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Volatility Determination in an Ambit Process Setting

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Abstract

The probability limit behaviour of normalised quadratic variation is studied for a simple tempo-spatial ambit process, with particular regard to the question of volatility memorylessness.

1 Introduction

Dynamic stochastic phenomena frequently involve a significant element of randomness beyond the most basic types of stochastic innovations. Additional variations of this kind are often referred to as volatility or intermittency, and they are of key importance particularly in finance and turbulence.

In many cases the volatility is expressed in stochastic modelling by a multiplicative term specified as a stationary positive process σ. Thus, for example, one considers stochastic processes

$$Y_t = \int_{A_t} g\left(t-s\right)\sigma_s \mathrm{d}W_s \tag{1}$$

where A is a t-dependent interval of $\mathbb{R}$, g is a deterministic function and W is Brownian motion. The question of what can be learned about σ from observations of the process is then often of central interest and the main tool to study that is (realised) multipower variations, in particular (realised) quadratic variation, see [8], [9], [5], [6], [7], [3], [11], [1], [2], and references given there.

There are two main types of (1). In case g is constant and $A_t = [0,t]$ we are in the framework of Brownian semimartingales while if g is nontrivial and A_t is of the form $[a,t]$ for some $a \in [-\infty, 0]$ we have a Brownian semistationary process (note that if $a = -\infty$ then Y is a strictly stationary process on $\mathbb{R}$). These two types are substantially different. In particular, Brownian semistationary processes are generally not semimartingales, and this, in particular, implies major differences between the theory of multipower variations for the two types, see [2]. To exemplify,

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in the Brownian semimartingale case the realised quadratic variation over $[0,t]$ will converge in probability to $\sigma^{2+}_{[0,t]}$ where for $a<b$

$$\sigma^{2+}_{(a,b]} = \int_a^b \sigma_s^2 \mathrm{d}s;$$

on the other hand, for Brownian semistationary processes, where a normalisation of the realised quadratic variation is generally required, it may, for instance, happen that the convergence is to $\lambda\sigma^{2+}_{[0,t]} + (1-\lambda)\sigma^{2+}_{(-1,t-1]}$ for some constant $\lambda \in (0,1)$ (cf. [7], [2]). When the limit is in fact $\sigma^{2+}_{[0,t]}$ we speak of a volatility memoryless process.

The Brownian semistationary processes constitute the null-spatial family of Brownian based ambit processes. The general form (except for a drift term that will not concern us here) is based on an ambit field Y, i.e. a stochastic field in space-time

$$Y(x,t) = \int_{A_t(x)} g(x,\xi,t-s)\,\sigma_s(\xi)\,W(\mathrm{d}\xi\mathrm{d}s) \tag{2}$$

where $A_t(x)$ is some subset of $\mathcal{X} \times (-\infty, t]$ for some spatial region $\mathcal{X}$ and where g is deterministic, σ is a positive stochastic field and W is two-dimensional white noise. Then an ambit process X is a process of the form $X = Y(\tau)$ where τ denotes a smooth curve in $\mathcal{X} \times \mathbb{R}$.

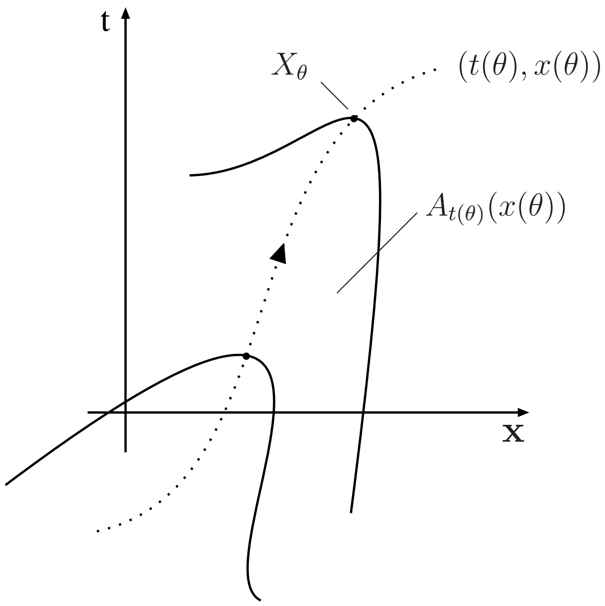

Figure 1: Ambit framework

The purpose of the present note is to explore the question of volatility memorylessness for a simple tempo-spatial setting and to draw some conclusions with respect to further related research questions.

Section 2 presents our main conclusions, while the proofs are given in Section 4. Section 3 summarises and provides a brief outlook.

2 Results and Examples

We restrict the discussion to the case $\mathcal{X} = \mathbb{R}$ and ambit fields of the form

$$Y(x,t) = \int_{A_t(x)} g(x-\xi, t-s)\,\sigma_s(\xi)\,W(\mathrm{d}\xi\mathrm{d}s) \tag{3}$$

where $A_t(x) = A + (x,t)$, $(\sigma_s(\xi))_{(\xi,s)\in\mathbf{R}^2}$ is a real valued continuous random field independent of W and g a Lebesgue square integrable function on $\mathbf{R}^2$. Here we are mainly interessted in the case where

$$A = \{(\xi,s)\in\mathbf{R}^2 \mid -M \le s \le 0,\ c_1(s)\le \xi \le c_2(s)\}$$

for some $M\in\mathbf{R}_+$ and smooth functions $c_1 : [-M,0]\to\mathbf{R}_-$ and $c_2 : [-M,0]\to\mathbf{R}_+$ such that $c_1(0) = c_2(0)$ and c_1 is increasing and c_2 is decreasing. Note that A is a closed set.

For a given smooth curve $\tau = (\tau_1,\tau_2) : \mathbf{R}\to\mathbf{R}^2$ consider the process

$$X_\theta = Y(\tau(\theta)) \quad \theta \ge 0.$$

The realised quadratic variation of X and its normalised version are, for $\delta > 0$ and $t > 0$, given by

$$[X_\delta]_t = \sum_{k=1}^{[t/\delta]} (X_{k\delta} - X_{(k-1)\delta})^2 \quad \text{and} \quad \overline{[X_\delta]}_t = \frac{\delta}{c(\delta)}\,[X_\delta]_t$$

where $c(\delta)$ is a positive constant depending only on δ, whose specific form will be defined below. We are interested in the asymptotic behavior of $\overline{[X_\delta]}_t$ for $\delta\to 0$. Up to now we can only satisfactorily handle the case of τ being a straight line or more generally a piecewise straight line. Therefore for ease of notation we will from now on assume that $\theta\mapsto\tau(\theta)$ is a straight line and thus $\triangle\tau(\delta) = \delta\triangle\tau$ where with obvious notation

$$\triangle\tau(\delta) = (\triangle\tau_1(\delta), \triangle\tau_2(\delta)) = (\tau_1(t+\delta)-\tau_1(t), \tau_2(t+\delta)-\tau_2(t)) \quad \text{for } t,\delta\ge 0.$$

We now introduce a probability measure π_δ which is determined by the kernel function g and whose behaviour as $\delta\to 0$ is of key importance for the probabilistic limit properties of $\overline{[X_\delta]}$.

Put

$$\psi_\delta(u,v) = (\,g_A(\triangle\tau_1(\delta)+u, \triangle\tau_2(\delta)+v) - g_A(u,v)\,)^2$$

$$= \begin{cases} (\,g(\triangle\tau_1(\delta)+u, \triangle\tau_2(\delta)+v) - g(u,v)\,)^2 \\ \qquad\qquad\qquad \text{for } (u,v)\in(-A)\cap(-A-\triangle\tau(\delta)) \\ g^2(u,v) \qquad\quad \text{for } (u,v)\in(-A)\setminus(-A-\triangle\tau(\delta)) \\ g^2(\triangle\tau_1(\delta)+u, \triangle\tau_2(\delta)+v) \\ \qquad\qquad\qquad \text{for } (u,v)\in(-A-\triangle\tau(\delta))\setminus(-A). \end{cases}$$

Observe that $\psi_\delta(u,v) = 0$ if $(u,v) \notin (-A) \cup (-A - \triangle\tau(\delta))$. Now define

$$\pi_\delta\,(\mathrm{d}u\mathrm{d}v) = \psi_\delta(u,v)/c(\delta)\,\lambda_2(du\,dv) \quad \delta > 0$$

where λ_2 denotes Lebesgue measure on $\mathbf{R}^2$ and

$$c(\delta) = \int_{\mathbf{R}^2} \psi_\delta(u,v)\,\lambda_2(du\,dv).$$

Then, by construction, π_δ is a probability measure and clearly all weak limit points of π_δ for $\delta \to 0$ will be probability measures concentrated on $-A$. Simple calculations together with the continuity assumption on σ imply that in case the limit $\pi_\delta \xrightarrow{w}_{\delta\to 0} \pi_0$ exists then

$$E[\,\overline{[Y_\delta]}_t \,|\, \sigma] \to_{\delta\to 0} \int_{\mathbf{R}^2}\int_0^t \sigma^2_{\tau_2(s)-v}(\tau_1(s) - u)\,ds\,\pi_0(du\,dv).$$

We are particularly interested in conditions on A and g ensuring that the limit π_0 exists and is concentrated on $\partial(-A) = -\partial A$. Because in this case we have furthermore that

$$\lim_{\delta\to 0} \mathrm{Var}(\overline{[X_\delta]}_t \,|\, \sigma) = 0.$$

(This result is established as Lemma 2 in Section 4). Consequently, under these conditions, we will have the central result that, as $\delta \to 0$,

$$\overline{[Y_\delta]}_t \xrightarrow{p} \int_{\mathbf{R}^2}\int_0^t \sigma^2_{\tau_2(s)+v}(\tau_1(s) + u)\,ds\,\pi(du\,dv). \tag{4}$$

Here π denotes the image measure of π_0 under the transformation $(u,v) \mapsto (-u,-v)$. Observe that π is concentrated on ∂A.

We can now state the key result of this paper. For proofs and further details see Section 4.

Theorem 1. *Suppose that τ is a straight line and A is a nonempty bounded closed convex set such that $A = \overline{A^\circ}$. Then there exists a probability measure π concentrated on the boundary ∂A of A such that formula (4) holds provided the following condition is satisfied for some $-\frac{1}{2} < \alpha < \frac{1}{2}$*

($\mathbf{i}_\alpha$) *$g = \varphi h_\alpha$ where φ is Lipschitz continuous and not identically vanishing on $-\partial A$, and*

$$h_\alpha(-x) = \begin{cases} d(x,\partial A)^\alpha & x \in A \\ 0 & x \notin A \end{cases} \tag{5}$$

where $d(x,\partial A)$ denotes the Euclidean distance between x and ∂A.

Remark. Note especially that π may be situated on ∂A even if the function g tends rather rapidly to 0 as its argument tends to the boundary. □

From this Theorem it is evident that by suitable choice of g one can arrange that the measure π is concentrated on a specified part of the boundary. In particular, if the set A has a unique top point then π may be the delta measure at that point, in which case there is volatility memorylessness.

3 Conclusion and outlook

We have discussed the probabilistic limit behaviour of (normalised) realised quadratic variation for a class of ambit processes, where the underlying ambit field is homogeneous provided the volatility/intermittency field σ is stationary, and where the mother ambit set A is a bounded, closed and convex set. In this setting a considerable variety of limits are possible, depending on the nature of the damping function g. All the limits are integrals of the squared volatility/intermittency field over the set A and with respect to a probability measure π on A. Under specified weak conditions the integrals are concentrated on the boundary of A. Volatility memorylessness is then ensured if A has a single top point.

There is a range of further questions of theoretical and applied interest in this context:

(o) What happens if A is not bounded, stretching to minus infinity in time, or if A is not convex. (Figure 2 shows a type of ambit sets that are of interest in turbulence studies and whose shape is motivated in Taylor's frozen field hypothesis (cf. [4]).)

(i) What is the situation in case the curve τ is not linear. (The linearity assumption is crucial in deriving formula (6).)

(ii) How is the probabilistic limit behaviour of multipower variations generally.

(iii) What type of central limit theorems can be established for the multipower variations. (Undoubtedly, as was the case for Brownian semimartingales see [2], Malliavin calculus will be a key tool.)

(iv) How may such central limit theorems be used to draw inference not only on σ but also on g (cf. [2]).

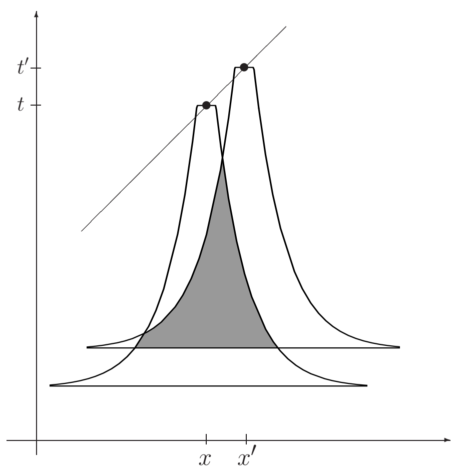

Figure 2: Ambit regions

4 Proofs

Maintaining the notation of Section 2, we write g_A for $g \cdot \mathbf{1}_{-A}$, for any Lebesgue square integrable function g on $\mathbf{R}^2$. Since A is bounded it is enough to assume that g is locally square integrable. Inserting this gives

$$[X_\delta]_t = \sum_{k=1}^{[t/\delta]} \left(\int_{\mathbf{R}^2} [\, g_A(\tau(k\delta) - (\xi, s)) - g_A(\tau((k-1)\delta) - (\xi, s)) \,]\, \sigma_s(\xi)\, W(d\xi\, ds) \right)^2$$

implying, by means of the independence between σ and W, that for all $\delta, t > 0$

$$E[\, [X_\delta]_t \,|\, \sigma] = \sum_{k=1}^{[t/\delta]} \int_{\mathbf{R}^2} [\, g_A(\tau(k\delta) - (\xi, s)) - g_A(\tau((k-1)\delta) - (\xi, s)) \,]^2\, \sigma_s^2(\xi)\, \lambda_2(d\xi\, ds).$$

Writing $\triangle\tau(k\delta)$ for $\tau(k\delta) - \tau((k-1)\delta)$ and using the linear substitution

$$(u, v) = \tau((k-1)\delta) - (\xi, s) = (\tau_1((k-1)\delta) - \xi, \tau_2((k-1)\delta) - s)$$

$E[\, [X_\delta]_t \,|\, \sigma]$ may be written as

$$\sum_{k=1}^{[t/\delta]} \int_{\mathbf{R}^2} [\, g_A(\triangle\tau(k\delta) + (u, v)) - g_A(u, v) \,]^2\, \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\, \lambda_2(du\, dv).$$

Thus if $\triangle\tau(k\delta) = \triangle\tau(\delta)$, that is independent of k, in particular if $\theta \mapsto \tau(\theta)$ is a straight line, we have

$$E[\, [Y_\delta]_t \,|\, \sigma] = \int_{\mathbf{R}^2} \psi_\delta(u, v) \sum_{k=1}^{[t/\delta]} \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\, \lambda_2(du\, dv) \qquad (6)$$

where for $\delta > 0$

$$\begin{aligned} \psi_\delta(u, v) &= (\, g_A(\triangle\tau_1(\delta) + u, \triangle\tau_2(\delta) + v) - g_A(u, v) \,)^2 \\ &= \begin{cases} (\, g(\triangle\tau_1(\delta) + u, \triangle\tau_2(\delta) + v) - g(u, v) \,)^2 \\ \qquad\qquad \text{for } (u, v) \in (-A) \cap (-A - \triangle\tau(\delta)) \\[1em] g^2(u, v) \qquad \text{for } (u, v) \in (-A) \setminus (-A - \triangle\tau(\delta)) \\[1em] g^2(\triangle\tau_1(\delta) + u, \triangle\tau_2(\delta) + v) \\ \qquad\qquad \text{for } (u, v) \in (-A - \triangle\tau(\delta)) \setminus (-A). \end{cases} \end{aligned}$$

Observe that $\psi_\delta(u, v) = 0$ if $(u, v) \notin (-A) \cup (-A - \triangle\tau(\delta))$. Formula (6) suggests that it is natural to put

$$c(\delta) = \int_{\mathbf{R}^2} \psi_\delta(u, v)\, \lambda_2(du\, dv)$$

since then

$$E[\,\overline{[X_\delta]}_t \,|\, \sigma] = \int_{\mathbf{R}^2} \delta \sum_{k=1}^{[t/\delta]} \sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\, \pi_\delta(du\, dv)$$

where π_δ denotes the probability measure

$$\pi_\delta\,(\mathrm{d}u\mathrm{d}v) = \psi_\delta(u,v)/c(\delta)\,\lambda_2(du\,dv) \quad \delta > 0.$$

Assume from now on that $\theta \mapsto \tau(\theta)$ is a straight line and thus $\triangle\tau(\delta) = \delta\triangle\tau$. As already observed

$$\forall\, \epsilon > 0 \;\; \exists\; \delta_\epsilon > 0 : \; \pi_\delta(\mathbf{R}^2 \setminus A_\epsilon) = 0 \quad \text{for all } 0 < \delta < \delta_\epsilon$$

where, using the notation $d((\xi,s),B) := \inf_{(u,v)\in B} |(\xi,s) - (u,v)|$ for any $B \subseteq \mathbf{R}^2$,

$$A_\epsilon = \{(\xi,s) \in \mathbf{R}^2 \,|\, d((\xi,s),-A) \leq \epsilon\}.$$

Thus all weak limit points of π_δ for $\delta \to 0$ will be probability measures concentrated on $-A$. Using the continuity assumption on σ we see that in case the limit $\pi_\delta \xrightarrow{w}_{\delta\to 0} \pi_0$ exists then

$$E[\,\overline{[Y_\delta]}_t \,|\, \sigma] \to_{\delta\to 0} \int_{\mathbf{R}^2} \int_0^t \sigma^2_{\tau_2(s)-v}(\tau_1(s) - u)\, ds\, \pi_0(du\, dv).$$

We are interested in conditions on A and g ensuring that the limit π_0 exists and is concentrated on $\partial(-A) = -\partial A$, implying of course that π is concentrated on ∂A. Before discussing specific conditions for this to happen we establish the following Lemma.

Lemma 2. *Under the assumption that π_0 exists and is concentrated on $-\partial A$ we have that*

$$\lim_{\delta\to 0} \mathrm{Var}(\overline{[X_\delta]}_t \,|\, \sigma) = 0. \tag{7}$$

Proof. For given $\delta, t > 0$, $\mathrm{Var}(\overline{[X_\delta]}_t \,|\, \sigma)$ equals $\delta^2/c(\delta)^2$ times

$$\sum_{k=1}^{[t/\delta]} \mathrm{Var}(\,(X_{k\delta} - X_{(k-1)\delta})^2 \,|\, \sigma) + 2 \sum_{1\leq k<l\leq [t/\delta]} \mathrm{Cov}((X_{k\delta} - X_{(k-1)\delta})^2, (X_{l\delta} - X_{(l-1)\delta})^2) \,|\, \sigma).$$

Applying that for any centered jointly Gaussian vector (U,V)

$$\mathrm{Cov}(U^2,V^2) = 2\,\mathrm{Cov}(U,V)^2 \quad \text{and} \quad \mathrm{Var}(U^2) = 2\,\mathrm{Var}(U)^2$$

we may write

$$\mathrm{Var}(\overline{[X_\delta]}_t \,|\, \sigma) = I_\delta + II_\delta$$

where

$$I_\delta = \frac{2\delta^2}{c(\delta)^2} \sum_{k=1}^{[t/\delta]} E[\,(X_{k\delta} - X_{(k-1)\delta})^2 \,|\, \sigma\,]^2$$

and

$$II_\delta = \frac{4\delta^2}{c(\delta)^2} \sum_{k=1}^{[t/\delta]} E[\,(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta})\,|\,\sigma\,]^2.$$

Simple manipulations show that for all $\delta > 0$

$$I_\delta \le \frac{2\delta}{c(\delta)} \max_{1\le k\le[t/\delta]} E[\,(X_{k\delta} - X_{(k-1)\delta})^2\,|\,\sigma\,]\ E[\,\overline{[X_\delta]}_t\,|\,\sigma]$$

and for all $1 \le k \le [t/\delta]$

$$\begin{aligned}
&E[\,(X_{k\delta} - X_{(k-1)\delta})^2\,|\,\sigma\,] \\
&\quad = \int_{\mathbf{R}^2} (\,g_A(\tau(k\delta) - (u,v)) - g_A(\tau((k-1)\delta) - (u,v))\,)^2 \sigma_v^2(u)\,\lambda_2(du\,dv) \\
&\quad \le \max_{(u,v)\in A_{|\triangle\tau|\delta}} \sigma_v^2(u) \int_{\mathbf{R}^2} (g_A(\triangle\tau(\delta) + (u,v)) - g_A(u,v))^2\,\lambda_2(du\,dv) \\
&\quad = \max_{(u,v)\in A_{|\triangle\tau|\delta}} \sigma_v^2(u)\ c(\delta).
\end{aligned}$$

Thus this shows that $\lim_{\delta\to 0} I_\delta = 0$. So it remains to verify that $\lim_{\delta\to 0} II_\delta = 0$.

For all $1 \le k < l \le [t/\delta]$

$$\begin{aligned}
&E[\,(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta})\,|\,\sigma\,] \\
&\quad = \int_{\mathbf{R}^2} (\,g_A(\tau(k\delta) - (u,v)) - g_A(\tau((k-1)\delta) - (u,v))\,) \\
&\qquad\qquad \cdot \big(g_A(\tau(l\delta) - (u,v)) - g_A(\tau((l-1)\delta) - (u,v))\,\big)\,\sigma_v^2(u)\,\lambda_2(du\,dv) \\
&\quad = \int_{\mathbf{R}^2} \big(\,g_A(\triangle\tau(\delta) + (u,v)) - g_A(u,v)\,\big) \\
&\qquad\qquad \cdot \big(g_A((l-k+1)\triangle\tau(\delta) + (u,v)) - g_A((l-k)\triangle\tau(\delta) + (u,v))\,\big) \\
&\qquad\qquad \cdot \mathbf{1}_{A_{|\triangle\tau|\delta}}(u,v)\,\sigma^2_{\tau_2((k-1)\delta)-v}(\tau_1((k-1)\delta) - u)\,\lambda_2(du\,dv).
\end{aligned}$$

Using the continuity of the σ-process and Cauchy-Schwarz's inequality this implies the existence of a constant M such that

$$\begin{aligned}
&E[\,(X_{k\delta} - X_{(k-1)\delta})(X_{l\delta} - X_{(l-1)\delta})\,|\,\sigma\,]^2 \\
&\quad \le M \int_{\mathbf{R}^2} (g_A(\triangle\tau(\delta) + (u,v)) - g_A((u,v))^2\,\lambda_2(du\,dv) \\
&\qquad \cdot \int_{\mathbf{R}^2} \big(g_A((l-k+1)\triangle\tau(\delta) + (u,v)) - g_A((l-k)\triangle\tau(\delta) + (u,v))\big)^2 \\
&\qquad\qquad\qquad \cdot \mathbf{1}_{A_{|\triangle\tau|\delta}}(u,v)\,\lambda_2(du\,dv) \\
&\quad = M\,c(\delta) \int_{\mathbf{R}^2} (g_A(\triangle\tau(\delta) + (u,v)) - g_A(u,v))^2 \\
&\qquad\qquad\qquad \cdot \mathbf{1}_{A_{|\triangle\tau|\delta}}((u,v) - (l-k)\triangle\tau(\delta))\,\lambda_2(du\,dv).
\end{aligned}$$

Thus $\lim_{\delta\to 0} II_\delta = 0$ if $\lim_{\delta\to 0} \widetilde{II}_\delta = 0$, where $\widetilde{II}_\delta$ denotes the expression

$$\begin{aligned}
&\sum_{1\le k<l\le[t/\delta]} \frac{\delta^2}{c(\delta)} \int_{\mathbf{R}^2} (g_A(\triangle\tau(\delta) + (u,v)) - g_A(u,v))^2 \\
&\qquad\qquad\qquad \cdot \mathbf{1}_{A_{|\triangle\tau|\delta}}((u,v) - (l-k)\triangle\tau(\delta))\,\lambda_2(du\,dv).
\end{aligned}$$

Given $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that for $0 < \delta < \delta_\epsilon$

$$\begin{aligned}\widetilde{II}_\delta &\le \sum_{1\le k<l\le[t/\delta]} \frac{\delta^2}{c(\delta)} \int_{\mathbf{R}^2} (g_A(\triangle\tau(\delta)+(u,v)) - g_A(u,v))^2 \\ &\qquad\qquad\qquad\qquad \cdot \mathbf{1}_{A_\epsilon}((u,v)-(l-k)\triangle\tau(\delta))\,\lambda_2(du\,dv) \\ &= \sum_{1\le k<l\le[t/\delta]} \delta^2 \int_{\mathbf{R}^2} \mathbf{1}_{A_\epsilon}((u,v)-(l-k)\delta\triangle\tau)\,\pi_\delta(du\,dv) \\ &\le \sum_{1\le k<l\le[t/\delta]} \delta^2 \int_{\mathbf{R}^2} (f_{\epsilon,1}+f_{\epsilon,2})((u,v)-(l-k)\delta\triangle\tau)\,\pi_\delta(du\,dv)\end{aligned}$$

where $f_{\epsilon,1}, f_{\epsilon,2} \in C_b(\mathbf{R}^2)_+$ are chosen such that $\mathbf{1}_{A_\epsilon^-} \le f_{\epsilon,1}+f_{\epsilon,2}$ and

$$\mathrm{supp}(f_{\epsilon,1}) \subseteq \{(u,v)\in -A \,|\, d((u,v),-\partial A) > \epsilon/2\}$$

and

$$\mathrm{supp}(f_{\epsilon,2}) \subseteq \{(u,v)\in \mathbf{R}^2 \,|\, d((u,v),-\partial A) < 2\epsilon\}.$$

By weak convergence

$$\limsup_{\delta\downarrow 0} \widetilde{II}_\delta \le \int_{\mathbf{R}^2}\int_0^t\int_s^t (f_{\epsilon,1}+f_{\epsilon,2})((u,v)-(r-s)\triangle\tau)\,dr\,ds\;\pi_0(du\,dv)$$

and so, since π_0 is concentrated on $-\partial A$, we find that

$$\begin{aligned}\limsup_{\delta\downarrow 0} \widetilde{II}_\delta &\le \int_{\mathbf{R}^2}\int_0^t\int_s^t f_{\epsilon,2}((u,v)-(r-s)\triangle\tau)\,dr\,ds\;\pi_0(du\,dv) \\ &\le \sup_{(u,v)\in-\partial A}\int_0^t\int_s^t f_{\epsilon,2}((u,v)-(r-s)\triangle\tau)\,dr\,ds \\ &\le \sup_{(u,v)\in-\partial A} \lambda_2(\{r,s)\,|\,0\le s\le r\le t,\,(u,v)-(r-s)\triangle\tau \in \mathrm{supp}(f_{\epsilon,2})\}).\end{aligned}$$

But for all $(u,v)\in -\partial A$ and all $0\le s\le t$

$$\lambda_1(\{r\,|\,0\le s\le r\le t,\,(u,v)-(r-s)\triangle\tau\in\mathrm{supp}(f_{\epsilon,2})\}) \le c_\tau\epsilon$$

for some constant c_τ depending only on τ and A. Thus

$$\limsup_{\delta\downarrow 0} \widetilde{II}_\delta \le c_\tau t\epsilon$$

and since ϵ was arbitrary this proves that $\widetilde{II}_\delta \to 0$ for $\delta\to 0$. That is (7) holds. □

Finally we turn to the proof of Theorem 1. We will consider only the case $\alpha = 0$. The other cases can be treated in a similar yet slightly more complicated way. The statement is a consequence of the two lemmas below.

Let in the following τ be a given vector in $\mathbf{R}^2$ and C a bounded closed convex subset of $\mathbf{R}^2$ satisfying

$$\underline{0}\in C \quad\text{and}\quad C=\overline{C^\circ}.$$

In particular $\lambda_{21}(\partial C) < \infty$ where λ_{21} is the 1-dimensional Hausdorff-measure in $\mathbf{R}^2$. Put

$$T(x) = \inf\{t>0\,|\,x\in tC\} \quad x\in\mathbf{R}^2.$$

That is T is the gauge function of C and so T is a convex function satisfying

i) $T(\lambda x) = \lambda T(x) \quad \lambda \geq 0,\, x \in \mathbf{R}^2$.

ii) $\exists\, r_1, r_2 \in (0, \infty) : \; r_1|x| \leq T(x) \leq r_2|x| \quad x \in \mathbf{R}^2$.

iii) $C = \{x \in \mathbf{R}^2 \,|\, T(x) \leq 1\}$ and $\partial C = \{x \in \mathbf{R}^2 \,|\, T(x) = 1\}$.

Thus $T : \mathbf{R}^2 \to \mathbf{R}$ is a non-negative a.e. smoothly regular 1-homogenous continuous function and so, using formula (8.25) in [10], we have

$$\int_{\mathbf{R}^2} \varphi \, d\lambda_2 = \int_0^\infty t \Big\{ \int_{\partial C} \frac{\varphi(tx)}{|T'(x)|} \, \lambda_{21}(dx) \Big\} \, dt = \int_{\partial C} \frac{1}{|T'(x)|} \Big\{ \int_0^\infty t\varphi(tx) \, dt \Big\} \, \lambda_{21}(dx) \tag{8}$$

for every non-negative Borel function $\varphi : \mathbf{R}^2 \to \mathbf{R}$. The use of Tonelli's Theorem is legitimate since $\lambda_{21}(\partial C) < \infty$. The properties of T ensure that $T'(x)$ exists and is non-zero for λ_{21}-almost all $x \in \partial C$. In the sequel we shall for $x \in \partial C$ use the notation

$$n(x) = \begin{cases} T'(x)/|T'(x)| & T'(x) \text{ exists and is non-zero} \\ \underline{0} & \text{otherwise} \end{cases}.$$

Set, for $\delta > 0$,

$$\nu_\delta = \delta^{-1} f_\delta \, d\lambda_2 \quad \text{for} \quad f_\delta(x) = (\mathbf{1}_C(x + \delta\tau) - \mathbf{1}_C(x))^2 \quad x \in \mathbf{R}^2.$$

Observe that the ν_δ's are all finite measures and that the ν_δ for $\delta \leq 1$ are all concentrated on a fixed compact set.

Lemma 3.

$$\nu_\delta \xrightarrow{w}_{\delta\downarrow 0} |\tau \cdot n| \, \mathbf{1}_{\partial C} \, d\lambda_{21}.$$

Proof. By the above observation it is enough to prove that

$$\lim_{\delta \downarrow 0} \delta^{-1} \int_{\mathbf{R}^2} h f_\delta \, d\lambda_2 = \int_{\partial C} h(x) \, |\tau \cdot n(x)| \, \lambda_{21}(dx)$$

for all Lipschitz continuous $h \in C_c(\mathbf{R}^2)_+$. Given such an h, we have according to (8),

$$\int_{\mathbf{R}^2} h \, d\nu_\delta = \delta^{-1} \int_{\mathbf{R}^2} h f_\delta \, d\lambda_2 = \int_{\partial C} \frac{1}{|T'(x)|} \Big\{ \frac{1}{\delta} \int_0^\infty t h(tx) f_\delta(tx) \, dt \Big\} \, \lambda_{21}(dx). \tag{9}$$

Fix $x \in \partial C$ with $n(x) \neq 0$ and consider the function $t \mapsto f_\delta(tx)$, that is the indicator function for the set

$$\{t \geq 0 \,|\, tx \in (C - \delta\tau) \triangle C\}.$$

We may and will assume that $T(\delta\tau) < 1$ as this is true for δ sufficiently small. Since $tx \in C$ if and only if $t \leq 1$ we have

$$(1, \infty) \cap \{t \geq 0 \,|\, f_\delta(tx) = 1\} = (1, \infty) \cap \{t \geq 0 \,|\, T(tx + \delta\tau) \leq 1\}$$

and similarly

$$(0, 1) \cap \{t \geq 0 \,|\, f_\delta(tx) = 1\} = (0, 1) \cap \{t \geq 0 \,|\, T(tx + \delta\tau) > 1\}.$$

Since $t \mapsto T(tx + \delta\tau)$ is convex, $\{t \geq 0 \,|\, T(tx + \delta\tau) \leq 1\}$ is an interval including 0. Thus

$$(1, \infty) \cap \{t \geq 0 \,|\, f_\delta(tx) = 1\} = (1, b_\delta(x)] \quad \text{for some} \quad b_\delta(x) \geq 1$$

and

$$(0, 1) \cap \{t \geq 0 \,|\, f_\delta(tx) = 1\} = (a_\delta(x), 1) \quad \text{for some} \quad 0 < a_\delta(x) \leq 1.$$

Suppose that $\tau \cdot T'(x) > 0$. Since

$$T(x + \delta\tau) = T(x) + \delta\underline{\tau} \cdot T'(x) + o(\delta^2) = 1 + \delta\tau \cdot T'(x) + o(\delta^2)$$

we have that $T(x + \delta\tau) > 1$ and so $T(tx + \delta\tau) > 1$ for δ small and t sufficiently close to 1. Thus $b_\delta(x) = 1$. Furthermore, since

$$\begin{aligned} T(tx + \delta\tau) &= T(tx) + \delta\tau \cdot T'(tx) + o(\delta^2) \\ &= tT(x) + t\delta\tau \cdot T'(x) + o(\delta^2) \\ &= t(1 + \delta\tau \cdot T'(x)) + o(\delta^2) \end{aligned}$$

we have

$$a_\delta(x) = \frac{1}{1 + \delta\tau \cdot T'(x)} + o(\delta^2) = 1 - \delta\,|\tau \cdot T'(x)| + +o(\delta^2).$$

Similarly, if $\tau \cdot T'(x) < 0$ we see that

$$a_\delta(x) = 1 \quad \text{and} \quad b_\delta(x) = 1 + \delta\,|\tau \cdot T'(x)| + +o(\delta^2);$$

and if $\tau \cdot T'(x) = 0$ we obtain

$$a_\delta(x) = 1 - o(\delta^2) \quad \text{and} \quad b_\delta(x) = 1 + o(\delta^2).$$

Inserting this in (9) we obtain by the Lipschitz continuity of h that

$$\lim_{\delta \downarrow 0} \delta^{-1} \int_{\mathbf{R}^2} h f_\delta \, d\lambda_2 = \int_{\partial C} \frac{h(x)}{|T'(x)|}\, |\tau \cdot T'(x)|\, \lambda_{2\,1}(dx) = \int_{\partial C} h(x)\, |\tau \cdot n(x)|\, \lambda_{2\,1}(dx).$$

□

Let now $g : \mathbf{R}^2 \to \mathbf{R}$ be a given Lipschitz continuous function. Set for $\delta > 0$

$$g_\delta(x) = ((g\,\mathbf{1}_C)(x + \delta\tau) - (g\,\mathbf{1}_C)(x))^2 \quad x \in \mathbf{R}^2.$$

Simple arithmetic shows that

$$\begin{aligned} g_\delta(x) = (g^2 f_\delta)(x) + (g(x) - g(x + \delta\tau)) \\ \cdot \mathbf{1}_C(x + \delta\tau) \big((g(x) + g(x + \delta\tau))\, \mathbf{1}_C(x + \delta\tau) \big) - 2(g\,\mathbf{1}_C)(x)\,) \end{aligned}$$

for all $x \in \mathbf{R}^2$. The assumptions on g and the above Lemma 3 therefore imply that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\mathbf{R}^2} g_\delta \, d\lambda_2 = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{\mathbf{R}^2} g^2 f_\delta \, d\lambda_2 = \int_{\partial C} g^2(x)\, |\tau \cdot n(x)|\, \lambda_{2\,1}(dx).$$

From this we may deduce the following result which proves Theorem 1.

Lemma 4. *Let $g : \mathbf{R}^2 \to \mathbf{R}$ be Lipschitz continuous such that*

$$\int_{\partial C} g^2(x) \, |\tau \cdot n(x)| \, \lambda_{2\,1}(dx) > 0.$$

Then maintaining the above notation

$$\mu_\delta \xrightarrow{w}_{\delta \downarrow 0} g^2 \, |\tau \cdot n| \, \mathbf{1}_{\partial C} \, d\lambda_{2\,1}$$

where, for each $\delta > 0$, μ_δ is the absolutely continuous Borel probability measure on $\mathbf{R}^2$ with density proportional to g_δ.

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