THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Lévy Bases and Extended Subordination

Ole E. Barndorff-Nielsen

Research Report

No. 12 | September 2010

Lévy Bases and Extended Subordination

Ole E. Barndorff-Nielsen

T.N. Thiele Centre; Department of Mathematical Sciences, Aarhus University

Abstract

The concept of subordination of Lévy processes is reinterpreted and then extended to a definition of subordination of Lévy bases. This is extended a step further, and then applied to give an alternative way of volatility/intermittency modulation in the context of ambit fields.

1 Introduction

Random time change of stochastic processes is a procedure of considerable interest, both theoretically and in various applications; see [BNShi10]. Of some special theoretical interest is the concept of subordination, [Bertoin99]. As regards modelling and inference, mathematical finance and financial econometrics provide important cases in point, see [BNS11].

Let X be a d-dimensional Lévy process and let T be a subordinator, i.e. a nonnegative Lévy process on $\mathbb{R}_+$. The subordination of X by T, denoted $Y = X \circ T$, is obtained by timewise composition of X by T, that is $Y_t = X_{T_t}$.

The present paper discusses extensions of the idea of time change to spatial and tempo-spatial settings, with special reference to ambit fields and processes. Section 2 provides some background material on Lévy bases and presents an alternative view of ordinary subordination of Lévy processes, as a lead up to the main part of this note, which introduces a subordination of Lévy bases by measure changes, in Section 3, and extends this to more general measure changes, with a view to volatilty/intermittency and the ambit approach in Section 4. Section 5 concludes.

2 Preliminaries

This Section has two parts. The first provides background material on Lévy bases, needed for the main part of this note, and the second presents an alternative view on subordination of Lévy processes, leading to the extended subordination concept introduced in Sections 3 and 4.

2.1 Lévy bases

This Section recalls basic definitions and properties of Lévy bases on $\mathbb{R}^d$. For detailed discusssions of the mathematical theory we refer to [RajRos89] and [Ped03].

Let $\mathcal{B}$ denote the family of Borel sets in $\mathbb{R}^d$ and let $\mathcal{B}_b$ be the subfamily consisting of the bounded elements of $\mathcal{B}$. An independently scattered random measure M on $\mathbb{R}^d$ is a collection $\{M(B) : B \in \mathcal{B}_b\}$ of random variables on some probability space $(\Omega, \mathcal{A}, P)$ such that for every sequence $\{B_n\}$ of disjoint sets in $\mathcal{B}_b$ with $\cup_{n=1}^{\infty} B_n \in \mathcal{B}_b$ the random variables $M(B_n)$, $n = 1, 2, \ldots$, are independent and

$$M\left(\cup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} M(B_n) \quad \text{a.s.}$$

Note that, in general, M may take both negative and positive values.

Definition. *A Lévy basis L on $\mathbb{R}^d$ is an independently scattered random measure on $\mathbb{R}^d$ such that for all $B \in \mathcal{B}_b$ the random variable $L(B)$ is infinitely divisible and its Lévy-Khintchine representation has the form*[1]

$$\mathrm{C}\{\zeta \ddagger L(B)\} = ia(B)\zeta - \frac{1}{2}m(B)\zeta^2 + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)\right) n(\mathrm{d}x; B) \quad (1)$$

where a and m are measures on $\mathbb{R}$ (a in general signed) and $n(\mathrm{d}x; B)$ is for fixed B a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ and for fixed $\mathrm{d}x$ a measure on $\mathbb{R}^d$.

The associated measure

$$c(B) = \|a\|(B) + m(B) + \int_{-\infty}^{\infty} \left(1 \wedge x^2\right) n(\mathrm{d}x; B),$$

where $\|a\|$ denotes the absolute variation of a, is called the control meaure of (1). We introduce the Radon-Nikodym derivatives

$$a(s) = \frac{\mathrm{d}a}{\mathrm{d}c}(s),$$
$$m(s) = \frac{\mathrm{d}m}{\mathrm{d}c}(s)$$

and

$$\nu(\mathrm{d}x; s) = \frac{n(\mathrm{d}x; \cdot)}{\mathrm{d}c}(s).$$

Thus, in particular,

$$n(\mathrm{d}x; \mathrm{d}s) = \nu(\mathrm{d}x; s)\, c(\mathrm{d}s).$$

There is no loss of generality in assuming that $\nu(\mathrm{d}x; s)$ is a Lévy measure for each fixed s and we do so.

In other words, any Lévy basis on $\mathbb{R}^d$ determines a quadruplet $(a, m, \nu(\mathrm{d}x; \cdot), c)$ or, written more explicitly, $\left(a(s), m(s), \{\nu(\mathrm{d}x; s)\}_{s \in \mathbb{R}^d}, c(\mathrm{d}s)\right)$ where a and $m \geq 0$ are functions on $\mathbb{R}^d$, $\nu(\mathrm{d}x; s)$ denotes for fixed s a Lévy measure on $\mathbb{R}$ and is for fixed $\mathrm{d}x$ a measurable function on $\mathbb{R}^d$, and c is a measure on $\left(\mathbb{R}^d, \mathcal{B}\right)$ such that the

[1] We denote the cumulant function of a random variable Y by $\mathrm{C}\{\zeta \ddagger Y\}$ and the cumulant function of Y conditional on another random variable X by $\mathrm{C}\{\zeta \ddagger Y | X\}$. Similarly, we write $\phi(\zeta \ddagger Y)$ and $\phi(\zeta \ddagger Y | X)$ for the associated characteristic functions.

integrals of a and m with repect to c exist and determine measures on $\left(\mathbb{R}^d, \mathcal{B}\right)$ (in the case of a possibly signed), and such that

$$\int_B \nu(\mathrm{d}x; s)\, c(\mathrm{d}s)$$

is a Lévy measure on $\mathbb{R}$ for each fixed $B \in \mathcal{B}$. Conversely, any such quadruplet determines, in law, a Lévy basis on $\mathbb{R}^d$. We refer to $(a, m, \nu(\mathrm{d}x; \cdot), c)$ as the *characteristic quadruplet* of the Lévy basis. Given such a quadruplet we denote $\nu(\mathrm{d}x; s)\, c(\mathrm{d}s)$ by $n(\mathrm{d}x; \mathrm{d}s)$ and we define $N(\mathrm{d}x; \mathrm{d}s)$ as the Poisson measure on $\mathbb{R}^d$ having compensator n. Generally, n with or without a suffix will stand for a compensator of this type and N with the same suffix denotes the corresponding Poisson measure

As is the case for Lévy processes, any Lévy basis has a Lévy-Ito representation

$$L(B) = a(B) + G(B) + \int_{|x|>1} xN(\mathrm{d}x; B) + \int_{|x|\le 1} x(N-n)(\mathrm{d}x; B) \tag{2}$$

where a is a, possibly signed measure, $G(B)$ is a Gaussian independently scattered random measure with $G(B) \sim N(0, m(A))$, N is a Poisson measure, independent of G and with compensator $n(\mathrm{d}x; \mathrm{d}s) = \mathrm{E}\{N(\mathrm{d}x; \mathrm{d}s)\}$. This result is due to [Ped03]. The notation used here is consistent with that of the Lévy-Khintchine representation (1).

Remark 1. The representation (2) may conveniently be expressed in infinitesimal form

$$L(\mathrm{d}s) = a(\mathrm{d}s) + G(\mathrm{d}s) + \int_{|x|>1} xN(\mathrm{d}x; \mathrm{d}s) + \int_{|x|\le 1} x(N-n)(\mathrm{d}x; \mathrm{d}s). \tag{3}$$

Correspondingly we may write the Lévy-Khintchine representation (1) infinitesimally as

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger L(\mathrm{d}s)\} &= ia(\mathrm{d}s)\zeta - \frac{1}{2}m(\mathrm{d}s)\zeta^2 + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)\right) n(\mathrm{d}x; \mathrm{d}s) \\ &= ia(\mathrm{d}s)\zeta - \frac{1}{2}m(\mathrm{d}s)\zeta^2 + \mathrm{C}\{\zeta \ddagger L'(s)\}c(\mathrm{d}s) \end{aligned}$$

where to each $s \in \mathbb{R}$ we have now associated an infinitely divisible random variable $L'(s)$ with Lévy-Khintchine representation

$$\mathrm{C}\{\zeta \ddagger L'(s)\} = \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)\right) \nu(\mathrm{d}x; s).$$

We refer to $L'(s)$ as the *Lévy seed* of L at s. By $\{L'_t(s)\}$ we denote the Lévy process generated by $L'(s)$, i.e. the Lévy process for which the law of $L'_1(s)$ equals that of $L'(s)$. □

In case $a = m = 0$, and thinking of the $L'(s)$ as independent, we may now formally represent the basis L by

$$L(B) = \int_B L'(s)\, c(\mathrm{d}s)$$

and then, for a general function f on $\mathbb{R}^d$ we have

$$f \bullet L = \int_{\mathbb{R}^d} f(s) L'(s) c(\mathrm{d}s). \tag{4}$$

When $\nu(\mathrm{d}x; s)$ does not depend on s, the Lévy basis is said to be *factorisable* and if, moreover, c is proportional to Lebesgue measure and $a(s)$ and $m(s)$ do not depend on s then L is *homogeneous*.

The basis L is said to be non-Gaussian if $G = 0$. We will initially treat the Gaussian case, where $L = G$ or $L = a + G$, and the non-Gaussian case separately, as these two cases are somewhat different in nature.

In the non-Gaussian case

$$L(B) = a(B) + \int_{|x|>1} xN(\mathrm{d}x; B) + \int_{|x|\leq 1} x(N - n)(\mathrm{d}x; B), \tag{5}$$

and so

$$\mathrm{C}\{\zeta \ddagger L(B)\} = ia(B)\zeta + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{(-1,1)}(x)\right) n(\mathrm{d}x; B). \tag{6}$$

For later reference we note that if $a = 0$ in (5) and if L is nonnegative then L can in fact be expressed more simply as

$$L(B) = \int_0^{\infty} xN(\mathrm{d}x; B). \tag{7}$$

With probability 1, an arbitrary realisation of a Lévy basis of this type is in fact an ordinary measure on $\mathbb{R}^d$. (This property does not hold generally for independently scattered random measures.)

Example 1 (Inverse Gaussian seeds)**.** We recall that the inverse Gaussian law, denoted $IG(\delta, \gamma)$, is infinitely divisible with probability density function

$$\frac{\delta}{\sqrt{2\pi}} e^{-\delta\gamma} x^{-3/2} \exp\{-\tfrac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\} \tag{8}$$

where $x > 0$ and the parameters satisfy $\delta > 0$ and $\gamma \geq 0$. This has Lévy density

$$\frac{1}{\sqrt{2\pi}} x^{-3/2} \exp\{-\tfrac{1}{2}\gamma^2 x\}$$

and cumulant function

$$\mathrm{C}\{\zeta\} = -\delta\gamma + \delta\left(\gamma^2 - 2i\zeta\right)^{1/2},$$

and a sum of independent observations from this law must consequently follow the $IG(n\delta, \gamma)$ distribution. An inverse Gaussian homogeneous Lévy basis L may now be specified by taking $L(A)$ to have the $IG(|A|\delta, \gamma)$ law, where $|A|$ is the Lebesgue measure of A. More generally, a non-Gaussian Lévy basis whose seeds are of the form

$$\nu(\mathrm{d}x; s) = \frac{\delta(s)}{\sqrt{2\pi}} x^{-3/2} \exp\{-\tfrac{1}{2}\gamma(s)^2 x\}\mathrm{d}x$$

will be referred to as an inverse Gaussian basis. □

Example 2 (Normal inverse Gaussian seeds)**.** The normal inverse Gaussian distribution $NIG\left(\alpha, \beta, \mu, \delta\right)$ ([BN98]) equals the law at time 1 of the process obtained by subordinating a Brownian motion of mean μ and drift β to the inverse Gaussian subordinator with law $IG\left(\delta, \gamma\right)$ at time 1. It is the distribution on $\mathbf{R}$ having probability density function

$$p(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q \left(\frac{x-\mu}{\delta}\right)^{-1} K_1 \left\{\delta \alpha q \left(\frac{x-\mu}{\delta}\right)\right\} e^{\beta x} \tag{9}$$

where $q(x) = \surd(1+x^2)$ and

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \alpha \exp\left\{\delta \surd(\alpha^2 - \beta^2) - \beta\mu\right\} \tag{10}$$

and where K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbf{R}$, $\delta \in \mathbf{R}_+$, and $0 \leq \beta < \alpha$. The Lévy density is

$$\frac{\delta\alpha}{\pi} |x|^{-1} K_1 \left(\alpha |x|\right) e^{\beta y}$$

and the cumulant function has the form

$$\mathrm{C}\{\zeta\} = \delta\{\surd(\alpha^2 - \beta^2) - \surd(\alpha^2 - (\beta + i\zeta)^2)\} + i\mu\zeta. \tag{11}$$

A non-Gaussian Lévy basis is then determined by having

$$L\left(A\right) \sim NIG\left(\left(\alpha, \beta, |A|\mu, |A|\delta\right)\right).$$

This is the homogeneous normal inverse Gaussian basis, the general form of normal inverse Gaussian bases having Lévy seeds

$$\nu\left(\mathrm{d}x; s\right) = \frac{\delta\left(s\right)\alpha\left(s\right)}{\pi} |x|^{-1} K_1 \left(\alpha\left(s\right)|x|\right) e^{\beta(s)x} \mathrm{d}x. \qquad \square$$

Integration of deterministic functions f with respect to an arbitrary Lévy basis L is defined in [RajRos89], where criteria for existence of the integral $f \bullet L$ are also given. We denote such an integral by $f \bullet L$. The resulting integral is infinitely divisible with Lévy-Khintchine representation provided by Proposition 2.6 in [RajRos89]. In the above notation this can be written as

$$\mathrm{C}\{\zeta \ddagger f \bullet L\} = -\frac{1}{2} \int_{\mathbb{R}^d} f^2\left(s\right) m\left(s\right) c\left(\mathrm{d}s\right) + \int_{\mathbb{R}^d} \mathrm{C}\{\zeta f\left(s\right) \ddagger a\left(s\right) + L'\left(s\right)\} c\left(\mathrm{d}s\right). \tag{12}$$

When $a = m = 0$ this becomes

$$\mathrm{C}\{\zeta \ddagger f \bullet L\} = \int_{\mathbb{R}^d} \mathrm{C}\{\zeta f\left(s\right) \ddagger L'\left(s\right)\} c\left(\mathrm{d}s\right),$$

a formula that also follows directly, though only formally, from (4).

Note finally, that any one-dimensional Lévy process $\{L_t\}$ determines a Lévy basis, given by $L\left(\left(u, v\right]\right) = L_v - L_u$ for $u \leq v$.

2.2 Timewise subordination: an alternative view

Let L be a one-dimensional non-Gaussian[2] Lévy process with compensator $n(\mathrm{d}x;\mathrm{d}s) = \nu(\mathrm{d}y)\mathrm{d}s$ and Lévy-Ito representation

$$L_t = at + \int_0^t \int_{|x|>1} xN\left(\mathrm{d}x;\mathrm{d}s\right) + \int_0^t \int_{|x|\leq 1} x\left(N-n\right)\left(\mathrm{d}x;\mathrm{d}s\right), \tag{13}$$

let T be a subordinator with compensator $n_0\left(\mathrm{d}u;\mathrm{d}s\right) = \nu_0\left(\mathrm{d}u\right)\mathrm{d}s$, Poisson basis N_0 and Lévy-Ito representation

$$T_t = \int_0^t \int_0^\infty xN_0\left(\mathrm{d}x;\mathrm{d}s\right),$$

and let $L \circ T$ be the subordination of L by T.

We denote by L_0 the Lévy basis determined from the subordinator T. In the terminology introduced above and denoting the Lebesgue measure by λ, the Lévy basis determined by the processes L and L_0 have characteristic quadruplets $\left(a, 0, \nu\left(\mathrm{d}x\right), \lambda\right)$ and $\left(0, 0, \nu_0\left(\mathrm{d}x\right), \lambda\right)$, respectively. Note that, in consequence of a remark above, the realisations of the Lévy basis L_0 are almost surely ordinary measures.

We now introduce a new random measure $\hat{L}$ on $\mathbb{R}$ by substituting $\lambda\left(\mathrm{d}s\right)$ in $\left(a, 0, \nu\left(\mathrm{d}x\right), \lambda\right)$ by $L_0\left(\mathrm{d}s\right)$. In other words, conditional on L_0, $\hat{L}$ is the Lévy basis with infinitesimal representation

$$\hat{L}\left(\mathrm{d}s\right) = aL_0\left(\mathrm{d}s\right) + \int_0^t \int_{|x|\geq 1} x\hat{N}\left(\mathrm{d}x;\mathrm{d}s\right) + \int_0^t \int_{|x|<1} x\left(\hat{N} - \hat{n}\right)\left(\mathrm{d}x;\mathrm{d}s\right)$$

where $\hat{n}\left(\mathrm{d}x;\mathrm{d}s\right) = \nu\left(\mathrm{d}x\right)L_0\left(\mathrm{d}s\right)$ and, as specified above, $\hat{N}$ is the Poisson measure with compensator $\hat{n}\left(\mathrm{d}y;\mathrm{d}s\right) = \nu\left(\mathrm{d}y\right)L_0\left(\mathrm{d}s\right)$.

The aim of this Section is to show

Proposition. *The (unconditional) law of $\hat{L}$ is the same as the law of $L \circ T$.*

Proof. On account of (12), the conditional cumulant functional of $\hat{L}$ given L_0 satisfies,

$$\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}|L_0\} = \int_0^\infty \mathrm{C}\{\zeta f\left(s\right) \ddagger a + L'\}L_0\left(\mathrm{d}s\right) = \mathrm{C}\{\zeta f\left(\cdot\right) \ddagger a + L'\} \bullet L_0$$

where L' is the Lévy seed of the Lévy basis determined by L; since that seed does not depend on s, in this proof we simply write L' for $L'\left(s\right)$. Further, on account

[2]The argument presented below is easily extended to include a Gaussian term.

again of (12),

$$
\begin{aligned}
\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} &= \int_0^\infty \mathrm{C}\{\mathrm{C}\{\zeta f(s) \ddagger a + L'\} \ddagger L_0'\}\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \left(e^{i\mathrm{C}\{\zeta f(s)\ddagger a+L'\}u} - 1\right) \nu_0(\mathrm{d}u)\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \left(e^{i\mathrm{C}\{\zeta f(s)\ddagger au+L_u'\}} - 1\right) \nu_0(\mathrm{d}u)\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \left(\phi\left(\zeta f(s) \ddagger au + L_u'\right) - 1\right) \nu_0(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}
$$

Introducing the law $P\{a + L_u' \in \mathrm{d}\xi\}$ of $a + L_u'$ the latter expression may be recast as

$$
\begin{aligned}
&\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} \\
&\quad = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left(e^{i\zeta f(s)\xi} - 1 - i\zeta f(s)\,\xi 1_{[-1,1]}(\xi)\right) P\{au + L_u' \in \mathrm{d}\xi\}\,\nu_0(\mathrm{d}u)\,\mathrm{d}s \\
&\qquad + i\zeta \int_0^\infty \int_0^\infty \int_{-1}^1 f(s)\,\xi P\{au + L_u' \in \mathrm{d}\xi\}\,\nu_0(\mathrm{d}u)\,\mathrm{d}s
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} \\
&\quad = i\zeta\hat{a} \int_0^\infty f(s)\,\mathrm{d}s + \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left(e^{i\zeta f(s)\xi} - 1 - i\zeta f(s)\,\xi 1_{[0,1]}(\xi)\right) \hat{\nu}(\mathrm{d}\xi)\,\mathrm{d}s
\end{aligned}
\tag{14}
$$

where

$$
\hat{a} = \int_0^\infty \int_{-1}^1 \xi P\{au + L_u' \in \mathrm{d}\xi\}\,\nu_0(\mathrm{d}u)
$$

and

$$
\hat{\nu}(\mathrm{d}x) = \int_0^\infty P\{au + L_u' \in \mathrm{d}x\}\,\nu_0(\mathrm{d}u).
$$

In other words, the random measure $\hat{L}$ is a Lévy basis which generates a Lévy process whose Lévy seed has Lévy-Khintchine representation

$$
\mathrm{C}\{\zeta \ddagger \hat{L}'\} = \hat{a} + \int_0^\infty \int_{-\infty}^\infty \left(e^{i\zeta\xi} - 1 - i\zeta\xi 1_{[0,1]}(\xi)\right) \hat{\nu}(\mathrm{d}\xi)
$$

and the characteristic triplet of $\hat{L}$ is given by $\hat{a}$ and $\hat{\nu}(\mathrm{d}x)$ above and by the Lebesgue measure λ. But, on noting that L_u' has the same law as L_u, one observes that the triplet $(\hat{a}, 0, \hat{\nu}\lambda)$ equals that of the Lévy-Khintchine representation of $L \circ T$, as follows from [Huff69] who by another, more analytical, type of reasoning, derived the characteristic triplet of any subordinated Lévy process (cf. also [Bertoin99], Section 8.4). □

3 Subordination of Lévy bases

Generalising the approach taken in Section 2, we now define the subordination of an arbitrary Lévy basis L on $\mathbb{R}^d$, with Lévy-Ito representation (5), by a nonnegative and dispersive Lévy basis L_0, also on $\mathbb{R}^d$, that is independent of L and has Lévy-Ito representation

$$L_0\left(\mathrm{d}s\right) = \int_0^\infty x N_0\left(\mathrm{d}x;\mathrm{d}s\right) \tag{15}$$

with compensator

$$n_0\left(\mathrm{d}x;\mathrm{d}s\right) = \nu_0\left(\mathrm{d}x;s\right) c_0\left(\mathrm{d}s\right). \tag{16}$$

As pointed out above, the realisations of L_0 are almost surely genuine measures on $\mathbb{R}^d$.

3.1 Non-Gaussian subordinand

Let L be a non-Gaussian Lévy basis with Lévy-Ito representation

$$L\left(B\right) = a\left(B\right) + \int_{|x|>1} xN\left(\mathrm{d}x;B\right) + \int_{|x|\le 1} x\left(N-n\right)\left(\mathrm{d}x;B\right).$$

The subordination of L by L_0 is carried out via substituting the factor $c\left(\mathrm{d}s\right)$, from the characteristic quadruplet of L, by $L_0\left(\mathrm{d}s\right)$. Provided $\nu\left(\mathrm{d}x;s\right) L_0\left(\mathrm{d}s\right)$ almost surely determines a σ-finite measure on $\mathbb{R}^d$, this construction yields a well-defined random measure $\hat{L} = \left\{\hat{L}\left(B\right)\right\}_{B\in\mathcal{B}_b}$ by the specification that conditionally on L_0 the random variable $\hat{L}\left(B\right)$ is infinitely divisible with infinitesimal Lévy-Khintchine representation

$$\mathrm{C}\{\zeta \ddagger \hat{L}\left(\mathrm{d}s\right)|L_0\} = i\zeta a\left(s\right) L_0\left(\mathrm{d}s\right) + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-\varepsilon,\varepsilon]}\left(x\right)\right)\nu\left(\mathrm{d}x;s\right) L_0\left(\mathrm{d}s\right)$$

and corresponding infinitesimal Lévy-Ito representation

$$\hat{L}\left(\mathrm{d}s\right)|L_0 = a\left(s\right) L_0\left(\mathrm{d}s\right) + \int_{|x|>1} x\hat{N}\left(\mathrm{d}x;\mathrm{d}s\right) + \int_{|x|\le 1} x\left(\hat{N}-\hat{n}\right)\left(\mathrm{d}x;\mathrm{d}s\right)$$

where $\hat{N}$ is a Poisson random measure with compensator $\hat{n}\left(\mathrm{d}x;\mathrm{d}s\right) = \nu\left(\mathrm{d}x;s\right) L_0\left(\mathrm{d}s\right)$.

Theorem. *The random measure $\hat{L}$ is a Lévy basis with infinitesimal Lévy-Khintchine representation*

$$\mathrm{C}\{\zeta \ddagger \hat{L}\left(\mathrm{d}s\right)\} = i\hat{a}\left(\mathrm{d}s\right)\zeta + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}\left(x\right)\right)\hat{n}\left(\mathrm{d}x;\mathrm{d}s\right)$$

where

$$\hat{a}\left(\mathrm{d}s\right) = \tilde{a}\left(s\right) c_0\left(\mathrm{d}s\right)$$

with

$$\tilde{a}\left(s\right) = \int_0^\infty \int_{-1}^{1} vP\left\{a\left(s\right)u + L'_u\left(s\right) \in \mathrm{d}v\right\}\nu_0\left(\mathrm{d}u;s\right)$$

and

$$\hat{n}(\mathrm{d}x;\mathrm{d}s)=\hat{\nu}(\mathrm{d}x;s)\,c_0(\mathrm{d}s)$$

with

$$\hat{\nu}(\mathrm{d}x;s)=\int_0^\infty P\left\{a(s)\,u+L_u'(s)\in\mathrm{d}x\right\}\nu_0(\mathrm{d}u;s)\,.$$

Furthermore, the functional cumulant transforms of L, L_0 and $\hat{L}$ are related by

$$\mathrm{C}\{f\bullet\hat{L}\}=\mathrm{C}\{\mathrm{C}\{f(s)\ddagger a(s)+L'(s)\}\ddagger L_0'(s)\}\bullet c_0(\mathrm{d}s)\,. \tag{17}$$

Remark 2. Formula (17) constitutes the generalisation to Lévy bases of the well known composition relation (cf. for instance [Bertoin99] Proposition 8.6) of the Laplace exponents for subordination of Lévy processes. □

Proof of the Theorem. By formula (12) we find

$$\mathrm{C}\{f\bullet\hat{L}|L_0\}=\int_{\mathbb{R}^d}\mathrm{C}\{f(s)\ddagger a(s)+L'(s)\}L_0(\mathrm{d}s)=\mathrm{C}\{f(\cdot)\ddagger a(\cdot)+L'(\cdot)\}\bullet L_0. \tag{18}$$

Consequently

$$\begin{aligned}
\mathrm{C}\{f\bullet\hat{L}\}&=\int_{\mathbb{R}^d}\mathrm{C}\{\mathrm{C}\{f(s)\ddagger a(s)+L'(s)\}\ddagger L_0'(s)\}c_0(\mathrm{d}s)\\
&=\int_{\mathbb{R}^d}\int_0^\infty\left(e^{i\mathrm{C}\{\zeta f(s)\ddagger a(s)+L'(s)\}u}-1\right)\nu_0(\mathrm{d}u;s)\,c_0(\mathrm{d}s)\\
&=\int_{\mathbb{R}^d}\int_0^\infty\left(e^{i\mathrm{C}\{\zeta f(s)\ddagger a(s)u+L_u'(s)\}}-1\right)\nu_0(\mathrm{d}u;s)\,c_0(\mathrm{d}s)\\
&=\int_{\mathbb{R}^d}\int_0^\infty\left(\phi\left(\zeta f(s)\ddagger a(s)\,u+L_u'(s)\right)-1\right)\nu_0(\mathrm{d}u;s)\,c_0(\mathrm{d}s)\\
&=\int_{\mathbb{R}^d}\int_0^\infty\int_{-\infty}^\infty\left(e^{if(s)v}-1-i\zeta f(s)\,v1_{[-1,1]}(v)\right)\\
&\qquad\qquad\times P\left\{a(s)\,u+L_u'(s)\in\mathrm{d}v\right\}\nu_0(\mathrm{d}u;s)\,c_0(\mathrm{d}s)\\
&\quad+i\zeta\int_{\mathbb{R}^d}f(s)\int_0^\infty\int_{-1}^1 vP\left\{a(s)\,u+L_u'(s)\in\mathrm{d}v\right\}\nu_0(\mathrm{d}u;s)\,c_0(\mathrm{d}s)\\
&=\int_{\mathbb{R}^d}\int_0^\infty\int_0^\infty\left(e^{i\zeta f(s)v}-1-i\zeta f(s)\,v1_{[0,1]}(v)\right)\hat{\nu}(\mathrm{d}v;s)\,c_0(\mathrm{d}s)\\
&\quad+i\zeta\int_{\mathbb{R}^d}f(s)\,\hat{a}(\mathrm{d}s)
\end{aligned}$$

where

$$\hat{\nu}(\mathrm{d}v;s)=\int_0^\infty P\left\{a(s)\,u+L_u'(s)\in\mathrm{d}v\right\}\nu_0(\mathrm{d}u;s) \tag{19}$$

and

$$\hat{a}(\mathrm{d}s)=\tilde{a}(s)\,c_0(\mathrm{d}s) \tag{20}$$

with

$$\tilde{a}(s)=\int_0^\infty\int_{-1}^1 vP\left\{a_\varepsilon(s)\,u+L_u'(s)\in\mathrm{d}v\right\}\nu_0(\mathrm{d}u;s)\,. \tag{21}$$

This proves the first part of the Theorem.

Formula (17) is just a reformulation of formula (18). □

Remark 3. The Lévy seeds of L, L_0 and $\hat{L}$ are related by $\hat{L}' = L' \circ L_0'$ where the latter formula is to be understood as saying that for (almost all) s$\in \mathbb{R}^d$ the Lévy process generated by $\hat{L}'(s)$ is equal (in law) to the subordination of the Lévy process generated from $L'(s)$ by the subordinator generated by $L_0'(s)$. □

3.2 Gaussian subordinand

Subordination of the Gaussian subordinand G by the independent subordinator L_0, given by (15)-(16), is now defined as the random measure $\hat{G}$ for which, conditionally on L_0, the law of $\hat{G}(B)$ is normal with mean 0 and variance $L_0(B)$. A direct calculation, using formula (12), shows that $\hat{G}$ is indeed a Lévy basis, and that it is non-Gaussian with compensator $\hat{\nu}(\mathrm{d}x;s)\,c_0(\mathrm{d}s)$ where

$$\hat{\nu}(\mathrm{d}x;s) = \int_0^\infty \varphi(x;u)\,v_0(\mathrm{d}u;s)\,\mathrm{d}x,$$

$\varphi(x;u)$ denoting the density of the Gaussian law of mean 0 and variance u.

Example 3. Suppose that G is the white noise basis and that L_0 is the homogeneous $IG(\delta,\gamma)$ basis. Then $\hat{G}$ is, in law, equal to the homogeneous $NIG(\delta,0,\gamma,0)$ basis. □

More generally (using the notation established in Section 2.1), subordination of a Gaussian Lévy basis $L = a + G$ by the independent L_0 is defined as the Lévy basis with compensator $\hat{\nu}(\mathrm{d}x;s)\,c_0(\mathrm{d}s)$ where

$$\hat{\nu}(\mathrm{d}x;s) = \int_0^\infty \varphi(x - a(s); m(s))\,\nu_0(\mathrm{d}u;s)\,\mathrm{d}x.$$

Example 4. Taking again L_0 to be the homogeneous $IG(\delta,\gamma)$ basis, the resulting subordinated random measure $\hat{G}$ is, in law, equal to the Lévy basis with characteristic quadruplet $(a, 0, \hat{\nu}(\mathrm{d}x,\cdot), c_0)$, the Lévy seeds $\hat{L}'(s)$ being $NIG(\alpha(s), a(s), \delta, 0)$ distributed where $\alpha(s) = \sqrt{\gamma^2 + a^2(s)}$. □

4 Extension and volatility modulation

Time changes of stochastic processes, other than subordination of Lévy processes, is a subject of considerable interest (cf. again [BNShi10]), and in line with this we now make some remarks extending the approach in the previous Section to modulation of Lévy bases by more general random measures.

So, let τ be a nonnegative random measure on $(\mathbb{R}^d, \mathcal{B})$ and given a Lévy basis L, independent of τ and with characteristic quadruplet $(a, m, \nu(\mathrm{d}x;\cdot), c)$, let $\hat{L}$ be the random, in general signed, measure that conditionally on τ is a Lévy basis with characteristic quadruplet $(a, m, \nu(\mathrm{d}x; \cdot), \tau)$. (Existence of $\hat{L}$ requires some mild regularity ssumptions.) Then the cumulant functional of $\hat{L}$ is determined by

$$\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} = \log \mathrm{E}\Big\{\exp\Big[-\frac{1}{2}\int_{\mathbb{R}^d} f^2(s)\, m(s)\, \tau(\mathrm{d}s) + \int_{\mathbb{R}^d} \mathrm{C}\{\zeta f(s) \ddagger a(s) + L'(s)\}\tau(\mathrm{d}s)\Big]\Big\} \quad (22)$$

cf. formula (12) and generalising (17). In particular, if $a = m = 0$ then

$$\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} = \mathrm{C}\{-i \ddagger \mathrm{C}\{\zeta f(\cdot) \ddagger L'(\cdot)\} \bullet \tau\}. \quad (23)$$

This may be used, in particular, in relation to ambit fields and processes. In complete generality, an ambit field $Y = \{Y_t(x) : (x, t) \in \mathbb{R}^d \times \mathbb{R}\}$, where t stands for time, was defined ([BNSch04], [BNSch07], [BNBV09]) as

$$Y_t(x) = \mu + \int_{A_t(x)} g(\xi, s; x, t)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, s; x, t)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s \quad (24)$$

where the *ambit sets* $A_t(x)$, and $D_t(x)$ are subsets of $\mathbb{R}^d \times (-\infty, 0]$, g and q are deterministic damping functions, $\sigma \geq 0$ is a stochastic field referred to as the *volatility*, and L is a Lévy basis. An ambit process is then the realisation of Y along a curve $(x(\theta), t(\theta))$ in $\mathbb{R}^d \times \mathbb{R}$, with $t(\theta)$ increasing in θ, from minus infinity to plus infinity.

The volatility field σ has a key role in various modelling contexts, particularly in turbulence (where it is referred to as *intermittency*), and in finance. See [BNSch07], [BNBV09], [BNBV10] and references given there.

Of particular interest are the cases where $Y_t(x)$ is stationary in t. Considering just the main term in (24), this occurs when Y is of the form

$$Y_t(x) = \int_{A+(x,1)} g(\xi, t-s; x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) \quad (25)$$

for some fixed ambit set A and provided the volatility field σ is stationary and L is homogeneous.

Now, while the multiplicative position of σ to the basis L in (24) goes naturally together with L when the homogeneous L is Gaussian or more generally stable, this is less so in general, and there are advantages in interpretation and calculation by, instead of having σL as the integrator, to use $\hat{L}$ obtained by subordinating L by τ in the sense defined here and with $\tau = \sigma^2$. In particular, the dependence structure in Y is then relatively simple to describe. Note that in the Gaussian and stable cases the result of the multiplicative approach σL can equally be achieved by subordination (in the Gaussian case by choosing $\tau(\mathrm{d}s) = \sigma^2(s)\,\mathrm{d}s$); but the subordination technique gives wider possibilties.

5 Concluding remarks

This paper has introduced a generalisation of the concept of subordination of Lévy processes. In summary, this consists of substituting the control measure c of a Lévy basis L with characteristic quadruplet $(a, m, \nu(\mathrm{d}x; s), c)$ by a random measure τ. In the case where τ is itself a Lévy basis L_0 the construction implies that the resulting random measure $\hat{L}$ is itself a Lévy basis whose Lévy seeds $\hat{L}'$, as defined here, are determined pointwise by the classical subordination of the Lévy process generated from the Lévy seed $L'(s)$ of L at s by the subordinator generated from the Lévy seed $L_0'(s)$ of L_0.

Part of the interest in this extended concept of subordination lies in the possibility for alternative modelling of the influence of volatility or intermittency, particularly in the context of ambit fields and processes.

The discussion given here has been on the level of distributional results for Lévy bases and volatility modulation of such. Development and strengthening of the results in stochastic process settings will be studied in a follow-up paper, joint with by Jan Pedersen.

References

[BN98] Barndorff-Nielsen, O.E. (1998): Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41–68.

[BNBV09] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A. (2009): Ambit processes and stochastic partial differential equations. In DiNunno, G. and Øksendal, B. (Eds.): *Advanced Mathematical Methods for Finance.* Berlin: Springer. (To appear.)

[BNBV10] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A. (2010b): Modelling electricity forward markets by ambit fields. (Submitted.)

[BNSch04] Barndorff-Nielsen, O.E. and Schmiegel, J. (2004): Lévy-based tempo-spatial modelling; with applications to turbulence. *Uspekhi Mat. NAUK* **59**, 65–91.

[BNSch07] Barndorff-Nielsen, O.E. and Schmiegel, J. (2007): Ambit processes; with applications to turbulence and cancer growth. In F.E. Benth, Nunno, G.D., Linstrøm, T., Øksendal, B. and Zhang, T. (Eds.): *Stochastic Analysis and Applications: The Abel Symposium 2005.* Heidelberg: Springer. Pp. 93–124.

[BNS11] Barndorff-Nielsen, O. E. and Shephard, N. (2011): *Financial Volatility in Continous Time.* Cambridge University Press. (To appear.)

[BNShi10] Barndorff-Nielsen, O.E. and Shiryaeev, A.N. (2010): *Change of Time and Change of Measure.* Singapore: World Scientific. (Forthcoming.)

[Bertoin99] Bertoin, J. (1999): Subordinators: Examples and Applications. In J. Bertoin, F. Martinelli and Y. Peres (Eds.): *Lectures on Probability Theory and Statistics* **LNM 1717**. Springer: Heidelberg.Pp. 4–91.

[Huff69] Huff, B.W. (1969): The strict subordination of differential processes. *Sankhyā A*, **31**, 403–412.

[Ped03] Pedersen, J. (2003): The Lévy-Ito decomposition of an independently scattered random measure. MaPhySto Research Report 2003-2

[RajRos89] Rajput, B.S. and Rosinski, J. (1989): Spectral representations of infinitely divisible processes. *Probab. Th. Rel. Fields* **82**, 451–487.