THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Some classes of proper integrals and generalized Ornstein-Uhlenbeck processes

Andreas Basse-O'Connor, Svend-Erik Graversen and Jan Pedersen

Research Report No. 09 | August 2010

Some classes of proper integrals and generalized Ornstein-Uhlenbeck processes

Andreas Basse-O'Connor[*†], Svend-Erik Graversen[◇‡] and Jan Pedersen[◇♯]

[*]Department of Mathematical Science, University of Tennessee, USA.

[†]Department of Mathematical Science, University of Aarhus, Denmark.

[†]E-mail: basse@imf.au.dk, [‡]E-mail: matseg@imf.au.dk, [♯]E-mail: jan@imf.au.dk.

Abstract

We give necessary and sufficient conditions for existence of proper integrals from 0 to infinity or from minus infinity to 0 of one exponentiated Lévy process with respect to another Lévy process. The results are related to the existence of stationary generalized Ornstein-Uhlenbeck processes. Finally, in the square integrable case the Wold-Karhunen representation is given.

Keywords: stochastic integration; Lévy processes; generalized Ornstein-Uhlenbeck processes.
AMS Subject Classification: 60G44; 60G57; 60H05.

1 Introduction

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t\in\mathbb{R}}$ denote a bivariate Lévy process indexed by $\mathbb{R}$ satisfying $\xi_0 = \eta_0 = 0$, that is, (ξ, η) is defined on a probability space $(\Omega, \mathcal{F}, P)$, has càdlàg paths and stationary independent increments. We are interested in the two integrals

$$(a): \int_0^\infty e^{-\xi_{s-}}\,\mathrm{d}\eta_s \qquad \text{and} \qquad (b): \int_{-\infty}^0 e^{\xi_{s-}}\,\mathrm{d}\eta_s. \tag{1.1}$$

The first of these has been thoroughly studied, see e.g. [4, 5, 8, 12, 14, 17], where it is treated as an improper integral, i.e. as the a.s. limit as $t \to \infty$ of $\int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s$. Recall that Erickson and Maller [8], Theorem 2, give necessary and sufficient conditions in terms of the Lévy-Khintchine triplet of (ξ, η) for the existence of (1.1)(a) in the improper sense. In the following this integral is considered as a semimartingale integral up to infinity in the sense of e.g. Cherny and Shiryaev [6] or Basse-O'Connor et al. [2], which we can think of as a proper integral. Theorem 3.1 shows that the conditions given by Erickson and Maller are also necessary and sufficient for the existence of (1.1)(a) in the proper sense.

To the best of our knowledge the second integral has previously only been studied in special cases, in particular when ξ is deterministic. As we shall see in the next section, η is a so-called increment semimartingale in the natural filtration of (ξ, η), that is, the least right-continuous and complete filtration to which (ξ, η) is adapted. Integration with respect to increment semimartingales has been studied in Basse-O'Connor et al. [2], and we use the results obtained there to give necessary and and sufficient conditions for the existence of (1.1)(b); see Theorem 3.1.

As an application, generalized Ornstein-Uhlenbeck processes (and some generalizations hereof) are considered. Recall that a càdlàg process $V = (V_t)_{t\in\mathbb{R}}$ is a generalized Ornstein-Uhlenbeck process if it satisfies

$$V_t = e^{-(\xi_t-\xi_s)}V_s + e^{-\xi_t}\int_s^t e^{\xi_{u-}}\,\mathrm{d}\eta_u \quad \text{for } s < t.$$

See Lindner and Maller [13] for a thorough study of these processes and references to theory and applications. Assuming $\xi_t \to \infty$ as $t \to \infty$ a.s., Theorem 3.4 shows that a necessary and sufficient condition for the existence of a stationary V is that (1.1)(b) exists, and in this case V is represented as

$$V_t = e^{-\xi_t}\int_{-\infty}^t e^{\xi_{u-}}\,\mathrm{d}\eta_u \quad \text{for } t \in \mathbb{R}.$$

This result complements Theorem 2.1 in [13] where the stationary distribution is expressed in terms of an integral from 0 up to infinity. Finally, assuming second moments, Theorem 3.5 gives the Wold-Karhunen representation of V.

2 Integration with respect to increment semimartingales

In this section we first recall a few general results related to integration with respect to increment semimartingales. Afterwards we specialize to integration with respect to η.

Let $(\mathcal{F}_t)_{t\in\mathbb{R}}$ denote a filtration satisfying the usual conditions of right-continuity and completeness. Recall from [2] that a càdlàg $\mathbb{R}$-valued process $Z = (Z_t)_{t\in\mathbb{R}}$ is called an increment semimartingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}}$ if for all $s \in \mathbb{R}$ the process $(Z_{t+s} - Z_s)_{t\geq 0}$ is an $(\mathcal{F}_{s+t})_{t\geq 0}$-semimartingale in the usual sense. Equivalently, by Example 4.1 in [2], Z is an increment semimartingale if and only if it induces an $L^0(P)$-valued Radon measure on the predictable σ-field $\mathcal{P}$. Note that in general an increment semimartingale is not adapted. Let $\mu^Z = \mu^Z(\omega; \mathrm{d}t \times \mathrm{d}x)$ denote the jump measure of Z defined as

$$\mu^Z(A) = \#\{s \in \mathbb{R} : (s, \Delta Z_s) \in A\} \quad \text{for } A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_0),$$

where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, and let (B, C, ν) denote the triplet of Z; see [2]. That is, $\nu = \nu(\omega; \mathrm{d}t \times \mathrm{d}x)$ is the predictable compensator of μ^Z in the sense of Jacod and Shiryaev

[11], Theorem II.1.8. Moreover, $B = B(\omega; \mathrm{d}t)$ is a random signed measure on $\mathbb{R}$ of finite total variation on compacts satisfying that $t \mapsto B((s, s+t])$ is an $(\mathcal{F}_{s+t})_{t\geq 0}$-predictable process for all $s \in \mathbb{R}$ and $C = C(\omega; \mathrm{d}t)$ is a random positive measure on $\mathbb{R}$ which is finite on compacts. Finally, for all $s < t$ we have

$$\begin{aligned} Z_t - Z_s = Z_t^c - Z_s^c &+ \int_{(s,t]\times\{|x|\leq 1\}} x\,[\mu^Z(\mathrm{d}u \times \mathrm{d}x) - \nu(\mathrm{d}u \times \mathrm{d}x)] \\ &+ \int_{(s,t]\times\{|x|>1\}} x\,\mu^Z(\mathrm{d}u \times \mathrm{d}x) + B((s,t]) \end{aligned}$$

where $Z^c = (Z_t^c)_{t\in\mathbb{R}}$ is a continuous increment local martingale and for all $s \in \mathbb{R}$ the quadratic variation of $(Z_{s+t}^c - Z_s^c)_{t\geq 0}$ is $C((s, s+t])$, $t \geq 0$. Choose a predictable random positive measure $\lambda = \lambda(\omega; \mathrm{d}t)$ on $\mathbb{R}$ which is finite on compacts, a real-valued predictable process $b = b_t(\omega)$, a positive predictable process $c = c_t(\omega)$, and a transition kernel $K = K(t, \omega; \mathrm{d}x)$ from $(\mathbb{R} \times \Omega, \mathcal{P})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\, K(t; \mathrm{d}x) < \infty$ and $K(t; \{0\}) = 0$ for all $t \in \mathbb{R}$ such that

$$B(\mathrm{d}t) = b_t\lambda(\mathrm{d}t), \quad C(\mathrm{d}t) = c_t\lambda(\mathrm{d}t), \quad \nu(\mathrm{d}t \times \mathrm{d}x) = K(t; \mathrm{d}x)\lambda(\mathrm{d}t).$$

As shown in [2] a necessary and sufficient condition for the existence of $\int_{\mathbb{R}} \phi_s\, \mathrm{d}Z_s$ is that we have the following:

$$\int_{\mathbb{R}} \Big| \phi_s b_s + \int_{\mathbb{R}} \big[\tau(\phi_s x) - \phi_s\tau(x)\big]\, K(s; \mathrm{d}x) \Big|\, \lambda(\mathrm{d}s) < \infty, \tag{2.1}$$

$$\int_{\mathbb{R}} \phi_s^2 c_s\, \lambda(\mathrm{d}s) < \infty, \qquad \int_{\mathbb{R}}\int_{\mathbb{R}} \big(1 \wedge (\phi_s x)^2\big)\, K(s; \mathrm{d}x)\, \lambda(\mathrm{d}s) < \infty, \tag{2.2}$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is a truncation function, i.e. it is bounded, measurable and satisfies $\tau(x) = x$ in a neighborhood of 0. Moreover, when these conditions are satisfied the process $\int_{-\infty}^t \phi_s\, \mathrm{d}Z_s$, $t \in \mathbb{R}$, is a semimartingale up to infinity. Here we use the usual convention that for a measurable subset A of $\mathbb{R}$, $\int_A \phi_s\, \mathrm{d}Z_s := \int_{\mathbb{R}} \phi_s 1_A(s)\, \mathrm{d}Z_s$, and $\int_s^t := \int_{(s,t]}$ for $s < t$.

Let us turn to integration with respect to η where (ξ, η) is a bivariate Lévy process indexed by $\mathbb{R}$ with $\xi_0 = \eta_0 = 0$, that is, η plays the role of Z from now on. Denote the Lévy-Khintchine triplet of ξ_1 by $(\gamma_\xi, \sigma_\xi^2, m_\xi)$, and let $\sigma_{\xi,\eta}$ denote the covariance of the Gaussian components of ξ and η at time $t = 1$. A similar notation will be used for all other Lévy processes. Let $(\mathcal{F}_t^{(\xi,\eta)})_{t\in\mathbb{R}}$ denote the natural filtration of (ξ, η). Note that $(\eta_t)_{t\geq 0}$ is a Lévy process in $(\mathcal{F}_t^{(\xi,\eta)})_{t\geq 0}$, i.e. for all $0 \leq s < t$, $\eta_t - \eta_s$ is independent of $\mathcal{F}_s^{(\xi,\eta)}$. Using this it is easily seen that the increment semimartingale triplet of η in $(\mathcal{F}_t^{(\xi,\eta)})_{t\in\mathbb{R}}$ is, for $t > 0$, given by $\lambda(\mathrm{d}t) = \mathrm{d}t$ and $(b_t, c_t, K(t; \mathrm{d}x)) = (\gamma_\eta, \sigma_\eta^2, m_\eta(\mathrm{d}x))$. Thus, (2.1)–(2.2) provide necessary and sufficient conditions that $\int_0^\infty \phi_s\, \mathrm{d}\eta_s$ exists for an arbitrary $(\mathcal{F}_t^{(\xi,\eta)})_{t\in\mathbb{R}}$-predictable process ϕ. This in particular includes (1.1)(a). When it comes to integrals involving the negative half axis, such as (1.1)(b), the situation is more complicated since η is not a Lévy process in $(\mathcal{F}_t^{(\xi,\eta)})_{t\in\mathbb{R}}$ (see [2], Section 5). In fact, a priori it is not even clear that η is an increment semimartingale in the natural filtration of (ξ, η) (or in any other filtration). However, using an enlargement of filtration result due to Jacod and Protter [10], Theorem 5.3 in [2] shows that this is indeed the case, and the triplet of η is calculated in an enlarged filtration. Theorem 5.5 in [2] provides sufficient conditions for integrals of the form $\int_{\mathbb{R}} \phi_s \, \mathrm{d}\eta_s$ to exist and as this result will be used throughout the paper we rephrase it as a remark.

Remark 2.1. Assume $E[\eta_1^2] < \infty$. Then for any $(\mathcal{F}_t^{(\xi,\eta)})_{t\in\mathbb{R}}$-predictable process having a.a. paths locally bounded the integral $\int_{\mathbb{R}} \phi_s \, \mathrm{d}\eta_s$ exists if $\int_{\mathbb{R}} (|\phi_s| + \phi_s^2) \, \mathrm{d}s < \infty$ a.s.

3 Main results

As above let (ξ, η) denote a bivariate Lévy process indexed by $\mathbb{R}$ with $\xi_0 = \eta_0 = 0$. To avoid trivialities assume that none of them are identically equal to 0. Often we will assume that $\xi_t \to \infty$ a.s. as $t \to \infty$ because if this fails, Theorem 2 in [8] shows that $\int_0^t e^{-\xi_{s-}} \, \mathrm{d}\eta_s$ does not converge as $t \to \infty$ a.s., implying that (1.1)(a) does not exist. We need the function A_ξ defined, cf. [8] and [13], as

$$A_\xi(x) = \max\{1, m_\xi((1,\infty))\} + \int_1^x m_\xi((y,\infty)) \, \mathrm{d}y, \quad x \geq 1.$$

To study (1.1)(b) we follow Lindner and Maller [13] and introduce $(L_t)_{t\geq 0}$ given by

$$L_t = \eta_t + \sum_{0<s\leq t} (e^{-\Delta \xi_s} - 1)\Delta \eta_s - t\sigma_{\xi,\eta} \quad \text{for } t \geq 0. \tag{3.1}$$

The process $(L_t, \xi_t)_{t\geq 0}$ is then a bivariate Lévy process in the $(\mathcal{F}_t^{(\xi,\eta)})_{t\geq 0}$-filtration (see [13], Proposition 2.3) and for all $t \geq 0$ we have

$$\int_{-t}^0 e^{\xi_{s-}} \, \mathrm{d}\eta_s \overset{\mathscr{D}}{=} e^{-\xi_t} \int_0^t e^{\xi_{s-}} \, \mathrm{d}\eta_s \overset{\mathscr{D}}{=} \int_0^t e^{-\xi_{s-}} \, \mathrm{d}L_s. \tag{3.2}$$

(Here $\overset{\mathscr{D}}{=}$ denotes equality in distribution.) Indeed, the second equality follows from [13], Proposition 2.3. To prove the first equality note that

$$\int_{-t}^0 e^{\xi_{s-}} \, \mathrm{d}\eta_s = e^{-(\xi_0 - \xi_{-t})} \int_{-t}^0 e^{\xi_{s-} - \xi_{-t}} \, \mathrm{d}\eta_s,$$

which by the stationary increments has the same law as

$$e^{-(\xi_t - \xi_0)} \int_0^t e^{\xi_{s-} - \xi_0} \, \mathrm{d}\eta_s = e^{-\xi_t} \int_0^t e^{\xi_{s-}} \, \mathrm{d}\eta_s.$$

The existence of (1.1)(a)–(b) is characterized in the following. All integrals over infinite intervals are defined in the proper sense of [2], implying in particular that they exist as improper integrals.

Theorem 3.1. *Assume $\xi_t \to \infty$ as $t \to \infty$ a.s.*

(1) *The following statements are equivalent:*

(a) *The integral* (1.1)(a) *exists.*

(b) $\int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s$ *converges in distribution as $t \to \infty$.*

(c) *We have*

$$\int_{[-e,e]^c} \frac{\log|y|}{A_\xi(\log|y|)}\, m_\eta(\mathrm{d}y) < \infty.$$

(2) *The following statements are equivalent:*

(a) *The integral* (1.1)(b) *exists.*

(b) $\int_{-t}^0 e^{\xi_{s-}}\,\mathrm{d}\eta_s$ *converges in distribution as $t \to \infty$.*

(c) *We have*

$$\int_{[-e,e]^c} \frac{\log|y|}{A_\xi(\log|y|)}\, m_L(\mathrm{d}y) < \infty. \tag{3.3}$$

If (1.1)(b) *exists then* $\int_{-\infty}^0 e^{\xi_{s-}}\,\mathrm{d}\eta_s \overset{\mathscr{D}}{=} \int_0^\infty e^{-\xi_{s-}}\mathrm{d}L_s$.

It should be noted that (1c) coincides with the condition in Erickson and Maller [8], Theorem 2, for (1.1)(a) to exist as an improper integral. In the case when $\xi_t \to \infty$ as $t \to \infty$ a.s, (1c) implies (2c); this is shown in [13], where also further interesting relations between these conditions can be found.

Proof. Throughout the proof assume $\xi_t \to \infty$ as $t \to \infty$ a.s. The equivalence between (1b) and (1c) is given in [13], Proposition 2.4. Using (3.2) it thus follows that (2b) and (2c) are equivalent. We prove the remaining assertions in a few steps.

Step 1. Assume there is an $\epsilon > 0$ such that $m_\eta(\{|x| > \epsilon\}) = 0$. (That is, η has no big jumps, implying in particular square integrability). We show that in this case (1.1)(a)–(b) both exist.

Note that if $E[|\xi_1|] < \infty$ then by the law of large numbers $\xi_t/t \to E[\xi_1]$ as $t \to \infty$ a.s. and in this case $E[\xi_1] > 0$. Recall that we assume $\xi_t \to \infty$ as $t \to \infty$ a.s. It follows from Kesten's trichotomy theorem (see e.g. [7], Theorem 4.4) that if $E[|\xi_1|] = \infty$ then $\lim_{t\to\infty} \xi_t/t = \infty$ a.s. Thus, there is a $\mu \in (0,\infty]$ such that $\xi_s/s \to \mu$ as $s \to -\infty$ and $\xi_t/t \to \mu$ as $t \to \infty$ a.s. In particular

$$\begin{aligned}
&\int_{-\infty}^0 (e^{\xi_s} + e^{2\xi_s})\,\mathrm{d}s + \int_0^\infty (e^{-\xi_t} + e^{-2\xi_t})\,\mathrm{d}t \\
&\quad = \int_{-\infty}^0 (e^{s(\xi_s/s)} + e^{2s(\xi_s/s)})\,\mathrm{d}s + \int_0^\infty (e^{-t(\xi_t/t)} + e^{-2t(\xi_t/t)})\,\mathrm{d}t < \infty
\end{aligned}$$

which by Remark 2.1 implies the existence of (1.1)(a)–(b).

Step 2. Assume η is a compound Poisson process, that is,

$$m_\eta(\mathbb{R}) < \infty, \quad \sigma_\eta^2 = 0, \quad \gamma_\eta = \int_{|x|\le 1} x\, m_\eta(\mathrm{d}x)$$

and η_t is given by $\eta_t = \sum_{0<s\le t} \Delta\eta_s$ for all $t > 0$ a.s. Assume in addition that (1c) holds. We show that in this case $\sum_{s>0} e^{-\xi_{s-}}|\Delta\eta_s| < \infty$ a.s. (This clearly implies that (1.1)(a) exists and equals $\sum_{s>0} e^{-\xi_{s-}}\Delta\eta_s$).

For this purpose we first prove

$$\int_0^\infty \int_{\mathbb{R}} 1 \wedge (e^{-\xi_s}|x|)\, m_\eta(\mathrm{d}x)\, \mathrm{d}s < \infty \quad \text{a.s.} \tag{3.4}$$

The integral in (3.4) can be written as $\int_0^\infty g(\xi_s)\,\mathrm{d}s$, where, for $y \in \mathbb{R}$, $g(y) = \int_{\mathbb{R}}(1 \wedge (e^{-y}|x|))\, m_\eta(\mathrm{d}x)$. Therefore, since g is non-increasing it follows from [8], Theorem 1, that (3.4) is satisfied if and only if

$$\int_{(1,\infty)} \frac{y}{A_\xi(y)}\, |\mathrm{d}g(y)| < \infty. \tag{3.5}$$

Simple manipulations show that

$$g(y) = m_\eta(\mathbb{R}) - \int_{-\infty}^{y} \int_{|x|\le e^z} |x|\, m_\eta(\mathrm{d}x) e^{-z}\,\mathrm{d}z$$

and hence the integral in (3.5) equals

$$\int_{(1,\infty)} \frac{y}{A_\xi(y)} \int_{|x|\le e^y} |x|\, m_\eta(\mathrm{d}x) e^{-y}\,\mathrm{d}y.$$

Since A_ξ is non-decreasing we can use Fubini to rewrite and dominate this integral as

$$\begin{aligned}
&\int_{[-e,e]^c} |x| \int_{\log|x|}^{\infty} \frac{ye^{-y}}{A_\xi(y)}\,\mathrm{d}y\, m_\eta(\mathrm{d}x)\\
&\quad\le \int_{[-e,e]^c} \frac{|x|}{A_\xi(\log|x|)} \int_{\log|x|}^{\infty} ye^{-y}\,\mathrm{d}y\, m_\eta(\mathrm{d}x)\\
&\quad= \int_{[-e,e]^c} \frac{|x|}{A_\xi(\log|x|)} e^{-\log|x|}(1+\log|x|)\, m_\eta(\mathrm{d}x),
\end{aligned}$$

which is finite by assumption. Now, by Cherny and Shiryaev [6], Lemma 3.4, (3.4) is equivalent to $\sum_{s>0}\big(1 \wedge (e^{-\xi_{s-}}|\Delta\eta_s|)\big) < \infty$ a.s. Hence $\sum_{s>0} e^{-\xi_{s-}}|\Delta\eta_s| < \infty$ a.s.

Step 3. Proof that (1c) implies (1a). Decompose $(\eta_t)_{t\ge 0}$ as $\eta_t = \eta_t^1 + \eta_t^2$ where $\eta_t^2 = \sum_{0<s\le t} \Delta\eta_s 1_{\{|\Delta\eta_s|>1\}}$, that is, η^2 contains all jumps of magnitude larger than 1. By Step 1, $\int_0^\infty e^{-\xi_{s-}}\,\mathrm{d}\eta_s^1$ exists and by Step 2, $\int_0^\infty e^{-\xi_{s-}}\,\mathrm{d}\eta_s^2$ exists if (1c) is fulfilled.

Step 4. We first prove that (2c) implies (2a). As in the proof of Step 2 we may and will assume that η is a compound Poisson process. In this case $(L_t)_{t\geq 0}$ in (3.1) is a compound Poisson process as well. By definition of $(L_t)_{t\geq 0}$ and Step 2 we have

$$\sum_{0<s<\infty} e^{-\xi_s}|\Delta\eta_s| = \sum_{0<s<\infty} e^{-\xi_{s-}}|\Delta L_s| < \infty \quad \text{a.s.}$$

Since $(\xi_t, \eta_t)_{t\geq 0} \overset{\mathscr{D}}{=} (-\xi_{(-t)-}, -\eta_{(-t)-})_{t\geq 0}$ it follows that

$$\sum_{0<s<\infty} e^{-\xi_s}|\Delta\eta_s| \overset{\mathscr{D}}{=} \sum_{-\infty<s<0} e^{\xi_{s-}}|\Delta\eta_s|.$$

Thus, the right-hand side is finite a.s., implying that the integral $\int_{-\infty}^0 e^{\xi_{s-}}\,\mathrm{d}\eta_s$ exists and equals $\sum_{-\infty<s<0} e^{\xi_{s-}}\Delta\eta_s$.

Finally, if (1.1)(b) exists then the condition in (3.3) is satisfied implying by (1) that $\int_0^\infty e^{-\xi_{s-}}\,\mathrm{d}L_s$ exists. From (3.2) it follows that

$$\int_{-\infty}^0 e^{\xi_{s-}}\,d\eta_s = \lim_{t\to\infty}\int_{-t}^0 e^{\xi_{s-}}\,d\eta_s \overset{\mathscr{D}}{=} \lim_{t\to\infty}\int_0^t e^{-\xi_{s-}}\,\mathrm{d}L_s = \int_0^\infty e^{-\xi_{s-}}\,\mathrm{d}L_s$$

where the first and third equality signs hold a.s.

□

Next we use the above theorem to study generalized Ornstein-Uhlenbeck processes. For this, consider a bivariate Lévy process $(U,\eta) = (U_t,\eta_t)_{t\in\mathbb{R}}$ with $U_0 = \eta_0 = 0$ and assume that none of them are identically equal to 0. Assume in addition that $m_U(\{-1\}) = 0$, meaning that U has no jumps of size -1. Following Basse-O'Connor et al. [2], Subsection 5.2, we introduce an extended filtration $(\mathcal{F}_t^{(U,\eta),\mathrm{ex}})_{t\in\mathbb{R}}$ which is defined as $\mathcal{F}_t^{(U,\eta),\mathrm{ex}} = \mathcal{F}_t^{(U,\eta)}$ for $t \geq 0$ and

$$\mathcal{F}_t^{(U,\eta),\mathrm{ex}} = \mathcal{F}_t^{(U,\eta)} \vee \sigma(\mu^{(U,\eta)}((t,0]\times A) : A \in \mathcal{B}(\mathbb{R}^2)) \quad \text{for } t<0,$$

where $\mu^{(U,\eta)}$ is the jump measure of (U,η). By [2], Theorem 5.3, η and U are increment semimartingales in the extended filtration, ensuring a well-defined integration theory with respect to these processes.

Since the index set is $\mathbb{R}$ rather than $\mathbb{R}_+$, we define the stochastic exponential of U, $\mathcal{E}(U) = (\mathcal{E}(U)_t)_{t\in\mathbb{R}}$, as the càdlàg process satisfying $\mathcal{E}(U)_0 = 1$ and

$$\frac{\mathcal{E}(U)_t}{\mathcal{E}(U)_s} = e^{(U_t-U_s)-\frac{(t-s)}{2}\sigma_U^2}\prod_{s<u\leq t}(1+\Delta U_u)e^{-\Delta U_u} \quad \text{for } s<t. \tag{3.6}$$

Put differently, $\mathcal{E}(U)$ is given by

$$\mathcal{E}(U)_t = \begin{cases} e^{U_t-\frac{t}{2}\sigma_U^2}\prod_{0<u\leq t}(1+\Delta U_u)e^{-\Delta U_u} & \text{for } t\geq 0, \\ e^{-U_t+\frac{t}{2}\sigma_U^2}\prod_{t<u\leq 0}(1+\Delta U_u)e^{-\Delta U_u} & \text{for } t\leq 0. \end{cases}$$

Equation (3.6) shows that for $s \in \mathbb{R}$, $(\mathcal{E}(U)_{t+s}/\mathcal{E}(U)_s)_{t\geq 0}$, is the usual stochastic exponential (with index set $\mathbb{R}_+$) of $(U_{t+s} - U_s)_{t\geq 0}$, cf. [11], II.8.

To describe $\mathcal{E}(U)$ in a more convenient way we follow [3] and introduce two important auxiliary processes $N = (N_t)_{t\in\mathbb{R}}$ and $\xi = (\xi_t)_{t\in\mathbb{R}}$ as follows. Let $N_0 = \xi_0 = 0$ and

$$N_t - N_s = \mu^U((s,t] \times (-\infty, 1)) \quad \text{for } s < t \tag{3.7}$$

$$\xi_t - \xi_s = -(U_t - U_s) + \tfrac{(t-s)}{2}\sigma_U^2 + \sum_{s<u\leq t} [\Delta U_u - \log|1 + \Delta U_u|] \quad \text{for } s < t. \tag{3.8}$$

These are essentially the definitions given in [3] except that the index set is $\mathbb{R}$ rather than $\mathbb{R}_+$, and our ξ corresponds to the process called $\widehat{U}$ there. As noted in [3], $N_t - N_s$ is the number of jumps in U of size less than -1 on the interval $(s,t]$. Moreover, N and ξ are both Lévy processes.

Lemma 3.2. *The process* $(\xi_t, N_t)_{t\in\mathbb{R}}$ *is* $(\mathcal{F}_t^{(U,\eta),\mathrm{ex}})_{t\in\mathbb{R}}$*-adapted.*

Proof. For $t \geq 0$ we let $s = 0$ in (3.7)–(3.8), which trivially shows that (N_t, ξ_t) is $\mathcal{F}_t^U$-measurable and hence also $\mathcal{F}_t^{(U,\eta),\mathrm{ex}}$-measurable. For $t < 0$ note that (use $t = 0$ and $s = t$ in (3.7)) $-N_t = \mu^U((t,0] \times (-\infty, 1])$ implying that N_t is $\mathcal{F}_t^{(U,\eta),\mathrm{ex}}$-measurable by definition of this σ-field. Moreover, by a standard argument,

$$\int_{(t,0]\times\mathbb{R}} \phi(x)\, \mu^U(\mathrm{d}u \times \mathrm{d}x) \tag{3.9}$$

is $\mathcal{F}_t^{(U,\eta),\mathrm{ex}}$-measurable for all measurable $\phi : \mathbb{R} \to \mathbb{R}$ for which (3.9) exists. In particular $\sum_{t<u\leq 0}[\Delta U_u - \log|1+\Delta U_u|]$ is measurable, implying by (3.8) that ξ_t is measurable with respect to $\mathcal{F}_t^{(U,\eta),\mathrm{ex}}$. □

The importance of N and ξ is due to the fact that $\mathcal{E}(U)$ is given as

$$\tfrac{\mathcal{E}(U)_t}{\mathcal{E}(U)_s} = (-1)^{N_t - N_s} e^{-(\xi_t - \xi_s)} \quad \text{for } s < t. \tag{3.10}$$

By Lemma 3.2 this shows that $\mathcal{E}(U)$ is $(\mathcal{F}_t^{(U,\eta),\mathrm{ex}})_{t\in\mathbb{R}}$-adapted. When U does not have jumps of size less than -1, we have

$$\tfrac{\mathcal{E}(U)_t}{\mathcal{E}(U)_s} = e^{-(\xi_t - \xi_s)} \quad \text{for } s < t.$$

In this case [11], II.8, shows that for all $s \in \mathbb{R}$ and $t \geq 0$, $U_{t+s} - U_s = \mathcal{L}\mathrm{og}(e^{-(\xi_{\cdot+s} - \xi_s)})_t$, where $\mathcal{L}\mathrm{og}$ denotes the stochastic logarithm.

Finally, we need the process $L^* = (L_t^*)_{t\in\mathbb{R}}$ defined as $L_0^* = 0$ and

$$\begin{aligned} L_t^* - L_s^* &= \eta_t - \eta_s + ([\eta, U]_t - [\eta, U]_s) \\ &= \eta_t - \eta_s + \sum_{s<u\leq t} \Delta U_u \Delta\eta_u + (t-s)\sigma_{U,\eta} \quad \text{for } s < t. \end{aligned} \tag{3.11}$$

It follows from (3.8) that when $\mathcal{E}(U_t) = e^{-\xi_t}$ for $t \in \mathbb{R}$ (that is, U has no jumps of size less than -1), then $L_t^* = L_t$, $t \geq 0$, where the latter is defined in (3.1). Clearly, L^* is determined by (U, η). Conversely, η is determined by (U, L^*) since

$$\eta_t - \eta_s = L_t^* - L_s^* - \sum_{s<u\leq t} \frac{\Delta U_u \Delta L_u^*}{1+\Delta U_u} - (t-s)\sigma_{U,L^*} \quad \text{for } s < t.$$

Note that since L^* differs from η only by a term which is of bounded variation on compacts and η is an increment semimartingale in the extended filtration, so is L^*. Similarly it follows that $(L_t)_{t\geq 0}$ is a semimartingale in $(\mathcal{F}_t^{(U,\eta)})_{t\geq 0}$.

In the following we consider càdlàg processes $V = (V_t)_{t\in\mathbb{R}}$ satisfying

$$V_t = \frac{\mathcal{E}(U)_t}{\mathcal{E}(U)_s}\left(V_s + \int_s^t \frac{\mathcal{E}(U)_s}{\mathcal{E}(U)_{u-}}\,\mathrm{d}\eta_u\right) \tag{3.12}$$

$$= \frac{\mathcal{E}(U)_t}{\mathcal{E}(U)_s}V_s + \mathcal{E}(U)_t \int_s^t [\mathcal{E}(U)_{u-}]^{-1}\,\mathrm{d}\eta_u \quad \text{for } s < t. \tag{3.13}$$

Remark 3.3. Assume $V = (V_t)_{t\in\mathbb{R}}$ is càdlàg and $(\mathcal{F}_t^{(U,\eta),\mathrm{ex}})_{t\in\mathbb{R}}$-adapted. Then V is given by (3.12) if and only if it satisfies the linear stochastic differential equation

$$V_t = V_s + (L_t^* - L_s^*) + \int_s^t V_{u-}\,\mathrm{d}U_u \quad \text{for } s < t. \tag{3.14}$$

For a proof, see [3], Proposition 3.2, or [9], Theoreme VI(6.8). A detailed study of stationary solutions to (3.14), including the nasty case $m_U(\{-1\}) > 0$, can be found in [3].

In the case when $\mathcal{E}(U_t) = e^{-\xi_t}$ for $t \in \mathbb{R}$, (3.12)–(3.13) reduce to so-called *generalized Ornstein-Uhlenbeck processes*:

$$V_t = e^{-(\xi_t - \xi_s)}\left(V_s + \int_s^t e^{\xi_{u-}-\xi_s}\,\mathrm{d}\eta_u\right) \tag{3.15}$$

$$= e^{-(\xi_t-\xi_s)}V_s + e^{-\xi_t}\int_s^t e^{\xi_{u-}}\,\mathrm{d}\eta_u \quad \text{for } s < t. \tag{3.16}$$

Theorem 3.4. (1) *Assume $\xi_t \to \infty$ as $t \to \infty$ a.s. The integral $\int_{-\infty}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ exists if and only if $\int_{-\infty}^0 e^{\xi_{u-}}\,\mathrm{d}\eta_u$ exists, and $\int_{-\infty}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u \overset{\mathscr{D}}{=} \int_0^\infty \mathcal{E}(U)_{u-}\,\mathrm{d}L_u^*$ in case of existence. Moreover, there is a stationary càdlàg process $V = (V_t)_{t\in\mathbb{R}}$ satisfying* (3.12) *if and only if $\int_{-\infty}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ exists. In this case $V = (V_t)_{t\in\mathbb{R}}$ is uniquely determined as*

$$V_t = \mathcal{E}(U)_t \int_{-\infty}^t \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u, \quad t \in \mathbb{R}. \tag{3.17}$$

(2) *Assume* $\xi_t \to -\infty$ *as* $t \to \infty$. *The integrals* $\int_0^\infty \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ *and* $\int_0^\infty e^{\xi_{u-}}\,\mathrm{d}\eta_u$ *exist at the same time. There is a stationary càdlàg process* $V = (V_t)_{t\in\mathbb{R}}$ *satisfying* (3.12) *if and only if* $\int_0^\infty \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ *exists. In this case* $V = (V_t)_{t\in\mathbb{R}}$ *is uniquely determined as*

$$V_t = -\mathcal{E}(U)_t \int_t^\infty \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u, \quad t \in \mathbb{R}. \tag{3.18}$$

Recall that necessary and sufficient conditions that $\int_0^\infty e^{\xi_{u-}}\,\mathrm{d}\eta_u$ and $\int_{-\infty}^0 e^{\xi_{u-}}\,\mathrm{d}\eta_u$ exist are given in Theorem 3.1. In particular, $\int_{-\infty}^0 e^{\xi_{u-}}\,\mathrm{d}\eta_u$ and $\int_0^\infty e^{-\xi_{u-}}\,\mathrm{d}L_u$ exist at the same time. This is also equivalent to the existence of $\int_0^\infty e^{-\xi_{u-}}\,\mathrm{d}L_u^*$; indeed, it is easily verified that $|\Delta L_t| = |\Delta L_t^*|$ for all $t > 0$, and thus the condition in Theorem 3.1(1c) with $\eta = L$ is equivalent to the one with $\eta = L^*$. Moreover, as in the proof of the first part of (1) below it follows that $\int_0^\infty e^{-\xi_{u-}}\,\mathrm{d}L_u^*$ and $\int_0^\infty \mathcal{E}(U)_{u-}\,\mathrm{d}L_u^*$ exist at the same time.

The above conditions for existence of V are also given in Behme et al. [3], Theorem 2.1, and so is the representation (3.18); thus, (2) is completely contained in [3] (except that we use proper integrals) but it is restated here for completeness. When $\xi_t = \lambda t$ for some non-zero constant λ, (3.15)–(3.16) simplify to a usual Ornstein-Uhlenbeck process; see e.g. [1] and [15]. In this case, (3.17) and (3.18) are well known representations of stationary Ornstein-Uhlenbeck processes cf. e.g. [15], Theorem 55.

The case when ξ_t does not converge to $\pm\infty$ as $t \to \infty$ is treated in Theorem 2.1 of [3].

Proof. (1) Since $\mathcal{E}(U)_{u-}^{-1}$ and $e^{\xi_{u-}}$ only differ by a factor of absolute value 1 the two proper integrals exist at the same time. The identity in distribution follows as in the proof of the last assertion of Theorem 3.1(2), where instead of (3.2) we use that for $t \geq 0$,

$$\int_{-t}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u \overset{\mathscr{D}}{=} \mathcal{E}(U)_t \int_0^t \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u \overset{\mathscr{D}}{=} \int_0^t \mathcal{E}(U)_{u-}\,\mathrm{d}L_u^*, \tag{3.19}$$

where the first equality follows as in the proof of (3.2) and the second comes from Lemma 3.1 in [3].

Assume V is stationary and satisfies (3.12). Letting $s \to -\infty$ and using that $\xi_s \to -\infty$ a.s. it follows from (3.13) and (3.10) that $\int_s^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ converges in distribution. From (3.19) it follows that $\int_0^t \mathcal{E}(U)_{u-}\,\mathrm{d}L_u^*$ converges in distribution as $t \to \infty$. Theorem 3.6 in [3] shows that the condition in Theorem 3.1(2c) (with L replaced by L^*) is satisfied, implying that $\int_{-\infty}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ exists.

Conversely, assuming that $\int_{-\infty}^0 \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u$ exists and defining V by (3.17) it is easily seen that (3.12) is satisfied. Moreover, since V_t is given a.s. as

$$V_t = \lim_{h\to\infty} \mathcal{E}(U)_t \int_{t-h}^t \mathcal{E}(U)_{u-}^{-1}\,\mathrm{d}\eta_u = \lim_{h\to\infty} \frac{\mathcal{E}(U)_t}{\mathcal{E}(U)_{t-h}} \int_{t-h}^t \frac{\mathcal{E}(U)_{t-h}}{\mathcal{E}(U)_{u-}}\,\mathrm{d}\eta_u$$

and the distribution of the right-hand side does not depend on t, the distribution of V_t is also independent of t. The variable V_t is moreover determined by $(\eta_t - \eta_u, U_t - U_u)_{u\leq t}$

and is hence in particular independent of $(\eta_{u+t}-\eta_t, U_{u+t}-U_t)_{u\geq 0}$. From (3.12) it follows that V is stationary.

The proof of (2) is similar except that in (3.12)–(3.13) we fix s, rearrange terms to isolate V_s, and let $t \to \infty$. □

In the next theorem we study integrability properties and the Wold-Karhunen representation of generalized Ornstein-Uhlenbeck processes. As above we consider the bivariate Lévy process (U, η) as well as ξ and L^* defined in (3.8) and (3.11). From now on we assume $m_U((-\infty, -1]) = 0$, that is, U has no jumps of size -1 or smaller. We will thus not need the process N. Whenever U is integrable let $\lambda := E[-U_1]$ and define $\overline{U} = (\overline{U}_t)_{t\in\mathbb{R}}$ as $\overline{U}_0 = 0$ and $\overline{U}_t - \overline{U}_s = U_t - U_s + \lambda(t-s)$ for $s < t$.

Theorem 3.5. *Assume* $m_U((-\infty, -1]) = 0$.

(1) *For* $r > 0$ *we have* $U_1 \in L^r(P)$ *if and only if* $E[e^{-r\xi_1}] < \infty$. *When* $U_1 \in L^1(P)$ *we have*

$$\exp(E[U_1]) = E[e^{-\xi_1}] = \exp\Big[-\gamma_\xi + \tfrac{1}{2}\sigma_\xi^2 + \int_{\mathbb{R}} (e^{-x} - 1 + x1_{\{|x|\leq 1\}})\, m_\xi(\mathrm{d}x)\Big]. \quad (3.20)$$

(2) *Assume* $L_1^* \in L^2(P)$ *and* $E[e^{-2\xi_1}] < 1$. *Then* $\xi_t \to \infty$ *as* $t \to \infty$ *a.s and the integral* $\int_0^\infty e^{-\xi_t}\,\mathrm{d}L_t^*$ *exists and is in* $L^2(P)$. *Moreover,* λ *is strictly positive, the generalized Ornstein-Uhlenbeck process satisfying* (3.15) *exists and is square integrable, and*

$$V_t = \int_{-\infty}^t e^{-\lambda(t-s)}\,\mathrm{d}L_s^* + \int_{-\infty}^t e^{-\lambda(t-s)} V_{s-}\,\mathrm{d}\overline{U}_s, \quad t \in \mathbb{R}. \quad (3.21)$$

Remark 3.6. Assume $L_1^* \in L^2(P)$, $E[e^{-2\xi_1}] < 1$ and $E[L_1^*] = 0$. By (3.21),

$$V_t = \int_{-\infty}^t e^{-\lambda(t-s)}\,\mathrm{d}\Xi_s, \quad t \in \mathbb{R}, \quad (3.22)$$

where, for $s < t$, $\Xi_t - \Xi_s = L_t^* - L_s^* - \int_s^t V_{u-}\,\mathrm{d}\overline{U}_u$ and $\Xi_0 = 0$. The process Ξ is square integrable with zero mean and stationary orthogonal increments, implying that (3.22) is the Wold-Karhunen representation of V. To verify this fix $s \in \mathbb{R}$. It was noted in the proof of Theorem 3.4 that V_s is independent of $(L_{s+t}^* - L_s^*, U_{s+t} - U_s)_{t\geq 0}$. Using that the distribution of V_s as well as of $(L_{s+t}^* - L_s^*, U_{t+s} - U_s)_{t\geq 0}$ does not depend on s, it follows from (3.15) that Ξ has stationary increments. Since L^* as well as $\overline{U}$ are zero mean square integrable Lévy processes and V is square integrable and stationary it follows by definition of Ξ that $(\Xi_{t+s} - \Xi_s)_{t\geq 0}$ is a square integrable martingale in the filtration generated by V_s and $(L_{s+t}^* - L_s^*, U_{t+s} - U_s)_{t\geq 0}$. In particular this implies that Ξ has zero mean and orthogonal increments.

Proof. (1) Since $U_t = \mathcal{L}\mathrm{og}(-\xi_t)$ for $t > 0$, [11], Corollary II.8.16, shows that

$$\int_{u>1} u^r\, m_U(\mathrm{d}u) = \int_{e^{rx}-1>1} (e^{rx} - 1)\, m_{-\xi}(\mathrm{d}x) = \int_{x>\frac{\log 2}{r}} (e^{rx} - 1)\, m_{-\xi}(\mathrm{d}x).$$

Thus, the left-hand side is finite if and only if $\int_{|x|>1} e^{rx}\, m_{-\xi}(\mathrm{d}x)$ is finite. On the other hand, since $m_U((-\infty,-1]) = 0$, [16], Corollary 25.8, shows that finiteness of the left-hand side is equivalent to $U_1 \in L^r(P)$, and similarly finiteness of $\int_{|x|>1} e^{rx}\, m_{-\xi}(\mathrm{d}x)$ is equivalent to $E[e^{-r\xi_1}] < \infty$.

Now assume $E[e^{-\xi_1}] < \infty$. The second equality in (3.20) follows from [16], Theorem 25.17. Moreover, since $U_t = \mathcal{L}\mathrm{og}(-\xi_t)$ for $t > 0$, [11], Theorem II.8.10, implies that

$$U_t = -\xi_t + \tfrac{t}{2}\sigma_\xi^2 + [(e^{-x} - 1 + x) * \mu^\xi]_t, \quad t \geq 0.$$

Recalling the Lévy-Itô decomposition of ξ:

$$\xi_t = x1_{\{|x|\leq 1\}} * (\mu^\xi - \nu^\xi)_t + [x1_{\{|x|>1\}} * \mu^\xi]_t + t\gamma_\xi + G_t, \quad t \geq 0,$$

where G denotes the (mean zero) Gaussian component of ξ and $\nu^\xi(\mathrm{d}t \times \mathrm{d}x) = m_\xi(\mathrm{d}x)\,\mathrm{d}t$, it follows that

$$U_t = -x1_{\{|x|\leq 1\}} * (\mu^\xi - \nu^\xi)_t - t\gamma_\xi - G_t + \tfrac{t}{2}\sigma_\xi^2 + [(e^{-x} - 1 + x1_{\{|x|\leq 1\}}) * \mu^\xi]_t, \quad t \geq 0.$$

All terms on the right-hand side have finite mean, and the first and third term have mean zero, implying

$$E[U_1] = -\gamma_\xi + \tfrac{1}{2}\sigma_\xi^2 + \int_{\mathbb{R}} (e^{-x} - 1 + x1_{\{|x|\leq 1\}})\, m_\xi(\mathrm{d}x).$$

In particular this gives (3.20).

(2) Since $E[e^{-\xi_1}] \leq \sqrt{E[e^{-2\xi_1}]} < 1$ it follows from (3.20) that $\lambda > 0$. It is shown in [13], Proposition 4.1, that $\xi_t \to \infty$ a.s. Recall ([16], Theorem 25.17) that $E[e^{-2\xi_t}] = (E[e^{-2\xi_1}])^t$ for $t \geq 0$. Using that L_1^* is square integrable we can decompose $(L_t^*)_{t\geq 0}$ as $L_t^* = M_t + c_1 t$ where c_1 is a constant and $(M_t)_{t\geq 0}$ is a square integrable martingale with $\langle M\rangle_t = c_2 t$ for some $c_2 \geq 0$. Since

$$E\Big[\int_0^\infty e^{-2\xi_t}\,\mathrm{d}\langle M\rangle_t\Big] = \int_0^\infty (E[e^{-2\xi_1}])^t\,\mathrm{d}(c_2 t) < \infty$$

the integral $\int_0^\infty e^{-\xi_t}\,\mathrm{d}M_t$ exists and is square integrable. Moreover, since ξ_s is independent of $\xi_u - \xi_s$ for all $0 \leq s < u$ and $E[e^{-\xi_1}], E[e^{-2\xi_1}] < 1$, we get

$$\begin{aligned} E\Big[\Big(\int_0^\infty e^{-\xi_s}\,\mathrm{d}s\Big)^2\Big] &= 2\int_0^\infty \int_s^\infty E[e^{-\xi_s - \xi_u}]\,\mathrm{d}u\,\mathrm{d}s \\ &= 2\int_0^\infty \int_s^\infty E[e^{-2\xi_s - (\xi_u - \xi_s)}]\,\mathrm{d}u\,\mathrm{d}s = 2\int_0^\infty \int_s^\infty (E[e^{-2\xi_1}])^s (E[e^{-\xi_1}])^{u-s}\,\mathrm{d}u\,\mathrm{d}s < \infty \end{aligned}$$

and hence the integrals $\int_0^\infty e^{-\xi_s}\,\mathrm{d}(c_1 s)$ and $\int_0^\infty e^{-\xi_s}\,\mathrm{d}L_s^*$ exist and are square integrable. By Theorem 3.4 and the remarks following it the generalized Ornstein-Uhlenbeck process $V = (V_t)_{t\in\mathbb{R}}$ exists and is square integrable. From (3.14) we have

$$V_t = V_s + [(L_t^* - L_s^*) + \int_s^t V_{u-}\,\mathrm{d}\overline{U}_u] - \lambda \int_s^t V_u\,\mathrm{d}u \quad \text{for } s < t.$$

Thus, by [9], Theoreme VI(6.8),

$$V_t = e^{-\lambda(t-s)} V_s + \int_s^t e^{-\lambda(t-u)} \, \mathrm{d}L_u^* + \int_s^t e^{-\lambda(t-u)} V_{u-} \, \mathrm{d}\overline{U}_u \quad \text{for } s < t. \tag{3.23}$$

Since V is a stationary square integrable process, λ is positive and $\overline{U}$ and L^* are square integrable, it follows from Remark 2.1 that $(V_{u-}e^{\lambda u}1_{\{u\leq t\}})_{u\in\mathbb{R}}$ is integrable with respect to $\overline{U}$ and $(e^{\lambda u}1_{\{u\leq t\}})_{u\in\mathbb{R}}$ is integrable with respect to L^*. Letting $s \to -\infty$ in (3.23) it follows that

$$V_t = \int_{-\infty}^t e^{-\lambda(t-u)} \, \mathrm{d}L_u^* + \int_{-\infty}^t e^{-\lambda(t-u)} V_{u-} \, \mathrm{d}\overline{U}_u \quad \text{for } t \in \mathbb{R}. \qquad \square$$

References

[1] Barndorff-Nielsen, O. E. and N. Shephard (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol. 63*(2), 167–241.

[2] Basse-O'Connor, A., S.-E. Graversen, and J. Pedersen (2010). A unified approach to stochastic integration on the real line. Thiele Research report 2010-08, `http://www.imf.au.dk/publs?publid=880`.

[3] Behme, A., A. Lindner, and R. A. Maller. (2010). Stationary Solutions of the Stochastic Differential Equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy Noise. Preprint.

[4] Bertoin, J., A. Lindner, and R. Maller (2008). On continuity properties of the law of integrals of Lévy processes. In *Séminaire de Probabilités XLI*, Volume 1934 of *Lecture Notes in Math.*, pp. 137–159. Berlin: Springer.

[5] Carmona, P., F. Petit, and M. Yor (2001). Exponential functionals of Lévy processes. In *Lévy processes*, pp. 41–55. Boston, MA: Birkhäuser Boston.

[6] Cherny, A. and A. Shiryaev (2005). On stochastic integrals up to infinity and predictable criteria for integrability. In *Séminaire de Probabilités XXXVIII*, Volume 1857 of *Lecture Notes in Math.*, pp. 165–185. Berlin: Springer.

[7] Doney, R. A. and R. A. Maller (2002). Stability and attraction to normality for Lévy processes at zero and at infinity. *J. Theoret. Probab. 15*(3), 751–792.

[8] Erickson, K. B. and R. A. Maller (2005). Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals. In *Séminaire de Probabilités XXXVIII*, Volume 1857 of *Lecture Notes in Math.*, pp. 70–94. Berlin: Springer.

[9] Jacod, J. (1979). *Calcul stochastique et problèmes de martingales*, Volume 714 of *Lecture Notes in Mathematics*. Berlin: Springer.

[10] Jacod, J. and P. Protter (1988). Time reversal on Lévy processes. *Ann. Probab. 16*(2), 620–641.

[11] Jacod, J. and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (Second ed.), Volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag.

[12] Kondo, H., M. Maejima, and K. Sato (2006). Some properties of exponential integrals of Lévy processes and examples. *Electron. Comm. Probab. 11*, 291–303 (electronic).

[13] Lindner, A. and R. Maller (2005). Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stochastic Process. Appl. 115*(10), 1701–1722.

[14] Lindner, A. and K. Sato (2009). Continuity properties and infinite divisibility of stationary distributions of some generalized Ornstein-Uhlenbeck processes. *Ann. Probab. 37*(1), 250–274.

[15] Rocha-Arteaga, A. and K. Sato (2003). *Topics in Infinitely Divisible Distributions and Lévy Processes*, Volume 17 of *Aportaciones Matemáticas: Investigación [Mathematical Contributions: Research]*. México: Sociedad Matemática Mexicana.

[16] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*, Volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press. Translated from the 1990 Japanese original, Revised by the author.

[17] Yor, M. (2001). *Exponential functionals of Brownian motion and related processes*. Springer Finance. Berlin: Springer-Verlag. With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson.