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Multiparameter processes with stationary increments: Spectral representation and integration

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Abstract

In this article, a class of multiparameter processes with second-order stationary increments is studied. The content is as follows. (1) The spectral representation is derived; in particular, necessary and sufficient conditions for a measure to be a spectral measure is given. The relations to a commonly used class of processes, studied e.g. by Yaglom, is discussed. (2) Some classes of deterministic integrands, here referred to as predomains, are studied in detail. These predomains consist of functions or, more generally, distributions. Necessary and sufficient conditions for completeness of the predomains are given. (3) In a framework covering the classical Walsh-Dalang theory of a temporal-spatial process which is white in time and colored in space, a class of predictable integrands is considered. Necessary and sufficient conditions for completeness of the class are given, and this property is linked to a certain martingale representation property.

Keywords: Multiparameter processes; stationary increments; spectral representation; integration.
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1 Introduction

Let $d \geq 1$ be an integer which is fixed throughout. In this article we consider a class of real valued processes $X = \{X_u : u \in \mathbb{R}^d\}$ indexed by $\mathbb{R}^d$ with so-called *zero-mean second-order stationary increments*. We refer to Section 2 for the precise definition so for now it suffices to say that this class is large and contains e.g. the d-parameter fractional Brownian sheet. The main purpose is to study different kinds of integrals with respect to such processes, focusing in particular on completeness of various sets of integrands.

In Section 3 we discuss classes of deterministic integrands, referred to as *predomains*. Predomains are not necessarily sets of functions but the corresponding integral takes values in the set of square-integrable random variables. On predomains we use the metric induced by the L^2-distance between corresponding integrals. If completeness is present, a predomain is referred to as a *domain*. In the one-dimensional case $d = 1$ several predomains have been studied for processes with stationary increments. A key reference in the case of fractional Brownian motion is Pipiras and Taqqu [5] where various (pre)domains consisting of functions is analyzed. These authors show that many natural predomains studied in the literature are in fact not complete and hence not domains. To remedy this, Jolis [4] introduced a larger predomain consisting of distributions in the case of a continuous processes with stationary increments. In particular she showed that this will often lead to a domain. In Section 3 we follow [4] and study predomains containing functions as well as distributions. Generalizing results of [4, 5], necessary and sufficient conditions on the spectral measure for a predomain to be a domain are given. Moreover, we show that the integral of an integrand φ belonging to any of the predomains considered is given by

$$\int_{\mathbb{R}^d} \varphi(u)\, X(du) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z)\, Z(dz), \tag{1.1}$$

where $\mathcal{F}$ denotes the Fourier transform and Z the random spectral measure of X.

As is obvious from (1.1) the integral is closely linked to the spectral representation of X. Therefore we study the spectral representation of X in detail in Section 2. Moreover, a comparison to the class of processes studied e.g. by Yaglom [10] is given.

Finally, in Section 4 we add a temporal component and thus consider Gaussian processes $X = \{X_u : u = (t, x) \in \mathbb{R}^{1+d}\}$ where $t \in \mathbb{R}$ is time and $x \in \mathbb{R}^d$ a spatial component. We assume that X is *white in time and colored in space*. A martingale integral with respect to X is constructed akin to the classical papers by Walsh [9] and Dalang [1] although it should be noticed that in the present situation, unlike these papers, X does in general not induce a martingale measure. For example, when $d = 1$, X could be fractional in space with Hurst exponent H in $(0, 1)$ in which case X only induces a martingale measure when $H \geq 1/2$. We show that the integral of a predictable integrand $\varphi_t(x)$ with respect to X is

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)\, X(d(t, x)) = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi_t(z)\, dZ_t(x),$$

where $\mathcal{F}$ denotes the Fourier transform in the space variable, and for fixed t, $Z_t(\cdot)$ is the random spectral measure of $X((0, t] \times \cdot)$ in the space variable. Necessary and sufficient conditions for completeness for a class of integrands are given and in particular this property is linked to a martingale representation property with respect to X.

Definitions and notation. For any measure μ, $L^2_{\mathbb{C}}(\mu)$ denotes the set of complex-valued μ-square integrable functions and $L^2_{\mathbb{R}}(\mu)$ the subset hereof taking values in $\mathbb{R}$. Likewise, for any $A \subseteq L^2_{\mathbb{C}}(\mu)$, $\overline{\mathrm{sp}}_{\mathbb{C}} A$ is the closed complex linear span and $\overline{\mathrm{sp}}_{\mathbb{R}} A$ the corresponding closed real linear span of A. Observe that $\overline{\mathrm{sp}}_{\mathbb{R}} A$ coincides with the real-valued elements in $\overline{\mathrm{sp}}_{\mathbb{C}} A$ if all elements in A are real-valued. According to usual notation the space of tempered distributions, that is the dual of the Schwartz space $\mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$ consisting of complex-valued C^∞–functions on $\mathbb{R}^d$ of rapid decrease, is denoted $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$. The subspace of $\mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$ consisting of real-valued functions is denoted $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$, and likewise $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ is the set of elements Ψ in $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ such that $\Psi(\phi) \in \mathbb{R}$ for all $\varphi \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$. Similarly, $\mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ denotes the set of complex-valued

C^∞-functions on $\mathbb{R}^d$ of compact support. For the general theory of distributions and especially tempered distributions we refer to Schwartz [7].

The Fourier transform of a distribution or a measurable function is, whenever it is well-defined, denoted by $\mathcal{F}$. That is, if e.g. $f \in L^1_{\mathbb{C}}(\lambda_d)$, where λ_d denotes Lebesgue measure on $\mathbb{R}^d$, then

$$\mathcal{F}f(z) = \int_{\mathbb{R}^d} e^{i\langle z,\cdot\rangle} f(\cdot)\, d\lambda_d = \int_{\mathbb{R}^d} e^{i\langle z,u\rangle} f(u)\, du, \quad \text{for } z \in \mathbb{R}^d.$$

Here, $\langle\cdot,\cdot\rangle$ is the canonical inner product on $\mathbb{R}^d$ with corresponding norm $\|\cdot\|$. The notation differs from the one used e.g. in [7] where, for $f \in L^1_{\mathbb{C}}(\lambda_d)$, $\mathcal{F}f(-2\pi\cdot)$ is used as the Fourier transform of f. But apart from a constant $(2\pi)^d$ appearing in Parseval's identity and the explicit form of the inverse $\mathcal{F}^{-1}$, all results from the general theory of distributions remain valid with the definition given above.

All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is fixed throughout. Equality in distribution is denoted $\overset{\mathscr{D}}{=}$. Finally, $\mathcal{B}_b(\mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}^d$.

2 Spectral representation

In Definition 2.3 the class of processes with zero-mean second-order stationary increments is defined and the spectral representation is given in Theorem 2.5. This representation is stated in terms of the following class of random measures.

Definition 2.1. Let F be a symmetric Borel measure on $\mathbb{R}^d$ finite on compacts. A set function $Z\colon \mathcal{B}_b(\mathbb{R}^d) \to L^2_{\mathbb{C}}(\mathbb{P})$ is said to be a *zero-mean $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure F* if

(1) $Z(A \cup B) = Z(A) + Z(B)$ $\mathbb{P}$-a.s. whenever $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ are disjoint;

(2) $Z(A) = \overline{Z(-A)}$ $\mathbb{P}$-a.s. for $A \in \mathcal{B}_b(\mathbb{R}^d)$;

(3) $\mathbb{E}[Z(A)\overline{Z(B)}] = F(A \cap B)$ for $A, B \in \mathcal{B}_b(\mathbb{R}^d)$;

(4) $\mathbb{E}[Z(A)] = 0$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$.

Remark 2.2. Let Z be a random measure as above. From (1) and (3) it follows that $Z(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty Z(A_n)$ in $L^2_{\mathbb{C}}(\mathbb{P})$ for any disjoint sequence $(A_n)_{n\geq 1}$ in $\mathcal{B}_b(\mathbb{R}^d)$ satisfying $\cup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbb{R}^d)$.

Decompose Z as $Z(A) = Z_1(A) + iZ_2(A)$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$; that is, Z_1 is the real part of Z, Z_2 the imaginary part, and $Z_1(A), Z_2(A) \in L^2_{\mathbb{R}}(\mathbb{P})$ for $A \in \mathcal{B}_b(\mathbb{R}^d)$. These parts are orthogonal in $L^2_{\mathbb{R}}(\mathbb{P})$ in the sense that

$$\mathbb{E}[Z_1(A)Z_2(B)] = 0, \quad \text{for } A, B \in \mathcal{B}_b(\mathbb{R}^d). \tag{2.1}$$

Indeed, by Definition 2.1(3),

$$\mathbb{E}[Z_1(A)Z_2(B)] = \mathbb{E}[Z_2(A)Z_1(B)], \quad \text{for } A, B \in \mathcal{B}_b(\mathbb{R}^d) \tag{2.2}$$

and similarly, with B replaced by $-B$,

$$\mathbb{E}[Z_1(A)Z_2(-B)] = \mathbb{E}[Z_2(A)Z_1(-B)], \quad \text{for } A, B \in \mathcal{B}_b(\mathbb{R}^d). \tag{2.3}$$

Adding (2.2) and (2.3) and using Definition 2.1(2) it follows that

$$2\mathbb{E}[Z_1(A)Z_2(B)] = 0, \quad \text{for } A, B \in \mathcal{B}_b(\mathbb{R}^d),$$

proving (2.1).

Let Z be a zero-mean $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure F. As usual, integration with respect to Z can be defined starting with simple functions and extending to $L^2_{\mathbb{C}}(F)$ using the isometry condition Definition 2.1(3). Thus, the integral $\varphi \mapsto \int \varphi \, dZ$ maps $L^2_{\mathbb{C}}(F)$ linearly isometrically onto a closed subset of $L^2_{\mathbb{C}}(\mathbb{P})$ consisting of zero-mean random variables, and satisfies, for $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $\varphi, \psi \in L^2_{\mathbb{C}}(F)$,

$$\int \mathbf{1}_A \, dZ = Z(A), \quad \text{and} \quad \mathbb{E}\Big[\int \varphi \, dZ \overline{\int \psi \, dZ}\Big] = \int \varphi \overline{\psi} \, dF.$$

Denoting by $\mathcal{R}_{\mathbb{C}}(Z)$ the set of integrals $\int \varphi \, dZ$, $\varphi \in L^2_{\mathbb{C}}(F)$, $\mathcal{R}_{\mathbb{R}}(Z)$ refers to the real-valued elements in $\mathcal{R}_{\mathbb{C}}(Z)$. With $\tilde{L}^2_{\mathbb{C}}(F)$ denoting the set of functions in $L^2_{\mathbb{C}}(F)$ satisfying $\varphi(x) = \overline{\varphi(-x)}$ for all $x \in \mathbb{R}^d$ we have

$$\mathcal{R}_{\mathbb{R}}(Z) = \Big\{ \int \varphi \, dZ : \varphi \in \tilde{L}^2_{\mathbb{C}}(F) \Big\}.$$

Indeed, the inclusion "$\supseteq$" follows from Definition 2.1(2) and "$\subseteq$" from the fact that for all $\varphi \in L^2_{\mathbb{C}}(F)$, $\frac{1}{2}(\varphi + \overline{\varphi(-\cdot)})$ is in $\tilde{L}^2_{\mathbb{C}}(F)$ with integral equal to the real part of $\int \varphi \, dZ$.

If $u_k = (u_{k1}, \dots, u_{kd}) \in \mathbb{R}^d$ for $k = 1, 2$, write $u_1 \le u_2$ if $u_{1j} \le u_{2j}$ for all j, and $u_1 < u_2$ if $u_{1j} < u_{2j}$ for all j. Let $(u_1, u_2] = \{v \in \mathbb{R}^d : u_1 < v \le u_2\}$. Consider a family $H = \{H_u \colon u \in \mathbb{R}^d\}$ with $H_u \in \mathbb{C}$. Define *the increment of H over* $(u_1, u_2]$, $H((u_1, u_2])$, as

$$H((u_1, u_2]) = \sum_{\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0,1\}^d} (-1)^{\epsilon_\cdot} H_{(c_1(\epsilon_1), \dots, c_d(\epsilon_d))},$$

where $\epsilon_\cdot = \epsilon_1 + \cdots \epsilon_d$, $c_j(0) = u_{2j}$ and $c_j(1) = u_{1j}$. That is, $H((u_1, u_2]) = H_{u_2} - H_{u_1}$ if $d = 1$ and

$$H((u_1, u_2]) = H_{(u_{21}, u_{22})} + H_{(u_{11}, u_{12})} - H_{(u_{11}, u_{22})} - H_{(u_{21}, u_{12})} \quad \text{if } d = 2.$$

Notice that $H((u, v]) = 0$ if $u \le v$ and $u \not< v$. Later we shall occasionally write $\triangle^h H(u)$ for $H((u, u + h])$ for $u \in \mathbb{R}^d$ and any $h \in \mathbb{R}^d_+$.

Definition 2.3. A real-valued process $X = \{X_u : u \in \mathbb{R}^d\}$ is said to have *zero-mean second-order stationary increments* if $X((u, v]) \in L^2_{\mathbb{R}}(\mathbb{P})$ with $\mathbb{E}[X((u, v])] = 0$ for all $u, v \in \mathbb{R}^d$, and

$$\begin{aligned} &\mathbb{E}\big[X\big((u_1 + h, v_1 + h]\big) X\big((u_2 + h, v_2 + h]\big)\big] \\ &= \mathbb{E}\big[X\big((u_1, v_1]\big) X\big((u_2, v_2]\big)\big], \quad \text{for all } h \in \mathbb{R}^d_+ \text{ and } u_1 \le v_1, u_2 \le v_2 \text{ in } \mathbb{R}^d. \end{aligned} \tag{2.4}$$

Remark 2.4. In the multiparameter case $d \geq 2$ there is an alternative definition of stationary increments studied e.g. by Yaglom [10]. Let us discuss the relations between the two definitions. A real-valued second-order process $X = \{X_u : u \in \mathbb{R}^d\}$ for which $\mathbb{E}[X_v - X_u] = 0$ for $u, v \in \mathbb{R}^d$ is said to have *second-order stationary increments in the strong sense* if

$$\begin{aligned} &\mathbb{E}[(X_{v_1+h} - X_{u_1+h})(X_{v_2+h} - X_{u_2+h})] \\ &= \mathbb{E}[(X_{v_1} - X_{u_1})(X_{v_2} - X_{u_2})], \quad \text{for all } h \in \mathbb{R}^d_+ \text{ and } u_1 \leq v_1, u_2 \leq v_2 \text{ in } \mathbb{R}^d. \end{aligned}$$

It is easily seen that this implies that X has zero-mean second-order stationary increments in the sense of (2.4). But conversely there are many processes with zero-mean second-order stationary increments that do not have stationary increments in the strong sense. One such example is the Brownian sheet, where increments over disjoint intervals are independent and $X((u, v]) \overset{\mathscr{D}}{=} N(0, \lambda_d((u, v]))$ for $u \leq v$, in the case $d \geq 2$. An even more restrictive definition of stationary increments is given in Samorodnitsky and Taqqu [6, Section 8.1]. However, when $d = 1$ the three definitions coincide.

In the following assume that X has zero-mean stationary increments in the strong sense. Yaglom [10], Remark 3, p. 295, shows that, up to addition of a random variable not depending on u, X_u is given by

$$X_u = \int_{\mathbb{R}^d} (e^{i\langle z, u\rangle} - 1)\, \tilde{Z}(dz) + \langle a, u\rangle, \quad \text{for } u \in \mathbb{R}^d, \tag{2.5}$$

where $\tilde{Z} = \{\tilde{Z}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a zero-mean $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure with control measure $\tilde{F}$ satisfying

$$\int_{\mathbb{R}^d} (\|z\|^2 \wedge 1)\, \tilde{F}(dz) < \infty$$

and $a \in \mathbb{R}^d$ is a random vector. After a few calculations it follows that when $d \geq 2$,

$$X((u, v]) = \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(z)\, Z(dz), \quad \text{for } u < v, \tag{2.6}$$

where $Z(dz) = i^d z_1 \cdots z_d \tilde{Z}(dz)$. That is, the control measure F of Z is $F(dz) = \prod_{j=1}^d z_j^2 \tilde{F}(dz)$ which satisfies

$$\int_{\mathbb{R}^d} \frac{1 \wedge \|z\|^2}{\prod_{j=1}^d z_j^2}\, F(dz) < \infty.$$

In the next result we give the corresponding spectral representation of processes with zero-mean second-stationary increments. In this case it is natural to seek for a representation as in (2.6) rather than (2.5). Recall that for $u, v \in \mathbb{R}^d$ with $u < v$,

$$\mathcal{F}\mathbf{1}_{(u,v]}(z) = \prod_{j=1}^d \Big(\frac{e^{iv_j z_j} - e^{iu_j z_j}}{iz_j}\Big), \quad \text{for } z = (z_1, \ldots, z_d) \in \mathbb{R}^d, \tag{2.7}$$

where the right hand side should be understood by continuity if $z_j = 0$ for some j, i.e. the j'th factor equals $v_j - u_j$ for $z_j = 0$.

Theorem 2.5. *Assume that $X = \{X_u : u \in \mathbb{R}^d\}$ is a real-valued square-integrable process with zero-mean second-order stationary increments satisfying that $u \mapsto X_u$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$. Then there is a symmetric measure F on $\mathbb{R}^d$ satisfying*

$$\int_{\mathbb{R}^d} \frac{1}{1+\prod_{j=1}^d z_j^2}\, F(dz) < \infty, \quad (\textit{where } z = (z_1, \dots, z_d)), \tag{2.8}$$

and a zero-mean $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure Z with control measure F such that

$$X((u,v]) = \int \mathcal{F}\mathbf{1}_{(u,v]}\, dZ, \quad \textit{for } u < v. \tag{2.9}$$

In particular, for $u_1 < v_1$ and $u_2 < v_2$,

$$\mathbb{E}[X((u_1,v_1])X((u_2,v_2])] = \int \mathcal{F}\mathbf{1}_{(u_1,v_1]}\overline{\mathcal{F}\mathbf{1}_{(u_2,v_2]}}\, dF. \tag{2.10}$$

The measures F and Z are uniquely determined by X. In addition, $\mathcal{R}_{\mathbb{C}}(Z) = \overline{\mathrm{sp}}_{\mathbb{C}}\{X((u,v]) : u \le v\}$ and $\mathcal{R}_{\mathbb{R}}(Z) = \overline{\mathrm{sp}}_{\mathbb{R}}\{X((u,v]) : u \le v\}$.

The measure F above is called the *spectral measure of* X and Z is the *random spectral measure of* X. The last statement in Theorem 2.5 shows that Z is Gaussian if X is Gaussian.

Proof. In the case $d = 1$ the result can be found e.g. in Itô [3], Theorem 6.1. In the general case we follow Itô's approach closely. Define $\{X(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ as

$$X(\varphi) = \int_{\mathbb{R}^d} X_u \varphi(u)\, du, \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

where the integral is constructed in the $L^2_{\mathbb{C}}(\mathbb{P})$-sense using that $u \mapsto X_u\varphi(u)$ is $L^2_{\mathbb{C}}(\mathbb{P})$–continuous with compact support. Clearly, $\{X(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ constitutes a random distribution in the sense of Itô [3] or Yaglom [10].

Denote by D the differential operator $\partial^d/\partial u_1 \cdots \partial u_d$ and define $\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ according to

$$X^{(1)}(\varphi) = (-1)^d \int_{\mathbb{R}^d} X_u D\varphi(u)\, du, \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

Since, with $e = (1, \dots, 1) \in \mathbb{R}^d$ denoting the vector of ones,

$$D\varphi(u) = \lim_{\epsilon \to 0} \varphi((u, u+\epsilon e])/\epsilon^d, \quad \text{for } u \in \mathbb{R}^d \text{ and } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

we get, using the assumptions and linear change of variables, that for $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} X_u D\varphi(u)\, du = \lim_{\epsilon \to 0} \epsilon^{-d} \int_{\mathbb{R}^d} X((u, u+\epsilon e])\varphi(u)\, du, \quad \text{in } L^2_{\mathbb{C}}(\mathbb{P}). \tag{2.11}$$

A key point is that $X^{(1)}$ is stationary in the sense that

$$\mathbb{E}[(\tau_h X^{(1)}(\varphi))(\tau_h X^{(1)}(\psi))] = \mathbb{E}[(X^{(1)}(\varphi))(X^{(1)}(\psi))], \quad \text{for } h \in \mathbb{R}^d,\ \varphi, \psi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

where

$$\tau_h X^{(1)}(\varphi) = X^{(1)}(\varphi(\cdot - h)), \quad \text{for } h \in \mathbb{R}^d, \ \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

To see this, fix $h \in \mathbb{R}^d$ and $\varphi, \psi \in \mathscr{D}_C(\mathbb{R}^d)$. Using (2.11) it follows that

$$\begin{aligned}
&\mathbb{E}[(\tau_h X^{(1)}(\varphi))(\tau_h X^{(1)}(\psi))] \\
&= \lim_{\epsilon \to 0} \epsilon^{-2d} \mathbb{E}[\int_{\mathbb{R}^d} X((u, u+\epsilon e])\varphi(u-h)\, du \int_{\mathbb{R}^d} X((v, v+\epsilon e])\psi(v-h)\, dv] \\
&= \lim_{\epsilon \to 0} \epsilon^{-2d} \mathbb{E}[\int_{\mathbb{R}^d} X((u+h, u+h+\epsilon e])\varphi(u)\, du \int_{\mathbb{R}^d} X((v+h, v+h+\epsilon e])\psi(v)\, dv] \\
&= \lim_{\epsilon \to 0} \epsilon^{-2d} \int_{\mathbb{R}^{2d}} \mathbb{E}[X((u+h, u+h+\epsilon e])X((v+h, v+h+\epsilon e])]\varphi(u)\psi(v)\, dudv \\
&= \lim_{\epsilon \to 0} \epsilon^{-2d} \int_{\mathbb{R}^{2d}} \mathbb{E}[X((u, u+\epsilon])X((v, v+\epsilon e])]\varphi(u)\psi(v)\, dudv \\
&= \mathbb{E}[(X^{(1)}(\varphi))(X^{(1)}(\psi))].
\end{aligned}$$

Applying [10], Theorem 3, there exists an $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure Z with symmetric control measure F satisfying

$$\int_{\mathbb{R}^d} \frac{1}{(1+|z|^2)^p}\, F(dz) < \infty, \quad \text{for some } p \geq 1,$$

such that

$$X^{(1)}(\varphi) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z)\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

To obtain (2.8), it suffices to show that

$$\int_C \frac{1}{\prod_{j=1}^d z_j^2}\, F(dz) < \infty, \tag{2.12}$$

where $C = \{z = (z_1, \ldots, z_d) : |z_j| \geq 1 \text{ for all } j\}$. Following Itô [3], set

$$X_1(\varphi) = \int_{\mathbb{R}^d} G\varphi(z)\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d), \tag{2.13}$$

where

$$G\varphi(z) = \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{e^{iu_j z_j} - 1_{C^c}(z)}{iz_j}\, \varphi(u)\, du.$$

Since $G\varphi$ is bounded and

$$G\varphi(z) = \mathcal{F}\varphi(z)\, (\prod_{j=1}^d iz_j)^{-1}, \quad \text{for } z \in C,$$

$G\varphi$ belongs to $L^2_{\mathbb{C}}(F)$, thus making (2.13) well-defined and $\{X_1(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ a random distribution. Maintaining the definition of the differential operator D from above and using integration by parts we get $G(D\varphi) = (-1)^d \mathcal{F}\varphi$ for $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$, implying that $X(D\varphi) = (-1)^d X^{(1)}(\varphi) = X_1(D\varphi)$ for $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$, or equivalently, $X_1(\varphi) = X(\varphi)$ for all $\varphi \in \mathscr{D}_0(\mathbb{R}^d) := \{\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \varphi(t)\, dt = 0\}$. Observe that

$\triangle^h\varphi \in \mathscr{D}_0(\mathbb{R}^d)$ for every $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d_+$. Thus, for every $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d_+$ we have

$$X(\triangle^h\varphi) = \int_{\mathbb{R}^d} G(\triangle^h\varphi)(z)\, Z(dz)$$

implying, since

$$G(\triangle^h\varphi)(z) = \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{1-e^{-ih_jz_j}}{iz_j},$$

that

$$X(\triangle^h\varphi) = (-1)^d \int_{\mathbb{R}^d} \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{e^{-ih_jz_j}-1}{iz_j}\, Z(dz).$$

Thus, for $\varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d_+$,

$$\begin{aligned} \|X(\triangle^h\varphi)\|^2_{L^2_{\mathbb{C}}(\mathbb{P})} &= \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \prod_{j=1}^d |\frac{1-e^{-ih_jz_j}}{iz_j}|^2\, F(dz) \\ &\geq \int_C \prod_{j=1}^d |1-e^{-ih_jz_j}|^2\, |\mathcal{F}\varphi(z)|^2\, \frac{F(dz)}{\prod_{j=1}^d z_j^2}. \end{aligned}$$

In particular this holds for every $\varphi_n \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)$, $n \geq 1$, of the form $\varphi_n(z) = \prod_{j=1}^d g_n(z_j)$ for $z \in \mathbb{R}^d$, where $g_n \in \mathscr{D}_{\mathbb{R}}(\mathbb{R})_+$ satisfies

$$g_n(x) = 0, \quad \text{for } |x| \geq 1/n, \text{ and } \int_{\mathbb{R}} g_n(x)\, dx = 1.$$

Using the same string of inequalities as [3], p. 221, we see that there exists a universal constant A such that for all n and $h \in \mathbb{R}^d_+$,

$$\|X(\triangle^h\varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} \geq A \int_C \prod_{j=1}^d (|1-e^{-ih_jz_j}|^2\, 1_{\{|z_j|\leq n\}}) \frac{F(dz)}{\prod_{j=1}^d z_j^2}.$$

Following [3] integrate both sides with respect to dh over the cube $[0,1]^d$. Using the product structure the integral of the right-hand side equals

$$A \int_C \prod_{j=1}^d \int_0^1 |1-e^{-ih_jz_j}|^2\, dh_j 1_{\{1\leq|z_j|\leq n\}} \frac{F(dz)}{\prod_{j=1}^d z_j^2}$$

which, again following [3], is

$$\geq A\, (\int_0^1 |1-e^{-\mathrm{i}u}|^2\, du)^d \int_C 1_{\{1\leq|z_j|\leq n,\, j=1,\ldots,d\}} \frac{F(dz)}{\prod_{j=1}^d z_j^2}.$$

Applying monotone convergence (2.12) follows if $\sup_{n\geq1,\, h\in[0,1]^d} \|X(\triangle^h\varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})}$ is finite. But using linear substitution and Jensen's inequality we have with obvious notation, for all $n \geq 1$ and $h \in [0,1]^d$,

$$\|X(\triangle^h \varphi_n)\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} = \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} X_u \triangle^h \varphi_n(t)\, dt\Big)^2\Big] = \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} \triangle^h X_{u-h} \varphi_n(u)\, dt\Big)^2\Big]$$
$$\leq \int_{\mathbb{R}^d} \mathbb{E}[(\triangle^h X_{u-h})^2] \varphi_n(u)\, du \leq \sup_{u \in \mathbb{R}^d,\, h \in [0,1]^d} \mathbb{E}[(\triangle^h X_{u-h})^2]$$

which is finite due to the $L^2_{\mathbb{R}}(\mathbb{P})$-continuity and the stationary increments.

To prove the representation (2.9) let $u < v$ be given. Setting $h = v - u$ we have from above that

$$X(\triangle^h \varphi) = (-1)^d \int_{\mathbb{R}^d} \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{1 - e^{-ih_j z_j}}{iz_j}\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d),$$

that is, using linear substitution,

$$\int_{\mathbb{R}^d} \triangle^h X_{x-h} \varphi(x)\, dx = (-1)^d \int_{\mathbb{R}^d} \mathcal{F}\varphi(z) \prod_{j=1}^d \frac{1 - e^{-ih_j z_j}}{iz_j}\, Z(dz), \quad \text{for } \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d).$$

In particular

$$\int_{\mathbb{R}^d} \triangle^h X_{x-h} \varphi_n(x)\, dx = (-1)^d \int_{\mathbb{R}^d} \mathcal{F}\varphi_n(z) \prod_{j=1}^d \frac{1 - e^{-ih_j z_j}}{z_j}\, Z(dz), \quad \text{for } n \geq 1, \tag{2.14}$$

where $(\varphi_n)_{n \geq 1} \subseteq \mathscr{D}_{\mathbb{R}}(\mathbb{R}^d)_+$ is such that

$$\int_{\mathbb{R}^d} \varphi_n(x)\, dx = 1 \text{ for } n \geq 1 \quad \text{and} \quad \varphi_n(x)\, dx \to \delta_v \text{ weakly.}$$

As n tends to infinity both sides of (2.14) converge in $L^2_{\mathbb{R}}(P)$ due to the continuity assumption on X and the integrability property (2.8) of F, giving the identity

$$X((u,v]) = \triangle^{v-u} X_u = (-1)^d \int_{\mathbb{R}^d} e^{i\langle z,v\rangle} \prod_{j=1}^d \frac{1 - e^{-i(v_j - u_j)z_j}}{iz_j}\, Z(dz)$$
$$= \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{e^{iv_j z_j} - e^{is_j z_j}}{iz_j}\, Z(dz) = \int_{\mathbb{R}^d} \mathcal{F}1_{(u,v]}(z)\, Z(dz)$$

which is (2.9).

To prove the last part notice that X and $X^{(1)}$ are in one-to-one correspondence, that $\overline{\mathrm{sp}}_{\mathbb{C}}\{X((u,v]) : u \leq v\} = \overline{\mathrm{sp}}_{\mathbb{C}}\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$, and that there is a similar result with subscript $\mathbb{C}$ replaced by $\mathbb{R}$. By construction (see [10] p. 281), Z is uniquely determined; moreover we have $\mathcal{R}_{\mathbb{C}}(Z) = \overline{\mathrm{sp}}_{\mathbb{C}}\{X^{(1)}(\varphi) : \varphi \in \mathscr{D}_{\mathbb{C}}(\mathbb{R}^d)\}$ as well as the corresponding result with subscript $\mathbb{C}$ replaced by $\mathbb{R}$. This concludes the proof. $\square$

Remark 2.6. Let X be a zero-mean second-order process for which the mapping $u \mapsto X((0,u])$ is continuous from $\mathbb{R}^d_+$ to $L^2_{\mathbb{R}}(\mathbb{P})$. Then one can construct an $L^2_{\mathbb{R}}(\mathbb{P})$-continuous process $\tilde{X} = \{\tilde{X}_u : u \in \mathbb{R}^d\}$ with zero-mean second-order stationary increments $\tilde{X} = \{\tilde{X}_u : u \in \mathbb{R}^d\}$ satisfying that $X((u,v]) = \tilde{X}((u,v])$ for all $0 \le u < v$. Thus, the conclusions in Theorem 2.5 remain valid if X is a real-valued process with zero-mean second-order stationary increments satisfying that $u \mapsto X((0,u])$ is continuous from $\mathbb{R}^d_+$ to $L^2_{\mathbb{R}}(\mathbb{P})$.

Remark 2.7. Let F denote a symmetric measure on $\mathbb{R}^d$ satisfying (2.8). As a converse to Theorem 2.5 there is a process $X = \{X_u : u \in \mathbb{R}^d\}$ with zero-mean second-order stationary increments process for which $u \mapsto X((0,u])$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$ and the spectral measure of X is F. Indeed, let Z denote a zero-mean $L^2_{\mathbb{C}}(\mathbb{P})$-valued random measure measure with control measure F, define the increments in X by (2.9) and notice that we have (2.10) as well. From (2.7) and (2.10) it follows that $u \mapsto X((0,u])$ is continuous in $L^2_{\mathbb{R}}(\mathbb{P})$. In addition, since, for arbitrary $u \in \mathbb{R}^d$ and $v_1, v_2 \in \mathbb{R}^d_+$,

$$\mathcal{F}\mathbf{1}_{(u,u+v_1]}\overline{\mathcal{F}\mathbf{1}_{(u,u+v_2]}} = \mathcal{F}\mathbf{1}_{(0,v_1]}\overline{\mathcal{F}\mathbf{1}_{(0,v_2]}},$$

it follows from (2.10) that (2.4) holds.

3 Deterministic integrands

Let $X = \{X_u : u \in \mathbb{R}^d\}$ be a real-valued process with zero-mean second-order stationary increments having spectral measure F satisfying (2.8) and random spectral measure Z. Assume furthermore that F is absolutely continuous with respect to λ_d with density f. In the following we study classes of deterministic integrands with respect to X.

Let $\mathscr{E}$ be the set of simple functions on $\mathbb{R}^d$ of the form

$$\varphi = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{(u_j,v_j]} \tag{3.1}$$

where $\{\alpha_j\} \subseteq \mathbb{R}$ and $\{u_j\}, \{v_j\} \subseteq \mathbb{R}^d$ satisfy $u_j \le v_j$ for all j. For $\varphi \in \mathcal{E}$ represented as in (3.1) define the *simple integral* as

$$\int \varphi \, dX := \sum_{j=1}^{n} \alpha_j X((u_j, v_j]), \tag{3.2}$$

and equip $\mathscr{E}$ with the norm $\|\varphi\|_{\mathscr{E}} := \|\int \varphi \, dX\|_{L^2_{\mathbb{R}}(\mathbb{P})}$. By Theorem 2.5,

$$\int_{\mathbb{R}^d} \varphi(u) \, dX_u = \int_{\mathbb{R}^d} \mathcal{F}\varphi(z) \, dZ_u \quad \text{and} \quad \|\varphi\|^2_{\mathscr{E}} = \int_{\mathbb{R}^d} |\mathcal{F}\varphi|^2 \, dF \quad \text{for } \varphi \in \mathscr{E}. \tag{3.3}$$

Definition 3.1. A pseudo normed linear space $(\Lambda, \|\cdot\|_\Lambda)$ containing $\mathscr{E}$ as a dense subspace and satisfying $\|\varphi\|_{\mathscr{E}} = \|\varphi\|_\Lambda$ for $\varphi \in \mathscr{E}$ is called a *predomain* for X. A *domain* is a complete predomain. Given a predomain Λ, there is a unique continuous linear mapping $\int \cdot \, dX \colon \Lambda \to L^2_{\mathbb{R}}(\mathbb{P})$, extending the simple integral (3.2). This mapping is called the *integral with respect to X*.

Notice that Λ is not assumed to be a function space. By definition, a domain is a completion of $\mathscr{E}$ and thus uniquely determined up to an isometric isomorphism. Below we give concrete examples of predomains and domains.

Remark 3.2. Using the completeness of $L^2_{\mathbb{R}}(\mathbb{P})$ we see that a predomain Λ is a domain if and only if

$$\Big\{ \int \varphi \, dX : \varphi \in \Lambda \Big\} = \overline{\mathrm{sp}}_{\mathbb{R}} \big\{ X((u,v]) : u, v \in \mathbb{R}^d, u \leq v \big\}. \tag{3.4}$$

This emphasizes why domains are more attractable than predomains since for the latter we only have "$\subseteq$" in (3.4).

For ease of reading we formulate two lemmas. For the second see [7], Chapter VII, Théorème VII.

Lemma 3.3. *Let $\varphi \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ be given such that $\mathcal{F}\varphi$ is a function. Then $\varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ if and only if $\mathcal{F}\varphi(-x) = \overline{\mathcal{F}\varphi(x)}$ for λ_d-a.a. x.*

Lemma 3.4. *Let μ be a signed Borel measure on $\mathbb{R}^d$. Then μ is a tempered measure, that is $\mu \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ if*

$$\int_{\mathbb{R}^d} (1 + \|u\|^2)^{-k} \, |\mu|(du) < \infty$$

for some positive integer $k \geq 1$. This condition is also necessary if μ is a positive measure. In particular, a real-valued Borel function h is a tempered distribution if, and in case h is non-negative only if,

$$\int_{\mathbb{R}^d} \frac{|h(u)|}{(1 + \|u\|^2)^k} \, du < \infty.$$

for some positive integer $k \geq 1$.

In view of (3.3) it is natural to look for predomains consisting of objects for which a Fourier transform can be defined, that is spaces of distributions. The absolute continuity of F allows us to introduce the following spaces

$$\Lambda_{\text{dist}} = \Big\{ \varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d) : \mathcal{F}\varphi \text{ is a function such that } \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \, F(dz) < \infty \Big\},$$

$$\Lambda_{\text{func}} = \Big\{ \varphi \in L^2_{\mathbb{R}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \, F(dz) < \infty \Big\},$$

equipped with the pseudo norms $\|\cdot\|_{\Lambda_{\text{func}}}$ and $\|\cdot\|_{\Lambda_{\text{dist}}}$ given by

$$\|\varphi\|^2_{\Lambda_{\text{func}}} = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \, F(dz), \qquad \|\varphi\|^2_{\Lambda_{\text{dist}}} = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(z)|^2 \, F(dz).$$

Notice that $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d) \subseteq \Lambda_{\text{func}} \subseteq \Lambda_{\text{dist}}$.

Theorem 3.5. *(1) Λ_{dist} is a predomain for X and the integral on Λ_{dist} is given by*

$$\int \varphi \, dX = \int \mathcal{F}\varphi \, dZ, \quad \varphi \in \Lambda_{\text{dist}}. \tag{3.5}$$

(2) Λ_{dist} is a domain for X if and only if

$$\forall g \in L^2_{\mathbb{R}}(F)\ \exists k \in \mathbb{N} : \int_{\{f>0\}} \frac{|g(u)|}{(1+\|u\|^2)^k}\,du < \infty. \tag{3.6}$$

In particular, Λ_{dist} is a domain for X if there exists $k \in \mathbb{N}$ such that

$$\int_{\{f>0\}} \frac{1}{f(u)(1+\|u\|^2)^k}\,du < \infty. \tag{3.7}$$

(3) Λ_{func} is a predomain, and it is a domain if and only if

$$L^2_{\mathbb{R}}(F) \subseteq L^2_{\mathbb{R}}(\mathbf{1}_{\{f(u)>0\}}\,du). \tag{3.8}$$

By Lemma 3.6 we further have that Λ_{dist} is complete if and only if $\mathcal{F}(\Lambda_{\mathrm{dist}}) = \tilde{L}^2_{\mathbb{C}}(F)$.

Proof. (1): Lemma 3.6 below implies that $\mathscr{E}$ is dense in Λ_{dist} showing together with (3.3) that Λ_{dist} is a predomain for X. The continuous linear mapping $\varphi \mapsto \int \varphi\,dX$ from Λ_{dist} to $L^2_{\mathbb{C}}(\mathbb{P})$ defined by (3.5) extends the simple integral by (3.3) and is hence the corresponding integral since $L^2_{\mathbb{R}}(\mathbb{P})$ is a closed subspace of $L^2_{\mathbb{C}}(\mathbb{P})$.

(2): Assume that for all $g \in L^2_{\mathbb{R}}(F)$, (3.6) holds for some k and let us show that Λ_{dist} is a domain for X. Let $\{\varphi_n\}$ be a Cauchy sequence in Λ_{dist}. By completeness of $L^2_{\mathbb{C}}(F)$ there exists $g \in L^2_{\mathbb{C}}(F)$ with $\overline{g} = g(-\cdot)$ such that $\mathcal{F}\varphi_n \to g$ in $L^2_{\mathbb{C}}(F)$. Since we may assume that $g = 0$ on $\{f = 0\}$, (3.6) and Lemma 3.4 shows that $g \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$. Hence, using Lemma 3.3, $\varphi := \mathcal{F}^{-1}g$ is in Λ_{dist} and $\varphi_n \to \varphi$ in Λ_{dist} which shows that Λ_{dist} is complete.

Conversely, assume that Λ_{dist} is complete. For contradiction consider an $h \in L^2_{\mathbb{R}}(F)$ which does not satisfy (3.6) with g replaced by h. Without loss of generality we may assume that $h \geq 0$ and $h = 0$ on $\{f = 0\}$. By Lemma 3.4, $h \notin \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$. Let $h_1 = \frac{1}{2}(h + h(-\cdot))$ and $h_2 = \frac{1}{2}(h - h(-\cdot))$ be the even and odd parts of h and set $g = h_1 + ih_2$. By linearity, $g \in L^2_{\mathbb{C}}(F)$ and if $g \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ then $h_1, h_2 \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ which implies that $h = h_1 + h_2 \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$. Thus $g \in L^2_{\mathbb{C}}(F) \setminus \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ and by construction $\overline{g} = g(-\cdot)$. Since F is a tempered measure, $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$ is dense in $L^2_{\mathbb{R}}(F)$ and therefore there exist sequences $\{g_{e,n}\}$ and $\{g_{o,n}\}$ in $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$ consisting of even and odd functions approximating h_1 and h_2 in $L^2_{\mathbb{R}}(F)$. Setting $g_n = g_{e,n} + ig_{o,n}$ for $n \geq 1$ we have $\{g_n\} \subseteq \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d) \subseteq \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ satisfying $\overline{g}_n = g_n(-\cdot)$ and $g_n \to g$ in $L^2_{\mathbb{C}}(F)$. Thus $\varphi_n := \mathcal{F}^{-1}g_n$ is a Cauchy sequence in Λ_{dist} which does not converge.

The last statement in (2) follows since for any measurable function $g\colon \mathbb{R}^d \to \mathbb{R}$, we have by the Cauchy-Schwarz inequality that

$$\int_{\{f>0\}} \frac{|g(u)|}{(1+\|u\|^2)^k}\,du \leq \Big(\int_{\{f>0\}} |g(u)|^2 f(u)\,du\Big)^{1/2}\Big(\int_{\{f>0\}} \frac{1}{f(u)}\frac{1}{(1+\|u\|^2)^{2k}}\,du\Big)^{1/2}.$$

(3): Assume (3.8) and let $\{\varphi_n\}$ be Cauchy in Λ_{func}. As in the proof of (2) there is a $g \in L^2_{\mathbb{C}}(F)$ with $\overline{g} = g(-\cdot)$ and satisfying $g = 0$ on $\{f = 0\}$ such that $\mathcal{F}\varphi_n \to g$ in $L^2_{\mathbb{C}}(\mathbb{R}^d)$. Since $g \in L^2_{\mathbb{C}}(\mathbb{R}^d)$ we have by Lemma 3.3 that $\varphi := \mathcal{F}^{-1}g$ is in Λ_{func} and $\varphi_n \to \varphi$ in Λ_{func}, showing that the latter space is complete.

Conversely assume that Λ_{func} is complete. As in the proof of (2), if (3.8) is not satisfied there is a function $g : \mathbb{R}^d \to \mathbb{C}$ satisfying $g = 0$ on $\{f = 0\}$ and $\overline{g} = g(-\cdot)$ such that $g \in L^2_{\mathbb{C}}(F) \setminus L^2_{\mathbb{C}}(\mathbf{1}_{\{f(u)>0\}} du)$. Again as in (2) we can construct a sequence $\{g_n\}$ in $L^2_{\mathbb{C}}(\mathbb{R}^d) \cap L^2_{\mathbb{C}}(F)$ satisfying $\overline{g_n} = g_n(-\cdot)$ such that $g_n \to g$ in $L^2_{\mathbb{C}}(F)$. Then $\varphi_n := \mathcal{F}^{-1} g_n$ is a non-converging Cauchy sequence in Λ_{func}. □

Lemma 3.6. *$\mathcal{F}(\mathscr{E})$ is a dense subspace of $\tilde{L}^2_{\mathbb{C}}(F)$.*

Proof. Let G be the real linear span of functions g of the form $g(u) = \prod_{j=1}^d g_j(u_j)$ where $\{g_j\} \subseteq \mathscr{D}_{\mathbb{R}}(\mathbb{R})$. Notice that $(\mathcal{F}g)(z) = \prod_{j=1}^d (\mathcal{F}g_j)(z_j)$. By arguments as in Itô [3], Theorem 4.1, it follows that $\mathcal{F}(G)$ is dense in $\tilde{L}^2_{\mathbb{C}}(F)$. To show that $\mathcal{F}(\mathscr{E})$ is dense in $\tilde{L}^2_{\mathbb{C}}(F)$ it is hence enough to show that for all $g \in G$ there exists a sequence $\{g_n\} \subseteq \mathscr{E}$ such that $\mathcal{F}g_n \to \mathcal{F}g$ in $L^2_{\mathbb{C}}(F)$. It suffices to consider $g \in G$ of the form $g(u) = \prod_{j=1}^d g_j(u_j)$ where for simplicity we assume that $\operatorname{supp}(g_j) \subseteq [0,1]$ for all j. Set for all $n \in \mathbb{N}$, $g_n(u) = \prod_{j=1}^d g_{n,j}(u_j)$ where $g_{n,j} = \sum_{k=1}^n g_j(\frac{k-1}{n}) \mathbf{1}_{(\frac{k-1}{n}, \frac{k}{n}]}$. For all n and j,

$$\begin{aligned}(\mathcal{F}g_{n,j})(t) &= \int_{\mathbb{R}} e^{ist} g_{n,j}(s)\, ds = \sum_{k=1}^n g_j(\tfrac{k-1}{n}) \Big(\frac{e^{i(k/n)t} - e^{i((k-1)/n)t}}{it} \Big) \\ &= \frac{1}{it} \Big(g_j(1) e^{it} - g_j(0) - \sum_{k=1}^{n-1} (g_j(\tfrac{k}{n}) - g_j(\tfrac{k-1}{n})) e^{i(k/n)t} \Big).\end{aligned}$$

Hence, denoting by $\mathrm{TV}(g_j; [0,1])$ the total variation of g_j on $[0,1]$,

$$|(\mathcal{F}g_{n,j})(t)| \le \frac{1}{|t|} \Big(|g_j(1)| + |g_j(0)| + \mathrm{TV}(g_j; [0,1]) \Big) \le C_{g_j}/|t|$$

for some constant C_{g_j} depending only on g_j. Furthermore, for all j,

$$\|\mathcal{F}g_{n,j} - \mathcal{F}g_j\|_\infty \le \|g_{n,j} - g_j\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } n \to \infty \tag{3.9}$$

implying that $C = \sup_{n \in \mathbb{N}} \|\mathcal{F}g_{n,j}\|_\infty < \infty$. Combining this we see that $\mathcal{F}g_n(z) = \prod_{j=1}^d \mathcal{F}g_{n,j}(z_j) \to \mathcal{F}g(z)$ pointwise by (3.9), and

$$|(\mathcal{F}g_n)(z)| \le \frac{C}{1 + \prod_{j=1}^d |z_j|},$$

which by dominated convergence implies that $\mathcal{F}g_n \to \mathcal{F}g$ in $L^2_{\mathbb{C}}(F)$. □

Remark 3.7. Theorem 3.5(2)–(3) underlines that it is easier for Λ_{dist} than for Λ_{func} to be a domain. As an illustration, consider the fractional Brownian sheet, which corresponds to X being Gaussian and $f(u) = c_H \prod_{j=1}^d |u_j|^{1-2H_j}$, where $H_1, \dots, H_d \in (0,1)$, see e.g. Terdik and Woyczyński [8]. Since f satisfies (3.7), Theorem 3.5 shows that Λ_{dist} is complete. Moreover, by Theorem 3.5 it follows that Λ_{func} is complete if and only if $H_1 = \cdots = H_d = \frac{1}{2}$, that is, X is a Brownian sheet. In the case $d = 1$, where X is a fractional Brownian motion, a quite long proof of the non-completeness of Λ_{func} can be founded in Pipiras and Taqqu [5], Theorem 3.1, and the completeness of Λ_{dist} is shown by Jolis [4], Proposition 4.1.

Remark 3.8. Λ_{dist} is not always a domain. For instance, if $d = 1$ and $f(u) = e^{-u^2}$ then $g(u) = e^u$ belongs to $L^2(F)$ but $\int_{\mathbb{R}} |g(u)|(1 + u^2)^{-k}\, du = \infty$ for all $k \in \mathbb{N}$. Hence, by Theorem 3.5(2) Λ_{dist} is not a domain.

4 Stochastic integrands for processes white in time and colored in space

In the following we add a temporal component; that is, we consider processes indexed by $\mathbb{R}^{1+d}$ rather than $\mathbb{R}^d$. A generic element $u \in \mathbb{R}^{1+d}$ will be decomposed as $u = (t, x)$ where $t \in \mathbb{R}$ is time and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ a space variable. Intervals in $\mathbb{R}^{1+d}$ will be written either as $(u, v]$ or $(s, t] \times (x, y]$ where $(s, t]$ is an interval in $\mathbb{R}$ and $(x, y]$ is an interval in $\mathbb{R}^d$. Functions on $\mathbb{R}^{1+d}$ will often be denoted by $\varphi_t(x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, and $\mathcal{F}\varphi_t(z)$ denotes the Fourier transform in the space variable for fixed t.

Let F denote a symmetric measure on $\mathbb{R}^d$ with density f with respect to λ_d satisfying $f(x) = f(-x)$ for all $x \in \mathbb{R}^d$. Assume throughout that F satisfies condition (2.8). The measure $\lambda_1 \times F$ on $\mathbb{R}^{1+d}$ then satisfies (2.8) as well. Consider an $L^2_{\mathbb{R}}(\mathbb{P})$-continuous *Gaussian* process $X = \{X_u : u \in \mathbb{R}^{1+d}\}$ with zero-mean second-order stationary increments and spectral measure $\lambda_1 \times F$. By Parseval's identity and (2.10) we have, for $s_i < t_i$ (in $\mathbb{R}$) and $x_i < y_i$ (in $\mathbb{R}^d$), $i = 1, 2$,

$$\begin{aligned}
&\mathbb{E}[X((s_1, t_1] \times (x_1, y_1])X((s_2, t_2] \times (x_2, y_2])] \\
&\quad = 2\pi \int_{\mathbb{R}} \mathbf{1}_{(s_1,t_1]}\mathbf{1}_{(s_2,t_2]}\, d\lambda_1 \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(x_1,y_1]}\overline{\mathcal{F}\mathbf{1}_{(x_2,y_2]}}\, dF.
\end{aligned}$$

Thus, there is independent increments in time but correlation in space determined by F. That is, X is *white in time and colored in space.*

From now on we consider only time points in $\mathbb{R}_+$. Notice that in general $X((0, t] \times \cdot)$ does not extend to an $L^2_{\mathbb{R}}(\mathbb{P})$-valued measure defined on $\mathcal{B}_b(\mathbb{R}^d)$. As an example, if $d = 1$ and X is fractional in space we have $f(x) = |x|^{1-2H}$ for $x \in \mathbb{R}$ (where $H \in (0, 1)$). In this case X extends to a measure in space if and only if $H \geq 1/2$.

Define the filtration $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$ as

$$\mathcal{G}_t := \sigma\{X((0, s] \times (u, v]) : s \leq t, u \leq v\} \vee \mathcal{N}, \quad t \geq 0,$$

where $\mathcal{N}$ denotes the set of $\mathbb{P}$-null sets. A standard argument based on the stationary independent increments in X shows that $\mathcal{G}$ is right-continuous. For fixed u and v the process $\{X((0, t] \times (u, v]) : t \geq 0\}$ is a $\mathcal{G}$-Wiener process.

We are now in a setting similar to [1] expect that X generally does not induce a martingale measure in the sense of [9] since $X((0, t] \times \cdot)$ does not extend to a measure. We shall see that one can nevertheless define a martingale integral with respect to X; moreover, we show that the set of integrands forms a complete space if and only if Λ_{dist} is complete.

For $t \geq 0$ let $Z_t = \{Z_t(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ denote the random spectral measure of $X((0, t]\times\cdot)$. The process $Z = \{Z_t(A) : t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$, being as earlier remarked also Gaussian, is then an orthogonal (and hence worthy) martingale measure in the sense that of [9]. (In fact, the only difference compared to [9] is that we use complex martingales rather than real ones.) That is, Z satisfies the following:

(a) For fixed $A \in \mathcal{B}_b(\mathbb{R}^d)$ the process $\{Z_t(A) : t \geq 0\}$ is a complex-valued Gaussian martingale with $Z_0(A) = 0$.

(b) For fixed $t \geq 0$ the mapping $A \mapsto Z_t(A)$ is σ-additive from $\mathcal{B}_b(\mathbb{R}^d)$ to $L^2_{\mathbb{C}}(\mathbb{P})$.

(c) For all $A, B \in \mathcal{B}_b(\mathbb{R}^d)$, $\langle Z_.(A), Z_.(B)\rangle = tF(A \cap B)$.

Here, for two complex square-integrable martingales M and N which are 0 at 0, $\langle M, N\rangle$ is the continuous complex process of bounded variation characterized by being 0 at 0 and $M\overline{N} - \langle M, N\rangle$ being a martingale.

To see that (a) is true, notice that for $s \leq t$, $\{Z_t(A) - Z_s(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is the random spectral measure of $X((s,t] \times \cdot)$. Since the latter process is independent of $\mathcal{G}_s$, this is by the last statement in Theorem 2.5 also true for the former implying the martingale property. Property (b) is simply the σ-additivity of the random spectral measure mentioned in Remark 2.2. Finally, (c) follows from the independent increments in Z and Definition 2.1(3). Using once more the last property in Theorem 2.5 it follows that

$$\mathcal{G}_t = \sigma\{Z_s(A) : s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\} \vee \mathcal{N}.$$

Denote by $\mathscr{P}$ the predictable σ-field on $\mathbb{R}_+ \times \Omega$. Set $\tilde{\mathscr{P}} := \mathscr{P} \times \mathcal{B}(\mathbb{R}^d)$ and

$$L^2_{\mathbb{C}}(Z) := \Big\{\varphi : \varphi \text{ is a } \tilde{\mathscr{P}}\text{-measurable mapping from } \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \text{ to } \mathbb{C} \\ \text{satisfying } \mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\varphi_t(x)|^2 \, dt\, F(dx)\Big] < \infty\Big\}.$$

This is clearly a complete space when equipped with the norm

$$\mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\varphi_t(x)|^2 \, dt\, F(dx)\Big]^{\frac{1}{2}}, \quad \varphi \in L^2_{\mathbb{C}}(Z).$$

Thus, also $\tilde{L}^2_{\mathbb{C}}(Z)$, the set of φ's in $L^2_{\mathbb{C}}(Z)$ satisfying $\varphi_t(x) = \overline{\varphi_t(-x)}$ for all (t, x) $\mathbb{P}$-a.s., is complete.

Standard martingale integration theory allows us to define stochastic integration with respect to Z, that is to construct the unique continuous linear mapping

$$(\varphi \in L^2_{\mathbb{C}}(Z)) \mapsto \int \varphi \, dZ \in L^2_{\mathbb{C}}(\mathbb{P})$$

determined by

$$\int \varphi \, dZ = c(Z_{s_2}((u,v]) - Z_{s_1}((u,v]))\mathbf{1}_G$$

if

$$\varphi_t(\omega, x) = c\mathbf{1}_F(\omega)\mathbf{1}_{(s_1,s_2]}(t)\mathbf{1}_{(u,v]}(x) \tag{4.1}$$

for some $c \in \mathbb{R}$, $s_1 < s_2$ (in $\mathbb{R}$), $G \in \mathcal{G}_{s_1}$ and $u \leq v$ (in $\mathbb{R}^d$), and

$$\Big\| \int \varphi \, dZ \Big\|^2_{L^2_{\mathbb{C}}(\mathbb{P})} = \mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^d} |\varphi_t(x)|^2 \, dtF(dx)\Big] \quad \text{for } \varphi \in L^2_{\mathbb{C}}(Z).$$

The real-valued integral processes correspond to integrands in $\tilde{L}^2_{\mathbb{C}}(Z)$. For $\varphi \in L^2_{\mathbb{C}}(Z)$ the process $M^\varphi_t := \int_0^t \int_{\mathbb{R}^d} \varphi \, dZ$, $t \geq 0$, is by construction a complex square-integrable martingale up to infinity which is 0 at 0; moreover,

$$\langle M^\varphi, M^\psi\rangle_t = \int_0^t \int_{\mathbb{R}^d} \varphi_s(y)\overline{\psi_s(x)} \, dsF(dy), \quad \text{for } \varphi, \psi \in L^2_{\mathbb{C}}(Z).$$

To define an integral with respect to X introduce the set

$$\Lambda_X = \Big\{\varphi\colon \mathbb{R}_+ \times \Omega \to \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d) : \varphi \text{ is predictable, } \mathcal{F}\varphi_t(\omega) \text{ is a function}$$
$$\text{for all } (\omega, t), \text{ and } \mathbb{E}\Big[\int_{\mathbb{R}^{1+d}} |\mathcal{F}\varphi_t(x)|^2\, dt\, F(dx)\Big] < \infty\Big\}.$$

On $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ we use the cylindrical σ-algebra $\sigma(\Psi \mapsto \Psi(\psi) : \psi \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d))$, that is, $\varphi\colon \mathbb{R}_+ \times \Omega \to \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ is predictable if and only if

$$\mathbb{R}_+ \times \Omega \ni (t, \omega) \mapsto \varphi_t(\omega)(\psi) \in \mathbb{R}$$

is predictable for all $\psi \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$. Furthermore, the following lemma shows that $\mathcal{F}\varphi_t(x)$ can be chosen bimeasurable making Λ_X well-defined.

Lemma 4.1. *Let $\varphi\colon \mathbb{R}_+\times\Omega \to \mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ be predictable such that $\mathcal{F}\varphi_t(\omega)$ is a function for all (ω, t). Then there exists a $\tilde{\mathscr{P}}$-measurable mapping $\Phi\colon \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{C}$ such that for all (t, ω), $\Phi(t, \omega, \cdot) = \mathcal{F}\varphi_t(\omega)(\cdot)$ λ_d-a.e.*

Proof. Since $\mathcal{F}$ maps $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$ continuously into $\mathscr{S}'_{\mathbb{C}}(\mathbb{R}^d)$

$$\Phi_\psi\colon \mathbb{R}_+ \times \Omega \to \mathbb{C}, \quad (t, \omega) \mapsto \int_{\mathbb{R}^d} (\mathcal{F}\varphi_t(\omega))(x)\psi(x)\, dx$$

is predictable, that is $\mathscr{P}$-measurable for all $\psi \in \mathscr{S}_{\mathbb{C}}(\mathbb{R}^d)$. Hence by a Monotone Class Lemma argument, Φ_ψ is predictable for all bounded measurable functions $\psi\colon \mathbb{R}^d \to \mathbb{C}$ with compact support. In particular, for all compact sets $K \subseteq \mathbb{R}^d$, the mapping $\mathbb{R}_+ \times \Omega \to L^1_{\mathbb{C}}(K)\colon (t, \omega) \mapsto \mathcal{F}\varphi_t(\omega)_{|K}$ is weakly measurable and hence (strongly) measurable by Pettis' theorem since $L^1_{\mathbb{C}}(K)$ is a separable Banach space. By applying [2], Exc. 1.75, there exists a $\tilde{\mathscr{P}}$-measurable mapping $\Phi_K\colon \mathbb{R}_+\times\Omega\times\mathbb{R}^d \to \mathbb{C}$ such that for all (t, ω), $\Phi(t, \omega, \cdot) = \mathcal{F}\varphi_t(\omega)(\cdot)$ $\lambda_{d|K}$-a.e., which shows the existence of Φ since K was arbitrary and $\mathbb{R}^d$ is a countable union of compact sets. ☐

Notice that Λ_X is Dalang's space $\overline{\mathcal{P}}$ considered in [1], page 9, with a few modifications: We consider the time interval $[0, \infty)$ rather than $[0, T]$ and as mentioned above our X does in general not induce a martingale measure. For $\varphi \in \Lambda_X$ define

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX := \int_0^\infty \int_{\mathbb{R}^d} (\mathcal{F}\varphi_t)(x)\, dZ.$$

By the above lemma, the integral is well-defined and maps Λ_X into $L^2_{\mathbb{R}}(\mathbb{P})$. On Λ_X define the norm $\|\cdot\|_{\Lambda_X}$ as

$$\|\varphi\|^2_{\Lambda_X} := \Big\| \int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX \Big\|^2_{L^2_{\mathbb{R}}(\mathbb{P})} = \mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^d} |(\mathcal{F}\varphi_t)(x)|^2\, dt\, F(dx)\Big].$$

The integral with respect to X just defined extends the simple integral since if φ is given by (4.1) then, by definition,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} \varphi\, dX &= c\mathbf{1}_G\Big(\int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(x)\, Z_{s_2}(dx) - \int_{\mathbb{R}^d} \mathcal{F}\mathbf{1}_{(u,v]}(x)\, Z_{s_1}(dx)\Big)\\ &= c1_F\big(X((0, s_2] \times (u, v]) - X((0, s_1] \times (u, v])\big) = c1_F X((s_1, s_2] \times (u, v]),\end{aligned}$$

where the second equality is due to (2.9). Moreover, if $\psi : \mathbb{R} \to \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$ is measurable, then the integral $\int_0^\infty \int_{\mathbb{R}^d} \psi \, dX$ exists if and only if $\psi \in \Lambda_X$, that is, $\mathcal{F}\psi_t$ is a function satisfying $\int_{\mathbb{R}^{1+d}} |\mathcal{F}\psi_t(x)|^2 dt F(dx) < \infty$. Thus, in view of the the first part of Theorem 4.2 below, this improves Theorem 3 in [1].

Theorem 4.2. *The real linear span of processes given by* (4.1) *is dense in* Λ_X*. Moreover, the following three statements are equivalent:*

(a) f satisfies (3.6),

(b) Λ_X equipped with the norm $\|\cdot\|_{\Lambda_X}$ is complete,

(c) to every $\mathcal{G}_\infty$-measurable random variable $V \in L^2_{\mathbb{R}}(\mathbb{P})$ there is a $\varphi \in \Lambda_X$ such that

$$V = \mathbb{E}[V] + \int_0^\infty \int_{\mathbb{R}^d} \varphi \, dX. \tag{4.2}$$

Proof. The first part: Using Lemma 3.3 and Lemma 4.1 it suffices to show that whenever $\varphi \in L^2_{\mathbb{C}}(Z)$ is of the form $\varphi_t(\omega, x) = \mathbf{1}_F(\omega) 1_{(s_1,s_2]}(t) \psi(x)$, where $\psi \in \tilde{L}^2_{\mathbb{C}}(F)$ then there is a sequence $(\varphi_n)_{n\geq 1}$ in $L^2_{\mathbb{C}}(Z)$ of the form $\varphi_{n,t}(\omega, x) = \mathbf{1}_F(\omega) 1_{(s_1,s_2]}(t) \mathcal{F}\psi_n(x)$ where $\psi_n \in \mathscr{E}$ (see (3.1)) approximating φ in Λ_X. However, this follows since by Lemma 3.6 the ψ_n's can be chosen such that $\mathcal{F}\psi_n$ approximate ψ in $L^2_{\mathbb{C}}(F)$.

(b) implies (a): Assume that Λ_X is complete. The set of φ given by $\varphi_t(\omega) = \mathbf{1}_{(0,1]}(t)\psi$, where $\psi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^d)$is such that $\mathcal{F}\psi$ is a function, forms a closed and hence complete subset of Λ_X that can be identified with Λ_{dist}. Hence, the latter is complete, showing by Theorem 3.5 that we have (3.6).

(a) implies (c): Assume (3.6). Every $\psi \in \tilde{L}^2_{\mathbb{C}}(Z)$ is given by $\psi_t(\omega, x) = \mathcal{F}\rho_t(\omega, x)$ for some $\rho \in \Lambda_X$. Indeed, by disregarding a null set if necessary we may and do assume that $\psi_t(\omega) \in \tilde{L}^2_{\mathbb{C}}(F)$ for all (t, ω). By Theorem 3.5, $\mathcal{F}(\Lambda_{\text{dist}}) = \tilde{L}^2_{\mathbb{C}}(F)$ so we can use $\rho_t(\omega) := \mathcal{F}^{-1}\psi_t(\omega)$. Hence, it suffices to show that V can be written as

$$V = \mathbb{E}[V] + \int_0^\infty \int_{\mathbb{R}^d} \psi \, dZ \quad \text{for some } \psi \in \tilde{L}^2_{\mathbb{C}}(Z).$$

In the following fix $n \geq 1$ and $A_1, \ldots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ satisfying $A_j \cap (-A_j) = \emptyset$ for all j and $(A_1 \cup (-A_1)) \cap \ldots \cap (A_n \cup (-A_n)) = \emptyset$. Define M^j and N^j as

$$M^j_t = Z_t(A_j) \quad \text{and} \quad N^j_t = Z_t(-A_j).$$

Decompose M^j_t in the real and imaginary parts as $M^j_t = M^{1,j}_t + iM^{2,j}_t$. The two processes $M^{1,j}$ and $M^{2,j}$ are independent Brownian motions by Remark 2.2. Thus, if V^j is any real-valued square integrable random variable measurable with respect to the σ-algebra generated by $(M^{1,j}, M^{2,j})$ there are two $(\mathcal{G}_t)$-predictable processes $\alpha^{1,j}, \alpha^{2,j} \colon \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying

$$V^j = \mathbb{E}[V^j] + \int_0^\infty \alpha^{1,j}_t \, dM^{1,j}_t + \int_0^\infty \alpha^{2,j}_t dM^{2,j}_t \quad \mathbb{P}\text{-a.s.}$$

(These two processes are even predictable in the filtration generated by $(M^{1,j}, M^{2,j})$.) Using that, by Definition 2.1(2), $N^j_t = \overline{M^j_t}$, it is readily seen that the right-hand side equals

$$\mathbb{E}[V^j] + \int_0^\infty \beta^j_t \, dM^j_t + \int_0^\infty \overline{\beta^j_t} \, dN^j_t,$$

where $\beta_t^j = (\alpha_t^{1,j} - i\alpha_t^{2,j})/2$. Thus,

$$V^j = \mathbb{E}[V^j] + \int_0^\infty \int_{\mathbb{R}^d} \varphi^j dZ,$$

where $\varphi_t^j(x) = \mathbf{1}_{A_j}(x)\beta_t^j + \mathbf{1}_{-A_j}(x)\overline{\beta_t^j}$. By the assumptions on the A_js, the martingales $M^{1,j}, M^{2,j}$, $j = 1, \ldots, n$, are orthogonal, so Itô's formula implies

$$\prod_{j=1}^n V^j = \prod_{j=1}^n \mathbb{E}[V^j] + \sum_{j=1}^n \int_0^\infty \int_{\mathbb{R}^d} \Big(\prod_{k:k\neq j} M_s^{\varphi^k}\Big)\varphi_s^j(y)\, Z(ds, dy).$$

This gives (4.2) when $V = \prod_{j=1}^n V^j$ from which the general case follows using the Monotone Class Lemma.

(c) implies (b): Follows from completeness of $L^2_{\mathbb{R}}(\mathbb{P})$. $\square$

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