L^0.

Let $f(x)=r|x|1_{\{|x|>\epsilon\}}$ where $r,\epsilon>0$ are chosen such that $|x-\tau_1(x)|\le f(x)$. Using that f is symmetric and increasing on $\mathbb{R}_+$ it follows that

$$|(\langle x,\psi_s\phi_s\rangle-\tau_1(\langle x,\psi_s\phi_s\rangle))*\mu|\le f(\langle x,\psi_s\phi_s\rangle)*\mu\le f(\langle x,\phi_s\rangle)*\mu.$$

By (c) and Lemma 4.8, $(|\langle x,\phi_s\rangle|^2\wedge 1)*\mu<\infty$ a.s. In particular, we find that the sum $\sum_{s\in\mathbb{R}}1_{\{|\langle\Delta m_s,\phi_s\rangle|>\epsilon\}}<\infty$ a.s. and hence

$$f(\langle x,\phi_s\rangle)*\mu=r\sum_{s\in\mathbb{R}}|\langle\Delta m_s,\phi_s\rangle|1_{\{|\langle\Delta m_s,\phi_s\rangle|>\epsilon\}}<\infty\qquad\text{a.s.}$$

implying that $\{(\langle x,\psi\phi\rangle-\tau_1(\langle x,\psi\phi\rangle)))*\mu:\psi\in D\}$ is bounded in L^0.

By (4.4) with $\psi\phi$ playing the role of ϕ we have

$$\begin{aligned}\int(\psi\phi)\cdot\mathrm{d}m=&\int(\psi\phi)\cdot\mathrm{d}m^c+\tau_1\big(\langle x,\psi_s\phi_s\rangle\big)*(\mu-\nu)\\&+\big(\langle x,\psi_s\phi_s\rangle-\tau_1(\langle x,\psi_s\phi_s\rangle)\big)*\mu+B(\psi),\end{aligned}$$

and the above shows that the right-hand side is bounded in L^0 as ψ runs through D. Therefore, $\{\int(\psi\phi)\cdot\mathrm{d}m:\psi\in D\}$ is bounded in L^0 and the proof is complete. □

Before proving Theorem 4.6 we study the relations between convergence in $\mathcal{SM}$ and convergence of triplets. Let $X^k=(X^k_t)_{t\in\mathbb{R}}$ and $X=(X_t)_{t\in\mathbb{R}}$ be real-valued semimartingales up to infinity with $X^k_{-\infty}=X_{-\infty}=0$ and let

$$\left(\begin{pmatrix}B^X\\B^{X^k}\end{pmatrix},\begin{bmatrix}C^X & C^{X,X^k}\\C^{X,X^k} & C^{X^k}\end{bmatrix},\nu^{(X,X^k)}\right)$$

denote the triplet of the $\mathbb{R}^2$-valued semimartingale (X,X^k). Write e.g. B^k_t instead of $B^k((-\infty,t])$ and B^k_∞ for $B^k(\mathbb{R})$.

Theorem 4.10. *$X^k\to X$ in $\mathcal{SM}$ if and only if for $k\to\infty$,*

$$C^{X^k}_\infty+C^X_\infty-2C^{X^k,X}_\infty\longrightarrow 0 \tag{4.8}$$

$$(1\wedge|x^k-x|^2)*\nu^{(X^k,X)}\longrightarrow 0 \tag{4.9}$$

$$\mathrm{Var}\big(B^{X^k}-B^X+[\tau(x^k-x)-(\tau_1(x^k)-\tau_1(x))]*\nu^{(X^k,X)}\big)_\infty\longrightarrow 0, \tag{4.10}$$

where the convergences are in L^0.

In particular,

Corollary 4.11. *$X^k\to 0$ in $\mathcal{SM}$ if and only if* $\mathrm{Var}(B^{X^k})_\infty\to 0, C^{X^k}_\infty\to 0$ *and* $(1\wedge x^2)*\nu^{X^k}\to 0$ *in L^0.*

For the proof we need the following.

Lemma 4.12. *Let $W = W(s, \omega, x)$ be a positive, predictable and bounded function. Then, $W * \mu^{X^k} \to 0$ in L^0 if and only if $W * \nu^{X^k} \to 0$ in L^0.*

Proof of Lemma 4.12. Assume that $W * \mu^{X^k} \to 0$ in L^0. For all $k \geq 1$ let

$$\sigma_k = \inf\{t \in \mathbb{R} : W * \mu_t^{X^k} > 1\}.$$

Assume $W \leq c$, then for all $k \geq 1$, $W * \mu_{\sigma_k}^{X^k} \leq 1 + c$ and therefore,

$$0 = \lim_{k\to\infty} \mathrm{E}[W * \mu_{\sigma_k}^{X^k}] = \lim_{k\to\infty} \mathrm{E}[W * \nu_{\sigma_k}^{X^k}]. \tag{4.11}$$

For all $\delta > 0$ we have by (4.11),

$$\begin{aligned}\limsup_{k\to\infty} \mathrm{P}(W * \nu^{X^k} > \delta) &\leq \limsup_{k\to\infty} \mathrm{P}(W * \nu_{\sigma_k}^{X^k} > \delta) + \limsup_{k\to\infty} \mathrm{P}(\sigma_k < \infty) \\ &= \limsup_{k\to\infty} \mathrm{P}(\sigma_k < \infty) \leq \limsup_{k\to\infty} \mathrm{P}(W * \mu^{X^k} > 1) = 0,\end{aligned}$$

which shows that $W * \nu^{X^k} \to 0$ in L^0. The reverse implication follows similarly. □

Proof of Theorem 4.10. First assume $X = 0$. To show the *if*-implication let (4.8)–(4.10) be satisfied. For all $t \in \mathbb{R}$,

$$X_t^k = X_t^{k,c} + \tau_1(x) * (\mu^{X^k} - \nu^{X^k})_t + (x - \tau_1(x)) * \mu_t^{X^k} + B_t^{X^k} \tag{4.12}$$

where $X^{k,c}$ denotes the continuous martingale part of X^k. We shall show that each term on the right-hand side converges to 0 in $\mathcal{SM}$. The total variation of the last term converges to 0 in L^0, which, since it is predictable, by Mémin (1980), Théorème IV.7, is equivalent to convergence in $\mathcal{SM}$. By Lenglart's inequality, for all predictable processes ϕ with $|\phi| \leq 1$ and $\theta, \delta > 0$,

$$\sup_{|\phi|\leq 1} \mathrm{P}\Big(|\int_{-\infty}^{\infty} \phi_s \,\mathrm{d}X_s^{k,c}| \geq \theta\Big) \leq \delta/\theta^2 + \mathrm{P}(C_\infty^{X^k} \geq \delta)$$

and with $M_t^k := \tau_1(x) * (\mu^{X^k} - \nu^{X^k})_t$ and using that $(\tau_1(x))^2 \leq a(1 \wedge x^2)$ for some $a > 0$,

$$\begin{aligned}&\sup_{|\phi|\leq 1} \mathrm{P}\Big(|\int_{-\infty}^{\infty} \phi_s \,\mathrm{d}M_s^k| \geq \theta\Big) \\ &\qquad \leq \delta/\theta^2 + \mathrm{P}(\tau_1(x)^2 * \nu^{X^k} \geq \delta) \leq \delta/\theta^2 + \mathrm{P}((1 \wedge x^2) * \nu^{X^k} \geq \delta/a).\end{aligned}$$

With $A_t^k := (x - \tau_1(x)) * \mu_t^{X^k}$ and if $\epsilon, v > 0$ are chosen such that $|x - \tau_1(x)| \leq v|x|1_{\{|x|>\epsilon\}}$ we have

$$\sup_{|\phi|\leq 1} \Big| \int_{-\infty}^{\infty} \phi_s \,\mathrm{d}A_s^k \Big| \leq \sup_{|\phi|\leq 1} \sum_{s\in\mathbb{R}} v|\phi_s \Delta X_s^k| 1_{\{|\Delta X_s^k|>\epsilon\}} \leq v \sum_{s\in\mathbb{R}} |\Delta X_s^k| 1_{\{|\Delta X_s^k|>\epsilon\}}. \tag{4.13}$$

By Lemma 4.12 and (4.9), $(1 \wedge x^2) * \mu^{X^k} \to 0$ in L^0, and therefore for all $\theta \in (0, \epsilon \wedge 1)$,

$$\mathrm{P}\Big(\sum_{s\in\mathbb{R}} |\Delta X_s^k| 1_{\{|\Delta X_s^k|>\epsilon\}} > \theta\Big) = \mathrm{P}\Big(\sum_{s\in\mathbb{R}} 1_{\{|\Delta X_s^k|>\epsilon\}} > \theta\Big) \to 0.$$

Thus, all terms on the right-hand side of (4.12) converge to 0 in $\mathcal{SM}$, which completes the proof of the *if*-part.

To show the *only if*-part assume that $X^k \to 0$ in $\mathcal{SM}$. By Emery (1979), cf. also Dellacherie and Meyer (1982), p. 307, this implies that the quadratic variation $[X^k]_\infty = C_\infty^{X^k} + x^2 * \mu^{X^k}$ converges to 0 in L^0. Thus $C_\infty^{X^k} \to 0$ and $(1 \wedge x^2) * \mu^{X^k} \to 0$ and therefore by Lemma 4.12 $(1 \wedge x^2) * \nu^{X^k} \to 0$ in L^0. It remains to be shown that $\mathrm{Var}(B^{X^k})_\infty \to 0$ in L^0. As in (4.13) there exists a constant $l > 0$ only depending only on the truncation function τ_1 such that for A^k, the third term on the right-hand side of (4.12),

$$\|A^k\|_{\mathcal{SM}} \leq l\mathrm{E}[[X^k]_\infty \wedge 1].$$

Thus A^k, and therefore by linearity the special semimartingales $X^k - A^k$, converges to 0 in $\mathcal{SM}$. Since, cf. Emery (1979), p. 273, the map $X \mapsto A(X)$ is continuous in the space of special semimartingales ($A(X)$ denotes the predictable bounded variation component of X) this implies that $B^{X^k} = A(X^k - A^k) \to 0$ in $\mathcal{SM}$. But as noted above this means that $\mathrm{Var}(B^{X^k})_\infty \to 0$ in L^0.

The general case now follows by observing that

$$\begin{aligned}
\nu^{X^k - X} &= \nu^{(X^k, X)} \circ \big((x,y) \mapsto x - y\big)^{-1}, \\
C_t^{X^k - X} &= \langle (X^k - X)^c \rangle_t = \langle X^{k,c} - X^c \rangle_t = C_t^{X^k} + C_t^X - 2C_t^{X,X^k}, \\
B_t^{X^k - X} &= B_t^{X^k} - B_t^X + [\tau_1(x-y) - (\tau_1(x) - \tau_1(y))] * \nu_t^{(X,X^k)}.
\end{aligned}$$ □

As an application of Theorem 4.10 we get the following simple condition for $\mathcal{SM}$-convergence for Lévy processes.

Corollary 4.13. *For all $k \geq 1$ let $(X_t^k)_{t\in[0,1]}$ be a Lévy process with respect to $(\mathcal{F}_t)_{t\in[0,1]}$ with Lévy-Khintchine triplet $(\gamma_k, \sigma_k^2, \kappa_k)$. Set $X_t^k = X_0^k = 0$ for $t \leq 0$ and $X_t^k = X_1^k$ for $t \geq 1$. Then, $X^k \to 0$ in $\mathcal{SM}$ if and only if $X_1^k \to 0$ in L^0 (or equivalently, $\int_\mathbb{R} (x^2 \wedge 1)\, \kappa_k(\mathrm{d}x) \to 0$, $\sigma_k^2 \to 0$ and $\gamma_k \to 0$).*

Proof. By Jacod and Shiryaev (2003), Theorem II.5.15, we have $B_t^k = \gamma_k t$, $C_t^k = t\sigma_k^2$ and $\nu^k(\mathrm{d}s \times \mathrm{d}x) = \kappa_k(\mathrm{d}x)\,\mathrm{d}s$ for $s, t \in [0,1]$. Hence by Corollary 4.11, $X^k \to 0$ in $\mathcal{SM}$ if and only if $\int_\mathbb{R} (x^2 \wedge 1)\, \kappa_k(\mathrm{d}x) \to 0$, $\sigma_k^2 \to 0$ and $|\gamma_k| \to 0$. Moreover by Sato (1999), Theorem 8.7, the latter conditions are equivalent to $X_1^k \to 0$ in L^0. □

Proof of Theorem 4.6. Assume $\phi^k \to 0$ in $L(m)$, which, as previously noted, is equivalent to $\phi^k \bullet m \to 0$ in $\mathcal{SM}$. If (B^k, C^k, ν^k) denotes the triplets of $\phi^k \bullet m$ we get by

Proposition 4.4 and Corollary 4.11 that for $k \to \infty$,

$$\int_{\mathbb{R}} \Big| \langle \phi_s^k, b_s \rangle + \int_{\mathbb{R}^n} [\tau_1(\langle x, \phi_s^k \rangle) - \langle \phi_s^k, \tau_n(x) \rangle] \, K(s; \mathrm{d}x) \Big| \, \lambda(\mathrm{d}s) \longrightarrow 0 \quad \text{in } L^0, \tag{4.14}$$

$$\int_{\mathbb{R}} \langle \phi_s^k, c_s \phi_s^k \rangle \, \lambda(\mathrm{d}s) \longrightarrow 0 \quad \text{in } L^0, \tag{4.15}$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \Big(1 \wedge |\langle x, \phi_s^k \rangle|^2 \Big) \, K(s; \mathrm{d}x) \, \lambda(\mathrm{d}s) \longrightarrow 0 \quad \text{in } L^0, \tag{4.16}$$

implying $\Psi_{\Phi,0}(\phi^k) \to 0$. On the other hand if $\|\phi^k\|_{\Phi,0} \to 0$ then (4.14)–(4.16) are satisfied. By Corollary 4.11, $\phi^k \bullet m \to 0$ in $\mathcal{SM}$, showing that $\|\phi^k\|_m \to 0$.

For $x \in \mathbb{R}^n$ and $A \in \mathscr{P}$ the above shows in particular that $\|x 1_A\|_{\Phi,0} = 0$ if and only if $\|x 1_A\|_m = 0$. Thus if $(\mathrm{P} \otimes \lambda)(A) = 0$ then for all $x \in \mathbb{R}^n$, $\|x 1_A\|_{\Phi,0} = 0$ and hence $\|x 1_A\|_m = 0$, showing that A is a null-set for m. On the other hand, if A is a null-set for m then $\|e_i 1_A\|_m = 0$ for all $i = 1, \ldots, n$ (e_i denote the ith Euclidean standard basic vector) and hence $\|e_i 1_A\|_{\Phi,0} = 0$. Therefore, for all $i = 1, \ldots, n$, a.s.

$$0 = \int 1_A(s) c_s^{i,i} \, \lambda(\mathrm{d}s), \qquad 0 = \int_{\mathbb{R}} \int_{\mathbb{R}^n} (1 \wedge |1_A(s) x^i|^2) \, K(s; \mathrm{d}x) \, \lambda(\mathrm{d}s). \tag{4.17}$$

Since c is symmetric and positive semidefinite and $K(s; \{0\}) = 0$, (4.17) shows that $\lambda(s \in \mathbb{R} : s \in A, c_s \neq 0) = 0$ and $\lambda(s \in \mathbb{R} : s \in A, K(s; \mathbb{R}^n) \neq 0) = 0$ a.s., and hence, for all $i = 1, \ldots, n$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \Big| \langle e_i 1_A(s), b_s \rangle + \int_{\mathbb{R}^n} [\tau_1(\langle e_i 1_A(s), x \rangle) - \langle e_i 1_A(s), \tau_n(x) \rangle] \, K(s; \mathrm{d}x) \Big| \, \lambda(\mathrm{d}s) \\ &= \int_{\mathbb{R}} |b_s^i 1_A(s)| \, \lambda(\mathrm{d}s) \qquad \text{a.s.} \end{aligned}$$

Thus if $\lambda(s : b_s = 0, c_s = 0, K(s; \mathbb{R}^n) = 0) = 0$ a.s., it follows that $(\lambda \otimes \mathrm{P})(A) = 0$ and the proof is complete. □

4.4 Quadratic variation and local martingales

As above let m be an n-dimensional σ-finite measure with triplet (B, C, ν) and jump measure μ. Define the optional and $\mathscr{P}$-σ-finite random signed measure on $\mathbb{R}$ $[m] = ([m]^{i,j})_{i,j \leq n}$ as

$$\int 1_A(s) \, [m](\mathrm{d}s) = \int 1_A(s) \, C(\mathrm{d}s) + 1_A(s) x x^{\intercal} * \mu, \quad A \in \mathscr{P}_m,$$

where $\intercal$ denotes the transpose and we use column vectors as default; that is, $xx^{\intercal}$ is $n \times n$. For each $A \in \mathscr{P}_m$, by Jacod and Shiryaev (2003), Theorem I.4.52,

$$\int_{-\infty}^{t} 1_A(s) \, [m](\mathrm{d}s), \quad t \in \mathbb{R},$$

is the quadratic variation of $(m((A\cap\Omega_t))_{t\in\mathbb{R}}$ which is an $\mathbb{R}^n$-valued semimartingale up to infinity. Thus we call $[m]$ *the quadratic variation measure* of m. As in Theorem 4.2 choose an optional and $\mathscr{P}$-σ-finite random positive measure $\pi=\pi(\omega;\mathrm{d}s)$ on $\mathbb{R}$ and an optional process $q=\{(q_t^{i,j}):t\in\mathbb{R}\}$ with values in the symmetric positive semidefinite $n\times n$ matrices such that $[m](\mathrm{d}s)=q_s\,\pi(\mathrm{d}s)$.

Proposition 4.14. *Let $Z=(Z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^n$-valued increment local martingale with associated measure m described in Example 4.1 and let ϕ denote an $\mathbb{R}^n$-valued predictable process. Then $\phi\in L(Z)$ and the process $(\int_{\Omega_t}\phi\cdot\mathrm{d}m)_{t\in\mathbb{R}}$ is a local martingale up to infinity if and only if there is a localizing sequence $(\sigma_k)_{k\geq 1}$ with $\mathrm{P}(\sigma_k=\infty)\to 1$ and*

$$\mathrm{E}\Big[\Big(\int_{-\infty}^{\sigma_k}\langle\phi_s,q_s\phi_s\rangle\,\pi(\mathrm{d}s)\Big)^{1/2}\Big]<\infty\qquad\textit{for all } k\geq 1.$$

Proof. Recall that by definition $\int_{-\infty}^{t}\phi\cdot\mathrm{d}m=(\phi\bullet m)(\Omega_t)$ whenever $1_{\Omega_t}\in L^1(\phi\bullet m)$. In this case $O_k=[-k,k]\times\Omega$ is m-feasible. If ξ is an $\mathbb{R}^n$-valued predictable process then by Proposition 4.4

$$[\xi\bullet m](\mathrm{d}s)=\langle\xi_s,c_s\xi_s\rangle\lambda(\mathrm{d}s)+\langle\xi_s,x\rangle^2*\mu=\langle\xi_s,q_s\xi_s\rangle\pi(\mathrm{d}s).$$

Moreover, if ξ is also bounded and $\xi_t(\omega)=0$ for all $(t,\omega)\notin O_k$ for some k then the process $((\xi\bullet m)(\Omega_t))_{t\in\mathbb{R}}$ is a local martingale up to infinity.

For $\psi\in D$ defined in (4.7) let $X_t^\psi=(\psi\phi)\bullet m(\Omega_t)$. By the above

$$|\Delta X_s^\psi|=[\psi\phi\bullet m](\{s\})^{1/2}=|\psi_s|(\langle\phi_s,q_s\phi_s\rangle\pi(\{s\}))^{1/2}\leq(\langle\phi_s,q_s\phi_s\rangle\pi(\{s\})^{1/2}.$$

That is, with L defined by

$$L_t=\Big(\int_{-\infty}^{t}\langle\phi_s,q_s\phi_s\rangle\,\pi(\mathrm{d}s)\Big)^{1/2},\qquad t\in\mathbb{R},$$

we have $|\Delta X_s^\psi|\leq L_s$ for all $\psi\in D$.

Assume there is a sequence $(\sigma_k)_{k\geq 1}$ with the properties in the *if*-part of the proposition. We then have $\mathrm{E}[L_{\sigma_k}]<\infty$. By the above, for all $\psi\in D$ for which $\psi\phi$ is a bounded process satisfying $(\psi\phi)_t(\omega)=0$ for all $(t,\omega)\notin O_k$ for some k the process X^ψ is a local martingale up to infinity and it is in the set denoted by $\mathcal{S}_L''$ by Stricker (1981), p. 505. By his Theorem 1.11, X^ψ is a local martingale up to infinity for all $\psi\in D$. Finally, using

$$[(\psi\phi)\bullet m]((-\infty,t])\leq[\phi\bullet m]((-\infty,t])$$

it follows from Davis' inequality that the set $\{\int(\psi\phi)\cdot\mathrm{d}m:\psi\in D\}$ is bounded in L^0, implying $\phi\in L(Z)$. Another application of Stricker (1981), Theorem 1.11, shows that $(\phi\bullet m(\Omega_t))_{t\in\mathbb{R}}$ is a local martingale up to infinity.

The converse implication follows from Jacod and Shiryaev (2003), Corollary 4.55(a). □

5 Lévy processes

In this section $Z = \{(Z_t^i)_{i\leq n} : t \in \mathbb{R}\}$ is an n-dimensional Lévy process indexed by $\mathbb{R}$ satisfying $Z_0 = 0$ a.s. Let (γ, Σ, κ) denote the Lévy-Khintchine triplet of Z_1. Our concern is integration with respect to Z in a filtration $\mathscr{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ in which Z is an increment semimartingale, that is, induces a Radon measure. Two such filtrations are $\mathscr{F}^{Z,\mathrm{inc}}$ and $\mathscr{F}^Z$, given up to completion as

$$\mathcal{F}_t^{Z,\mathrm{inc}} = \sigma(Z_u - Z_s : -\infty < s < u \leq t), \qquad \mathcal{F}_t^Z = \sigma(Z_u : -\infty < u \leq t), \qquad t \in \mathbb{R}.$$

(The superscript *inc* is short for *increment*). However, a major difference between the two is that Z is a Lévy process in $\mathscr{F}^{Z,\mathrm{inc}}$ but (except in trivial cases) not in $\mathscr{F}^Z$. Here we recall that by definition Z is a Lévy process in a filtration $\mathscr{F}$ if, for all $s < t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$, measurable with respect to $\mathcal{F}_t$, the distribution of $Z_t - Z_s$ depends only on $t - s$, and Z has càdlàg paths with $Z_0 = 0$. Thus, it is obvious that Z is a Lévy process in $\mathscr{F}^{Z,\mathrm{inc}}$. To see that it is not Lévy in $\mathscr{F}^Z$ note that for all $s < 0$, $Z_0 - Z_s = -Z_s$ is $\mathcal{F}_s^Z$-measurable and hence not independent of $\mathcal{F}_s^Z$ except when it is deterministic. Therefore we first consider the case when Z is a Lévy process in $\mathscr{F}$ and then turn to $\mathscr{F}^Z$.

5.1 The case when Z is a Lévy process in $\mathscr{F}$

Let $\mathscr{F}$ be a filtration and assume Z is a Lévy process in $\mathscr{F}$. Note that in this case Z is an increment semimartingale in $\mathscr{F}$ and hence induces an L^0-valued Radon measure as described in Example 4.1. Furthermore, it is easily seen that the triplet of Z is given by

$$b_s = \gamma, \qquad c_s = \Sigma, \qquad K(s; \mathrm{d}x) = \kappa(\mathrm{d}x), \qquad \lambda(\mathrm{d}s) = \mathrm{d}s,$$

and therefore, by Theorem 4.5, we have the following:

Corollary 5.1. *For all $\mathbb{R}^n$-valued $\mathscr{F}$-predictable processes $\phi = (\phi_t)_{t\in\mathbb{R}}$, $\phi \in L(Z)$ if and only if the following (a)–(c) hold almost surely*

$$(a)\ \int_{\mathbb{R}} \langle \phi_s, \Sigma\phi_s\rangle\,\mathrm{d}s < \infty, \qquad (b)\ \int_{\mathbb{R}}\int_{\mathbb{R}^n} \big(1 \wedge |\langle\phi_s, x\rangle|^2\big)\,\kappa(\mathrm{d}x)\,\mathrm{d}s < \infty,$$
$$(c)\ \int_{\mathbb{R}} \Big|\langle\gamma, \phi_s\rangle + \int_{\mathbb{R}^n} \big[\tau_1(\langle x, \phi_s\rangle) - \langle\phi_s, \tau_n(x)\rangle\big]\,\kappa(\mathrm{d}x)\Big|\,\mathrm{d}s < \infty.$$

Consider the special case where Z is an $\mathbb{R}^n$-valued strictly α-stable Lévy process with $\alpha \in (0, 2)$. In this case the conditions in Corollary 5.1 become particularly simple. Since $\Sigma = 0$ only (b) and (c) have to be verified. According to Sato (1999), Theorem 14.3, there exists a finite measure Θ, often referred to as the spherical part of κ, on $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, such that the Lévy measure κ of Z is of the form

$$\kappa(A) = \int_{S^{n-1}} \Big(\int_0^\infty \frac{1_A(rx)}{r^{1+\alpha}}\,\mathrm{d}r\Big)\,\Theta(\mathrm{d}x), \qquad A \in \mathscr{B}(\mathbb{R}^n). \tag{5.1}$$

The following corollary extends results by Rosiński and Woyczyński (1986) and Cherny and Shiryaev (2005) to processes indexed by $\mathbb{R}$ and with values in $\mathbb{R}^n$.

Corollary 5.2. *Let $Z = (Z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^n$-valued strictly α-stable Lévy process with Lévy measure given by* (5.1) *(for $\alpha = 1$, assume Z is symmetric, which in particular implies $\gamma = 0$). Then for all $\mathscr{F}$-predictable $\mathbb{R}^n$-valued processes ϕ, the integral $\int_{\mathbb{R}} \phi_s \cdot \mathrm{d}Z_s$ exists if and only if*

$$\int_{\mathbb{R}} \Big(\int_{S^{n-1}} |\langle \phi_s, x\rangle|^{\alpha}\, \Theta(\mathrm{d}x)\Big)\, \mathrm{d}s < \infty \qquad a.s. \tag{5.2}$$

Proof. For simplicity let τ_n and τ_1 be the truncation functions on $\mathbb{R}^n$ and $\mathbb{R}$ given by $\tau_n(x) = x1_{\{\|x\|\le 1\}}$ and $\tau_1(x) = x1_{\{|x|\le 1\}}$. Due to the fact that Z is strictly stable we have for $\alpha \in (1,2)$, $\gamma = \int_{\mathbb{R}^n}(\tau_n(x) - x)\,\kappa(\mathrm{d}x)$ and for $\alpha \in (0,1)$, $\gamma = \int_{\mathbb{R}^n} \tau_n(x)\,\kappa(\mathrm{d}x)$, cf. Sato (1999), Theorem 14.7. Using (5.1), a simple calculation shows

$$C_{1,\alpha} \int_{\mathbb{R}^n} \big(1 \wedge |\langle \phi_s, x\rangle|^2\big)\,\kappa(\mathrm{d}x) = \int_{S^{n-1}} |\langle \phi_s, x\rangle|^{\alpha}\, \Theta(\mathrm{d}x) \tag{5.3}$$

$$\ge C_{2,\alpha}\Big|\langle \phi_s, \gamma\rangle + \int_{\mathbb{R}^n} \big[\tau_1(\langle \phi_s, x\rangle) - \langle \phi_s, \tau_n(x)\rangle\big]\,\kappa(\mathrm{d}x)\Big|, \tag{5.4}$$

where the equality (5.3) holds for all $\alpha \in (0,2)$ and $C_{1,\alpha} = ((2-\alpha)\alpha)/2$, and the inequality (5.4) holds for $\alpha \neq 1$ and $C_{2,\alpha} = |\alpha - 1|$. In the case $\alpha = 1$ (5.4) remains true for some constant $C_{2,1}$. This shows that (a)–(c) of Corollary 5.1 are equivalent to (5.2) and completes the proof. □

5.2 Integrability in $\mathscr{F}^Z$

When considering integrals as (1.2)(β) the process $\{f(s, Z_{s-}) : s \in \mathbb{R}\}$ is usually not predictable (nor adapted) in $\mathscr{F}^{Z,\mathrm{inc}}$ (or in any other filtration in which Z is a Lévy process). Thus, (1.2)(β) can not be studied in $\mathscr{F}^{Z,\mathrm{inc}}$, forcing us to consider instead the filtration $\mathscr{F}^Z$. Although Z is an increment semimartingale in $\mathscr{F}^Z$ (by Theorem 5.3 below and Stricker's theorem), it is in general difficult to calculate its triplet in this filtration. Therefore we consider the larger filtration $\mathscr{F}^{Z,\mathrm{ex}}$ given by $\mathcal{F}_t^{Z,\mathrm{ex}} = \mathcal{F}_t^Z$ for $t \ge 0$ and

$$\mathcal{F}_t^{Z,\mathrm{ex}} = \mathcal{F}_t^Z \vee \sigma\big(\mu^Z\big((t,0] \times A\big) : A \in \mathscr{B}(\mathbb{R}^n)\big), \qquad \text{for } t < 0,$$

in which we are able to calculate the triplet explicitly. (The superscript *ex* is short for *extended*). Note that $\mathcal{F}_t^{Z,\mathrm{inc}} \subseteq \mathcal{F}_t^Z \subseteq \mathcal{F}_t^{Z,\mathrm{ex}}$ for all $t \in \mathbb{R}$. Let $Z^c = (Z_t^c)_{t\in\mathbb{R}}$ be the Gaussian component of Z chosen such that $Z_0 = 0$. Recall that μ^Z is the jump measure of Z, and set $\nu(\mathrm{d}s \times \mathrm{d}x) = \kappa(\mathrm{d}x)\,\mathrm{d}s$ and

$$M_t = Z_t^c - \int_t^0 \int_{\mathbb{R}^n} \tau_n(x)\,(\mu^Z - \nu)(\mathrm{d}s \times \mathrm{d}x), \qquad t < 0.$$

Due to the fact that $\mathcal{F}_t^Z \subseteq \mathcal{F}_t^{Z,\mathrm{ex}}$ for all $t \in \mathbb{R}$, the next result gives in particular sufficient conditions for the integral $\int_{\mathbb{R}} \phi_s \cdot \mathrm{d}Z_s$ to exist for any $\mathscr{F}^Z$-predictable process ϕ. It relies on an enlargement of filtration result essentially due to Jacod and Protter (1988); see Appendix.

Theorem 5.3. *In the filtration $\mathscr{F}^{Z,\mathrm{ex}}$, Z is an increment semimartingale with triplet*

$$b_s = \gamma + (M_s/s)1_{\{s<0\}}, \qquad c_s = \Sigma, \qquad \lambda(\mathrm{d}s) = \mathrm{d}s, \tag{5.5}$$

$$K(s;\mathrm{d}x) = \frac{\mu^Z((s,0]\times \mathrm{d}x)}{|s|}1_{\{s<0\}} + \kappa(\mathrm{d}x)1_{\{s\geq 0\}}. \tag{5.6}$$

Hence for all $\mathscr{F}^{Z,\mathrm{ex}}$-predictable processes ϕ, the integral $\int_{\mathbb{R}} \phi_s \cdot \mathrm{d}Z_s$ exists if and only if (4.5)–(4.6) *are satisfied with b, c, λ and K given by* (5.5)–(5.6).

Note that the triplet of Z with respect to $\mathscr{F}^{Z,\mathrm{ex}}$ is random, reflecting that Z does not have independent increments with respect to $\mathscr{F}^{Z,\mathrm{ex}}$. Note also that for all $s<0$, $\mu^Z((s,0]\times \mathrm{d}x)$ is a Poisson random measure with intensity measure $|s|\kappa(\mathrm{d}x)$.

Proof of Theorem 5.3. Fix an $r<0$ and define $Z^{(r)} = (Z_t^{(r)})_{t\geq r}$ as $Z_t^{(r)} = Z_t - Z_r$. Consider the filtration $\mathscr{F}^{Z^{(r)},\mathrm{ex}} = (\mathcal{F}_t^{Z^{(r)},\mathrm{ex}})_{t\geq r}$ where

$$\mathcal{F}_t^{Z^{(r)},\mathrm{ex}} = \sigma\Big(Z_u^{(r)} : u \in [r,t]\Big) \vee \sigma\big(Z_0^{(r)}\big) \vee \sigma\Big(\mu^{Z^{(r)}}\big((r,0]\times A\big) : A \in \mathscr{B}(\mathbb{R}^n)\Big).$$

Using that $Z_r = -(Z_0 - Z_r) = -Z_0^{(r)}$, we get for $t \geq r$,

$$\mathcal{F}_t^{Z,\mathrm{ex}} = \mathscr{F}_t^{Z^{(r)},\mathrm{ex}} \vee \mathcal{G}_r, \qquad \text{where} \qquad \mathcal{G}_r = \sigma(Z_r - Z_u : u < r). \tag{5.7}$$

By Theorem A.1 (where r plays the role of zero, and zero plays the role of t_0), $Z^{(r)}$ is a semimartingale with respect to $\mathscr{F}^{Z^{(r)},\mathrm{ex}}$. Let $b^{(r)}, c^{(r)}, K^{(r)}$ and $\lambda^{(r)}$ denote the quantities defining the triplet of $Z^{(r)}$ in (A.1) and (A.2). Let $Z^{(r),c}$ denote the Gaussian component of the Lévy process $Z^{(r)}$ and note that $Z_t^{(r),c} = Z_t^c - Z_r^c$. Thus, with

$$\begin{aligned}
M_t^{(r)} &:= Z_t^{(r),c} + \int_{(r,t]\times\mathbb{R}^n} \tau_n(x)\,(\mu^{Z^{(r)}} - \nu)(\mathrm{d}s\times\mathrm{d}x), \\
&= (Z_t^c - Z_r^c) + \int_{(r,t]\times\mathbb{R}^n} \tau_n(x)\,(\mu^{Z} - \nu)(\mathrm{d}s\times\mathrm{d}x), \qquad t\geq r,
\end{aligned}$$

we have by Theorem A.1 that

$$b_s^{(r)} = \gamma + \frac{M_0^{(r)} - M_s^{(r)}}{0-s}1_{\{r\leq s<0\}} = \gamma + (M_s/s)1_{\{r\leq s<0\}}.$$

Similarly,

$$\begin{aligned}
K^{(r)}(s;\mathrm{d}x) &= \frac{\mu^{Z^{(r)}}\big((s,0]\times\mathrm{d}x\big)}{0-s}1_{\{r\leq s\leq 0\}} + \kappa(\mathrm{d}x)1_{\{s\geq 0\}} \\
&= \frac{\mu^{Z}\big((s,0]\times\mathrm{d}x\big)}{|s|}1_{\{r\leq s\leq 0\}} + \kappa(\mathrm{d}x)1_{\{s\geq 0\}}.
\end{aligned}$$

Finally, $\lambda^{(r)}(\mathrm{d}s) = 1_{\{s\geq r\}}\,\mathrm{d}s$ and $c_s = \Sigma$. Since $\mathcal{G}_r$ is independent of $Z^{(r)} = (Z_t - Z_r)_{t\geq r}$ it follows by (5.7) that $Z^{(r)}$ is an $(\mathcal{F}_t^{Z,\mathrm{ex}})_{t\geq r}$-semimartingale with the above triplet. Furthermore, since this is true for all $r<0$, it follows that the triplet of Z is given by (5.5)–(5.6). □

Remark 5.4. Even though b_s and $K(s;\mathrm{d}x)$ are random for $s\in(-\infty,0)$ it follows by the strong law of large numbers for Lévy processes (see e.g. Sato (1999), Theorem 36.5) that they are both deterministic in the limit $s\to-\infty$; in fact, for all measurable functions $f\colon \mathbb{R}^n\to\mathbb{R}_+$, $\int_{\mathbb{R}^n} f(x)\,K(s;\mathrm{d}x)\to\int_{\mathbb{R}^n} f(x)\,\kappa(\mathrm{d}x)$ and $b_s\to\gamma$ a.s. as $s\to-\infty$.

In the next result we use Remark 5.4 to give a simple condition that $\int_{\mathbb{R}}\phi_s\cdot\mathrm{d}Z_s$ exists.

Theorem 5.5. *Assume* $\mathrm{E}[\|Z_1\|^2]<\infty$. *Then, for all* $\mathscr{F}^Z$*-predictable processes* ϕ *with a.s. locally bounded sample paths the integral* $\int_{\mathbb{R}}\phi_s\cdot\mathrm{d}Z_s$ *exists if*

$$\int_{\mathbb{R}}\big(\|\phi_s\|^2+\|\phi_s\|\big)\,\mathrm{d}s<\infty \quad a.s.$$

Proof. Let us show that (a)–(c) of Theorem 4.5 are satisfied with b,c,K and λ as in Theorem 5.3. Property (b) follows from the fact that $\langle\phi_s,\Sigma\phi_s\rangle\leq C_\Sigma\|\phi_s\|^2$ for some constant $C_\Sigma\geq 0$ depending only on Σ.

In the proof of (a) and (c) we will use the following consequences of Remark 5.4:

$$\int_{\mathbb{R}^n}\|x\|^2\,K(s;\mathrm{d}x)\xrightarrow[s\to-\infty]{}\int_{\mathbb{R}^n}\|x\|^2\,\kappa(\mathrm{d}x),\qquad\text{a.s.}\tag{5.8}$$

$$\int_{\{\|x\|>1\}}\|x\|\,K(s;\mathrm{d}x)\xrightarrow[s\to-\infty]{}\int_{\{\|x\|>1\}}\|x\|\,\kappa(\mathrm{d}x),\quad\text{a.s.}\tag{5.9}$$

$$b_s\xrightarrow[s\to-\infty]{}\gamma,\qquad\text{a.s.}\tag{5.10}$$

For simplicity let $\tau_n(x)=x1_{\{\|x\|\leq1\}}$ for $x\in\mathbb{R}^n$ and $\tau_1(y)=y1_{\{|y|\leq1\}}$ for $y\in\mathbb{R}$. To verify (c) it suffices, due to the fact that ϕ has a.s. locally bounded sample paths, to show that there exists a $k=k(\omega)\in\mathbb{N}$ such that

$$\int_{[-k,k]^c}\int_{\mathbb{R}^n}\big(1\wedge\langle\phi_s,x\rangle^2\big)\,K(s;\mathrm{d}x)\,\mathrm{d}s<\infty,\tag{5.11}$$

However, since

$$\int_{\mathbb{R}^n}\big(1\wedge|\langle\phi_s,x\rangle|^2\big)\,K(s;\mathrm{d}x)\leq\|\phi_s\|^2\int_{\mathbb{R}^n}\|x\|^2\,K(s;\mathrm{d}x),$$

equation (5.8) implies (5.11). To show (a) note that

$$\begin{aligned}|\tau_1(\langle\phi_s,x\rangle)-\langle\phi_s,\tau_n(x)\rangle| &= \Big|\langle\phi_s,x\rangle\Big(1_{\{|\langle\phi_s,x\rangle|\leq1\}}-1_{\{\|x\|\leq1\}}\Big)\Big|\\ &\leq\|\phi_s\|^2\|x\|^2+\|\phi_s\|\|x\|1_{\{\|x\|>1\}},\end{aligned}$$

and therefore (a) follows in the same way as (c) using (5.8)–(5.10) and the estimates

$$\begin{aligned}
&\Big|\langle\phi_s, b_s\rangle + \int_{\mathbb{R}^n}\big[\tau_1(\langle\phi_s, x\rangle) - \langle\phi_s, \tau_n(x)\rangle\big]\,K(s;\mathrm{d}x)\Big| \\
&\quad\le \|\phi_s\|\|b_s\| + \int_{\mathbb{R}^n}\Big|\tau_1(\langle\phi_s, x\rangle) - \langle\phi_s, \tau_n(x)\rangle\Big|\,K(s;\mathrm{d}x) \\
&\quad\le \|\phi_s\|\|b_s\| + \|\phi_s\|^2\int_{\mathbb{R}^n}\|x\|^2\,K(s;\mathrm{d}x) + \|\phi_s\|\int_{\{\|x\|>1\}}\|x\|\,K(s;\mathrm{d}x). \qquad \square
\end{aligned}$$

A Appendix

In this Appendix we consider an expansion of filtration result for $\mathbb{R}^n$-valued Lévy processes $Z = (Z_t)_{t\ge0}$ indexed by $\mathbb{R}_+$. Let $t_0 > 0$ be fixed and let $\mathscr{F}^{Z,\mathrm{ex}} = (\mathcal{F}_t^{Z,\mathrm{ex}})_{t\ge0}$ be the least filtration to which Z is adapted and for all Borel sets A that are bounded away from zero, $\mu_Z((0, t_0] \times A)$ and Z_{t_0} are $\mathcal{F}_t^{Z,\mathrm{ex}}$-measurable. For all $t \ge 0$ let $M_t = Z_t^c + \tau_n(x) * (\mu^Z - \nu)_t$, where we use the notation of Subsection 5.2. The following results extend Theorems 2.6 and 2.9 in Jacod and Protter (1988) to the multivariate case.

Theorem A.1. *Let $Z = (Z_t)_{t\ge0}$ be an $\mathbb{R}^n$-valued Lévy process on $\mathbb{R}_+$ with Lévy-Khintchine triplet (γ, Σ, κ). Then Z is a semimartingale with respect to $\mathscr{F}^{Z,\mathrm{ex}}$ with triplet*

$$b_s = \gamma + \frac{M_{t_0} - M_s}{t_0 - s}1_{\{0\le s<t_0\}}, \qquad c_s = \Sigma, \qquad \lambda(\mathrm{d}s) = \mathrm{d}s, \tag{A.1}$$

$$K(s;\mathrm{d}x) = \frac{\mu^Z((s, t_0] \times \mathrm{d}x)}{t_0 - s}1_{\{0\le s<t_0\}} + \kappa(\mathrm{d}x)1_{\{s\ge t_0\}}. \tag{A.2}$$

To prove Theorem A.1 we need the following two lemmas.

Lemma A.2. *Let Y_i for $i = 1, \dots, k$ be integrable, independent and identically distributed $\mathbb{R}^n$-valued random vectors. Then $\mathrm{E}[Y_1 | \sum_{i=1}^k Y_i] = \frac{1}{k}\sum_{i=1}^k Y_i$.*

Lemma A.2 follows by standard arguments and hence its proof is omitted. For fixed $t_0 > 0$ let $\mathscr{F}^{Z,t_0} = (\mathcal{F}_t^{Z,t_0})_{t\ge0}$ denote the least filtration for which Z_{t_0} is $\mathcal{F}_0^{Z,t_0}$-measurable and Z is adapted.

Lemma A.3. *Let $Z = (Z_t)_{t\ge0}$ be an $\mathbb{R}^n$-valued integrable Lévy process with mean zero. Then $N = (N_t)_{t\ge0}$ is an $\mathscr{F}^{Z,t_0}$-martingale, where N is given by*

$$N_t = Z_t - \int_0^{t\wedge t_0}\frac{Z_{t_0} - Z_s}{t_0 - s}\,\mathrm{d}s, \qquad t \ge 0.$$

Proof. We may and do assume that $t_0 = 1$. To show that $(N_t)_{t\ge0}$ is an $\mathscr{F}^{Z,1}$-martingale it is enough to show that for all $t > 0$ we have that $\lim_n A_t^n = A_t$ in L^1, where for all

$n \geq 1$

$$A_t^n = \sum_{i=1}^{[nt]} \mathrm{E}[Z_{t_i^n} - Z_{t_{i-1}^n} | \mathcal{F}_{t_{i-1}^n}^{Z,1}] \qquad \text{and} \qquad A_t = \int_0^{t\wedge 1} \frac{Z_1 - Z_s}{1-s}\,\mathrm{d}s,$$

where $t_i^n = i/n$. For $0 \leq s < t \leq 1$ with $t = k/n$ and $s = j/n$ we have by Lemma A.2 that

$$\mathrm{E}[Z_t - Z_s | \mathcal{F}_s^{Z,1}] = \mathrm{E}[Z_t - Z_s | Z_1 - Z_s] = \frac{Z_1 - Z_s}{1-s}(t-s),$$

and therefore

$$A_t^n = \sum_{i=1}^{a_n} \frac{Z_1 - Z_{t_{i-1}^n}}{1 - t_{i-1}^n}(t_i^n - t_{i-1}^n). \tag{A.3}$$

Moreover, using that Z is an integrable Lévy process there exist $c_1, c_2 > 0$ such that for all $s \in [0,1]$,

$$\Big\|\frac{Z_1 - Z_s}{1-s}\Big\|_1 = \frac{\|Z_{1-s}\|_1}{1-s} \leq \frac{c_1(1-s)^{1/2} + c_2(1-s)}{1-s} \leq \frac{c_1}{(1-s)^{1/2}} + c_2,$$

and hence

$$\int_0^1 \Big\|\frac{Z_1 - Z_s}{1-s}\Big\|_1 \,\mathrm{d}s < \infty.$$

Therefore by (A.3) and for $t > 0$, $\lim_n A_t^n = A_t$ in L^1, which completes the proof. $\square$

Proof of Theorem A.1. We may and do assume that $t_0 = 1$. To show (A.2) let $\mathscr{B}_0(\mathbb{R}^n)$ be the family of all bounded Borel sets bounded away from zero. Let $\rho(\mathrm{d}s \times \mathrm{d}x) = K(s;\mathrm{d}x)\,\mathrm{d}s$ where K is given in (A.2). Note that ρ is an $\mathscr{F}^{Z,\mathrm{ex}}$-predictable random positive measure on $\mathbb{R} \times \mathbb{R}^n$. For all $A \in \mathscr{B}_0(\mathbb{R}^n)$ let $Z_t^A = \mu_Z((0,t] \times A)$ and

$$U_t^A := \mu_Z((0,t]\times A) - \rho((0,t]\times A) = Z_t^A - \int_0^{t\wedge 1} \frac{Z_1^A - Z_s^A}{1-s}\,\mathrm{d}s - \int_1^{t\vee 1} \kappa(A)\,\mathrm{d}s.$$

Since Z^A is an integrable Lévy process, U^A is a $\mathscr{F}^{U^A,1}$-martingale by Lemma A.3. If $(A_i)_{i=1}^k \subseteq B_0(\mathbb{R}^n)$ are pairwise disjoint, $(U^{A_i})_{i=1}^k$ are independent and $\sum_{i=1}^k U^{A_i} = U^{\cup_{i=1}^k A_i}$. Therefore by the Monotone Class Lemma, U^A is an $\mathscr{F}^{Z,\mathrm{ex}}$-martingale for all $A \in \mathscr{B}_0(\mathbb{R}^n)$. For any positive function W of the form

$$W(s,\omega,x) = \sum_{i=1}^k \alpha_i 1_{(t_{i-1},t_i]}(s) 1_{A_i}(x) 1_{B_i}(\omega), \qquad B_i \in \mathcal{F}_{t_{i-1}}^{Z,\mathrm{ex}} \text{ for } i = 1,\ldots,n,$$

we have

$$W * (\mu^Z - \rho) = \sum_{i=1}^k \alpha_i 1_{B_i}(U_{t_i}^{A_i} - U_{t_{i-1}}^{A_i}).$$

Thus, since for all $i = 1, \dots, k$, U^{A_i} is an $\mathscr{F}^{Z,\mathrm{ex}}$-martingale,

$$\mathrm{E}[W * \mu^Z] = \mathrm{E}[W * \rho],$$

which shows that ρ is the predictable compensator of μ^Z. To show that b is given by (A.1) let N^c be given by

$$N_t^c = Z_t^c - \int_0^{t\wedge 1} \frac{Z_1^c - Z_s^c}{1-s}\,\mathrm{d}s, \qquad t \geq 0.$$

We have

$$Z_t = N_t^c + \tau_n(x) * (\mu^Z - \rho)_t + (x - \tau_n(x)) * \mu_t^Z + \Big(\gamma t + \int_0^t \frac{Z_1^c - Z_s^c}{1-s}\,\mathrm{d}s + \tau_n(x) * (\rho - \nu)_t\Big),$$

which shows that

$$B_t = \gamma t + \int_0^{t\wedge 1} \frac{Z_1^c - Z_s^c}{1-s}\,\mathrm{d}s + \tau_n(x) * (\rho - \nu)_t = \gamma t + \int_0^{t\wedge 1} \frac{M_1 - M_s}{1-s}\,\mathrm{d}s.$$

Since C is the continuous part of the matrix $[Z]$ we have $c_s = \Sigma$, completing the proof. □

References

Barndorff-Nielsen, O. E. and J. Schmiegel (2007). Ambit processes: with applications to turbulence and tumour growth. In *Stochastic analysis and applications*, Volume 2 of *Abel Symp.*, pp. 93–124. Berlin: Springer.

Barndorff-Nielsen, O. E. and J. Schmiegel (2008). A stochastic differential equation framework for the timewise dynamics of turbulent velocities. *Theory Prob. Its Appl. 52*, 372–388.

Barndorff-Nielsen, O. E. and J. Schmiegel (2009). Brownian semistationary processes and volatility/intermittency. In *Advanced Financial Modelling*, Volume 8 of *Radon Series Comp. Appl. Math.*, pp. 1–26. Walter de Gruyter.

Basse-O'Connor, A., S.-E. Graversen, and J. Pedersen (2010a). Martingale-type processes indexed by the real line. *ALEA Lat. Am. J. Probab. Math. Stat. 7*, 117–137.

Basse-O'Connor, A., S.-E. Graversen, and J. Pedersen (2010b). Some classes of proper integrals and generalized Ornstein-Uhlenbeck processes. (Preprint).

Bichteler, K. (1976). Measures with values in non-locally convex spaces. In *Measure theory (Proc. Conf., Oberwolfach, 1975)*, pp. 277–285. Lecture Notes in Math., Vol. 541. Berlin: Springer.

Bichteler, K. (1981). Stochastic integration and L^p-theory of semimartingales. *Ann. Probab. 9*(1), 49–89.

Bichteler, K. and J. Jacod (1983). Random measures and stochastic integration. In *Theory and application of random fields (Bangalore, 1982)*, Volume 49 of *Lecture Notes in Control and Inform. Sci.*, pp. 1–18. Berlin: Springer.

Cherny, A. and A. Shiryaev (2005). On stochastic integrals up to infinity and predictable criteria for integrability. In *Séminaire de Probabilités XXXVIII*, Volume 1857 of *Lecture Notes in Math.*, pp. 165–185. Berlin: Springer.

Curbera, G. P. and O. Delgado (2007). Optimal domains for L^0-valued operators via stochastic measures. *Positivity 11*(3), 399–416.

Dellacherie, C. and P.-A. Meyer (1982). *Probabilities and Potential. B*, Volume 72 of *North-Holland Mathematics Studies*. Amsterdam: North-Holland Publishing Co. Theory of martingales, Translated from the French by J. P. Wilson.

Doob, J. L. (1990). *Stochastic Processes*. Wiley Classics Library. New York: John Wiley & Sons Inc. Reprint of the 1953 original, A Wiley-Interscience Publication.

Emery, M. (1979). Une topologie sur l'espace des semimartingales. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, Volume 721 of *Lecture Notes in Math.*, pp. 260–280. Berlin: Springer.

Emery, M. (1982). A generalization of stochastic integration with respect to semimartingales. *Ann. Probab. 10*(3), 709–727.

Jacod, J. and P. Protter (1988). Time reversal on Lévy processes. *Ann. Probab. 16*(2), 620–641.

Jacod, J. and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (Second ed.), Volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag.

Kwapień, S. and W. A. Woyczyński (1991). Semimartingale integrals via decoupling inequalities and tangent processes. *Probab. Math. Statist. 12*(2), 165–200 (1992).

Kwapień, S. and W. A. Woyczyński (1992). *Random Series and Stochastic Integrals: Single and Multiple*. Probability and its Applications. Boston, MA: Birkhäuser Boston Inc.

Lenglart, E. (1977). Relation de domination entre deux processus. *Ann. Inst. H. Poincaré Sect. B (N.S.) 13*(2), 171–179.

Marcus, M. B. and J. Rosiński (2001). L^1-norms of infinitely divisible random vectors and certain stochastic integrals. *Electron. Comm. Probab. 6*, 15–29 (electronic).

Mémin, J. (1980). Espaces de semi martingales et changement de probabilité. *Z. Wahrsch. Verw. Gebiete 52*(1), 9–39.

Musielak, J. (1983). *Orlicz spaces and modular spaces*, Volume 1034 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag.

Rajput, B. S. and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields 82*(3), 451–487.

Rolewicz, S. (1985). *Metric linear spaces* (Second ed.), Volume 20 of *Mathematics and its Applications (East European Series)*. Dordrecht: D. Reidel Publishing Co.

Rosiński, J. and W. A. Woyczyński (1986). On Itô stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab. 14*(1), 271–286.

Ryll-Nardzewski, C. and W. A. Woyczyński (1975). Bounded multiplier convergence in measure of random vector series. *Proc. Amer. Math. Soc. 53*(1), 96–98.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*, Volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press. Translated from the 1990 Japanese original, Revised by the author.

Schwartz, L. (1981). Les semi-martingales formelles. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, Volume 850 of *Lecture Notes in Math.*, pp. 413–489. Berlin: Springer.

Stricker, C. (1981). Quelques remarques sur la topologie des semimartingales. Applications aux intégrales stochastiques. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, Volume 850 of *Lecture Notes in Math.*, pp. 499–522. Berlin: Springer.

Talagrand, M. (1981). Les mesures vectorielles à valeurs dans L^0 sont bornées. *Ann. Sci. École Norm. Sup. (4) 14* (4), 445–452 (1982).

Turpin, P. (1974). Convexités dans les espaces vectoriels topologiques généraux. Thesis.

Turpin, P. (1975). Intégration par rapport à une mesure à valeurs dans un espace vectoriel topologique non supposé localement convexe. In *Intégration vectorielle et multivoque (Colloq., Univ. Caen, Caen, 1975), Exp. No. 8*, pp. 22. Dép. Math., U. E. R. Sci., Univ. Caen, Caen.