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Abstract

Let $\{Z_n\}_{n\geq 0}$ be a random walk with a negative drift and i.i.d. increments with heavy-tailed distribution and let $M = \sup_{n\geq 0} Z_n$ be its supremum. Asmussen & Klüppelberg (1996) considered the behavior of the random walk given that $M > x$, for x large, and obtained a limit theorem, as $x \to \infty$, for the distribution of the quadruple that includes the time $\tau = \tau(x)$ to exceed level x, position Z_τ at this time, position $Z_{\tau-1}$ at the prior time, and the trajectory up to it (similar results were obtained for the Cramér-Lundberg insurance risk process). We obtain here several extensions of this result to various regenerative-type models and, in particular, to the case of a random walk with dependent increments. Particular attention is given to describing the limiting conditional behavior of τ. The class of models include Markov-modulated models as particular cases. We also study fluid models, the Björk-Grandell risk process, give examples where the order of τ is genuinely different from the random walk case, and discuss which growth rates are possible. Our proofs are purely probabilistic and are based on results and ideas from Asmussen, Schmidli & Schmidt (1999), Foss & Zachary (2002), and Foss, Konstantopoulos & Zachary (2007).

Keywords: Björk-Grandell model, Breiman's theorem, conditioned limit theorems, Markov-modulation, mean excess function, random walk, regenerative process, regular variation, ruin time, subexponential distribution

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1 Introduction

Let Z be a stochastic process with increments having a regenerative structure ([2]): there exist random times $T_0 = 0, T_1, T_2, \ldots$ splitting Z up into i.i.d. cycles

$$\bigl\{Z(t) - Z(0)\bigr\}_{0 \le t < R_1} = \bigl\{Z(t+T_0) - Z(T_0)\bigr\}_{0 \le t < R_1}, \bigl\{Z(t+T_k) - Z(T_k)\bigr\}_{0 \le t < R_{k+1}}, \ldots$$

with lengths $R_0 = T_0 = 0, R_1 = T_1 - T_0, R_2 = T_2 - T_1, \ldots$ (traditionally as in [2], one allows the first cycle to have a different distribution; we won't do this since our results are easily adapted to this setting). We will also assume $Z(0) = 0$. A main example we have in mind is the claims surplus process of an insurance company (accumulated claims minus premiums, cf. [3]). In this setting, $\tau = \tau(x) = \inf\{t : Z(t) > x\}$ is the ruin time with initial surplus x, $M = \sup_{t \ge 0} Z(t)$ is the maximal claims surplus, and

$$\mathbb{P}(\tau < \infty) = \mathbb{P}(M > x)$$

is the ruin probability, but τ and M are also of interest in many other contexts. For example, M could be the stationary waiting time in a single-server queue with i.i.d. service times whose input process is modulated by a Markov chain (say, this is an output process from another stationary single-server queue, see e.g. [10]).

Under heavy-tailed conditions implying, in particular, that the supremum

$$\sup_{0 \le t \le R_{k+1}} \bigl(Z(t+T_k) - Z_{T_k}\bigr)$$

over a typical regenerative cycle of the process increments has a heavy-tailed distribution, say F, on $[0, \infty)$ with mean $m_F < \infty$ whose *integrated tail* distribution

$$\overline{F^I}(x) = \min\Bigl\{1, \int_x^\infty \overline{F}(y)\,\mathrm{d}y\Bigr\}$$

is subexponential, it has been proved in a variety of settings that

$$\mathbb{P}(\tau < \infty) = \mathbb{P}(M > x) \sim b\overline{F^I}(x)\,, \qquad x \to \infty, \tag{1.1}$$

where b is a constant, thereby extending a classical result for random walks and the Cramér-Lundberg process due to (in alphabetical order) Borovkov, Cohen, Embrechts, Pakes, Veraverbeke, von Bahr. In particular, Asmussen, Schmidli & Schmidt (cf. [8]) proved the following (for background on subexponential distributions, see, e.g., [16], [3, X.1], or [20]):

Theorem 1.1. *In the regenerative setting, let*

$$\xi_k = Z(T_{k+1}) - Z(T_k)\,, \qquad \xi_k^* = \sup_{T_k \le t < T_{k+1}} Z(t) - Z(T_k)\,.$$

Assume that

$$\mathbb{P}(\xi_1 > x) \sim \mathbb{P}(\xi_1^* > x) \sim \overline{F}(x)\,, \qquad x \to \infty, \tag{1.2}$$

for some distribution F such that $\overline{F^I}$ is a subexponential tail and that $-a = \mathbb{E}\xi_1 < 0$. Then

$$\mathbb{P}(M > x) \sim \tfrac{1}{a}\overline{F^I}(x), \qquad x \to \infty.$$

As demonstrated by the examples in [8] (and later papers, of which Asmussen & Biard [4] is a recent instance), this result covers a large number of examples. Foss & Zachary [19] gave a similar result in the case of a modulated random walk.

The purpose of the present paper is to supplement Theorem 1.1 and the corresponding result from [19] with a description of the asymptotic behavior of τ given $M > x$, but in a more general setting that covers both scenarios (of regenerative structure and of modulation). Results of this type were given for the first time in Asmussen & Klüppelberg [5], assuming that Z is either the classical Cramér-Lundberg risk process, a Lévy process, or a discrete time random walk $Z_n = \xi_1 + \cdots + \xi_n$ with the ξ_k i.i.d. and having common distribution F and mean $-a < 0$. Note that there is a discrete time random walk imbedded in the regenerative setting: consider the process at times T_n.

In the random walk setting, the basic assumption of [5] is that there exists a function $e(x) \uparrow \infty$ such that, for any $t > 0$,

$$\lim_{x\to\infty} \frac{\overline{F^I}\big(x + te(x)\big)}{\overline{F^I}(x)} = \overline{G}(t) \tag{1.3}$$

for some distribution G. We assume in addition that function $e(x)$ is what could be called *weakly self-neglecting*, i.e.

$$\limsup_{x\to\infty} \frac{e\big(x + e(x)\big)}{e(x)} < \infty. \tag{1.4}$$

Both assumptions (1.3) and (1.4) are typical in the subexponential class, cf. [9]. In the regularly varying case $\overline{F}(x) = L(x)/x^\alpha$, a natural scaling is $e(x) = x$; then (1.4) is automatic and G is Pareto with $\overline{G}(t) = (1+t)^{-\alpha}$. For other subexponential distributions such as the lognormal and the heavy-tailed Weibull, one may take $e(x) = \mathbb{E}[X - x \,|\, X > x]$ and then G is standard exponential. Let W be a r.v. with distribution G. Then, with $\tau^{\mathrm{rw}}(x) = \inf\{n : Z_n > x\}$, it is shown in [5] (for later contributions in the same direction, see [21], [7]) that:

Theorem 1.2. *Given in the random walk setting that F^I is a subexponential distribution and that* (1.3) *holds, as $x \to \infty$, the conditional distribution of $\tau^{\mathrm{rw}}(x)/e(x)$ given $M > x$ converges to the distribution of W/a.*

Our first main result is the following extension. For a stochastic process with regenerative structure introduced earlier, for cycle i, let

$$t_i = t_i(x) = \inf\{t \le R_i : Z(t + T_{i-1}) - Z(T_{i-1}) > x\}$$

if $\xi_i^* > x$, and $t_i = R_i$, otherwise.

Theorem 1.3. *In the regenerative setting, assume in addition to the conditions of Theorem 1.1 and to conditions* (1.3)–(1.4) *that for any $y > 0$*

$$\mathbb{P}\big(t_1 > ye(x) \,\big|\, \xi_1 > x\big) = \mathrm{o}(1)\,, \qquad x \to \infty. \tag{1.5}$$

Then the conditional distribution of $\tau/e(x)$ given $M > x$ converges to the distribution of $\mu W/a$ where $\mu = \mathbb{E}R$.

The intuition behind Theorem 1.3 is the following. In [5], a number of supplementary results are given supporting the folklore that exceedance of level x occurs as result of one big ξ_k and that all the other ξ_k are 'typical'. In the regenerative setting, it is shown in [8] that the events $\tau < \infty$ and $\tau^{\mathrm{rw}} < \infty$ (where the random walk is the process observed at times T_n) essentially are equivalent, and that exceedance asymptotically occurs in cycle $\tau^{\mathrm{rw}}(x)$. Thus one expects by the LLN, by the 'typical' behavior before τ^{rw} and by (1.5) (which ensures that the length of the cycle in which ruin occurs can be neglected), that conditionally on $M > x$, $\tau/\tau^{\mathrm{rw}} \to \mu$. Given this, Theorem 1.2 then gives the desired conclusion.

The technical problem is to make this intuition precise in this and in more general settings. A difficulty is that conditioning on τ introduces some (presumably) small dependence between cycles $1, \ldots, \tau^{\mathrm{rw}} - 1$ as well as some bias in their distribution (expected to be small as well); this was realized in [4], with the consequent that some result there are heuristic. To overcome this difficulty, we present an approach to results of type Theorem 1.2 which is novel and combines the ideas from [5] and a sample-path analysis developed in [10, 17, 19]. The new approach is developed in Section 3 in the setting of random walks modulated by a regenerative process Y. For such a process, the asymptotics for $\mathbb{P}(M > x)$ is given in [17] (note that the setting allows Y to be a Markov process with a general state space, whereas [8] only can deal with the finite case). We supplement here with our second main result, Theorem 3.5, giving the conditional behaviour of τ. Compared to Theorem 1.3, it has the advantage that no conditions like (1.2) or (1.5) have to be verified, but it is also somewhat less general.

It is easy to construct examples where (1.5) fails as well as the conclusion of Theorem 1.3, see Section 7. The order of τ may remain $e(x)$ (then with a larger multiplier than $\mu W/a$) or be effectively larger. It is tempting to conjecture that any rate $h(x)$ with $h(x)/e(x) \to \infty$ may be attained. However, we shall show that $1/\overline{F}(x)$ is a critical upper bound.

2 Preliminaries

We need some notation.

Definition 2.1. *Let F be a distribution function and $\overline{F}(x) = 1 - F(x)$ its tail. Let $h(x)$ be a positive non-decreasing function. We say that F is h-*insensitive *if*

$$\overline{F}\big(x + h(x)\big) \sim \overline{F}(x), \qquad x \to \infty.$$

If (1.3) holds for F, one can take h as any function with $h(x) = \mathrm{o}\big(e(x)\big)$. One can find more about the h-insensitivity property in [20], Chapter 2.

Any subexponential distribution F is long-tailed, i.e. $\overline{F}(x + C) \sim \overline{F}(x)$, for any constant C. Therefore, for a subexponential F, one can choose a positive function $h \uparrow \infty$ such that F is also h-insensitive (clearly, the choice of h depends on F). If F is h-insensitive and if $0 \le g \le h$, then F is also g-insensitive.

Definition 2.2. *We say that two families of events A_x and B_x of positive probabilities, indexed by $x > 0$, are equivalent and write $A_x \sim B_x$, if $\mathbb{P}\big(A_x \Delta B_x\big) = \mathrm{o}\big(\mathbb{P}(A_x)\big)$, $x \to \infty$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.*

Note that if $A_x \sim B_x$, then also $\mathbb{P}(A_x) \sim \mathbb{P}(B_x)$.

3 Modulated random walk

Consider a discrete-time regenerative process $Y = \{Y_n, n \ge 1\}$ such that, for each n, Y_n takes values in some measurable space $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$. We say that a random walk $\{Z_n, n \ge 0\}$ defined by $Z_0 = 0$ and $Z_n = \xi_1 + \cdots + \xi_n$ for $n \ge 1$, is *modulated* by the process Y if

(i) conditionally on Y, the random variables $\xi_n, n \ge 1$, are independent;

(ii) for some family $\{F_y, y \in \mathcal{Y}\}$ of distribution functions such that, for each x, $F_y(x)$ is a measurable function of y, we have, for $n = 1, 2, \dots,$

$$\mathbb{P}(\xi_n \le x \,|\, Y) = \mathbb{P}(\xi_n \le x \,|\, Y_n) = F_{Y_n}(x) \quad \text{a.s.} \tag{3.1}$$

Let $M^{rw} = \sup_{n \ge 0} Z_n$. Under the conditions we give below, $Z_n \to -\infty$ a.s. as $n \to \infty$, and so the random variable M^{rw} is finite a.s.

The regenerative epochs of the modulating process Y are denoted by $0 = T_0 < T_1 < \dots$, with $R_k = T_k - T_{k-1}$. By definition, the cycles $\big(R_k, (Y_n, 0 < n \le T_k - T_{k-1})\big)$, $k \ge 1$, are i.i.d. We assume that

$$\mu = \mathbb{E}R_1 < \infty. \tag{3.2}$$

Let

$$\pi(B) = \frac{\mathbb{E}\sum_1^{R_1} \mathbf{1}(Y_n \in B)}{\mu}, \qquad B \in \mathcal{B}_{\mathcal{Y}}$$

be the stationary probability measure. We assume that each distribution $F_y, y \in \mathcal{Y}$ has a finite mean

$$a_y = \mathbb{E}[\xi_n \,|\, Y_n = y] = \int_{-\infty}^{\infty} x\, F_y(\mathrm{d}x) \in (-\infty, \infty), \tag{3.3}$$

and that

$$\text{the family of distributions } \{F_y, y \in \mathcal{Y}\} \text{ is uniformly integrable.} \tag{3.4}$$

In addition, we assume that this family of distributions satisfies the following additional assumptions with respect to some reference distribution F with finite mean and some measurable function $c : \mathcal{Y} \to [0, 1]$:

(C1) $\overline{F_y}(x) \le \overline{F}(x)$, for all $x \in \mathcal{R}, y \in \mathcal{Y}$,

(C2) $\overline{F_y}(x) \sim c(y)\overline{F}(x)$ as $x \to \infty$, for all $y \in \mathcal{Y}$,

(C3) $\kappa = \sup_{y \in \mathcal{Y}} a_y$ is finite and $a = -\int_{\mathcal{Y}} a_y \pi(\mathrm{d}y)$ is finite and strictly positive,

(C4) for some nonnegative $b > \kappa$,

$$\mathbb{P}(bR_1 > n) = \mathrm{o}\big(\overline{F}(n)\big), \quad n \to \infty.$$

Note that condition (C4) is redundant if $\kappa < 0$ – then one can take $b = 0$.

The following result is known (see Theorem 2.2 from [17] for a slightly more general version and also for discussion on importance of conditions).

Theorem 3.1. *Suppose that conditions* (3.1)–(3.4) *and* (C1)–(C4) *hold and that the distribution F^I is subexponential. Then $Z_n/n \to -a$ a.s. as $n \to \infty$; in particular, M^{rw} is an a.s. finite random variable. Furthermore,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(M^{rw} > x)}{\overline{F^I}(x)} = \frac{C}{a} \tag{3.5}$$

where $C = \int_{\mathcal{Y}} c(y)\pi(\mathrm{d}y) \in [0, 1]$ *and* $\overline{F^I}(x) = \min\Big(1, \int_x^\infty \overline{F}(z)\mathrm{d}z\Big)$.

The main idea in the proof of Theorem 3.1 that the supremum of the modulated random walk, M^{rw}, may be closely approximated by a sum of two *independent* random variables where one of them has a light-tailed distribution and the other is the supremum of an ordinary random walk with i.i.d. heavy-tailed increments with integrated tail distribution proportional to $\overline{F}^I$.

Based on Theorem 3.1, we obtain the following auxiliary result (see, e.g., Corollary 5 in [19] for an analogous statement in the case of an ordinary random walk).

Proposition 3.2. *Assume that the conditions of Theorem 3.1 hold. Assume that $C > 0$. Let the function $h(x) \uparrow \infty$, $h(x) = \mathrm{o}(x)$ be such that F^I is h-insensitive. By the strong law of large numbers (SLLN), one can choose a sequence $\varepsilon_n \downarrow 0$ such that*

$$\mathbb{P}\big(|Z_m + ma| \le m\varepsilon_m \ \ \forall m \ge n\big) \to 1, \qquad n \to \infty. \tag{3.6}$$

Then

$$\mathbb{P}\big(|Z_m + ma| \le m\varepsilon_m + h(x) \ \ \forall m\big) \to 1, \qquad x \to \infty. \tag{3.7}$$

Further, introduce the events:

$$K_{n,x} = \bigcap_{m\le n-1} \big\{|Z_m + ma| \le m\varepsilon_m + h(x)\big\};$$
$$A_{n,x} = \{\xi_n > x + na\}; \quad A^{\varepsilon,h}_{n,x} = \big\{\xi_n > x + na + n\varepsilon_n + h(x)\big\}.$$

Then the following equivalences hold:

$$\{M^{rw} > x\} \sim \bigcup_{n\ge1} \{M^{rw} > x\} \cap A_{n,x} \cap K_{n,x} \sim \bigcup_{n\ge1} \{M^{rw} > x\} \cap A^{\varepsilon,h}_{n,x} \cap K_{n,x} \tag{3.8}$$

$$\sim \bigcup_{n\ge1} \{M^{rw} > x\} \cap A_{n,x} \sim \bigcup_{n\ge1} \{M^{rw} > x\} \cap A^{\varepsilon,h}_{n,x} \tag{3.9}$$

$$\sim \bigcup_{n\ge1} A_{n,x} \sim \bigcup_{n\ge1} A^{\varepsilon,h}_{n,x} \tag{3.10}$$

and, therefore,

$$\mathbb{P}(M^{rw} > x) \sim \sum_{n\ge1} \mathbb{P}\left(A^{\varepsilon,h}_{n,x} \cap K_{n,x}\right) \sim \sum_{n\ge1} \mathbb{P}(A^{\varepsilon,h}_{n,x}) \sim \sum_{n\ge1} \mathbb{P}(A_{n,x}) \sim \frac{C}{a}\overline{F^I}(x).$$

Finally, since the function F^I is long-tailed, there exists a function $N = N(x) \to \infty$ such that $\overline{F}^I(x+aN) \sim \overline{F}^I(x)$ and, therefore, equivalences (3.8)–(3.10) *continue to hold if one replaces $n \geq 1$ by $n \geq N$.*

Proof. Indeed, (3.7) follows directly from (3.6) since $h(x) \to \infty$. Further, one can easily verify that

$$\bigcup_{n\geq 1} K_{n,x} \cap A_{n,x}^{\varepsilon,h} \subseteq \{M^{rw} > x\}.$$

The events $K_{n,x} \cap A_{n,x}^{\varepsilon,h}$ are disjoint and $\sum_{n\geq 1} \mathbb{P}(A_{n,x}^{\varepsilon,h} \setminus A_{n,x}) = o(\overline{F^I}(x))$, so by (3.7),

$$\mathbb{P}\Big(\bigcup_{n\geq 1} K_{n,x} \cap A_{n,x}^{\varepsilon,h}\Big) = \sum_{n\geq 1} \mathbb{P}(K_{n,x} \cap A_{n,x}^{\varepsilon,h}) \sim \sum_{n\geq 1} \mathbb{P}(K_{n,x} \cap A_{n,x}) \sim \sum_{n\geq 1} \mathbb{P}(A_{n,x}).$$

Since $\mathbb{P}(M^{rw} > x) \sim \frac{C}{a}\overline{F^I}(x)$ by Theorem 3.1 and since, by direct computations, $\sum_{n\geq 1} \mathbb{P}(A_{n,x}) \sim \frac{C}{a}\overline{F^I}(x)$, the results follow. □

A special case of a modulated random walk is an ordinary random walk with i.i.d. increments. Consider an auxiliary i.i.d. sequence $\{\xi_n^\sharp\}$ with distribution F and introduce the events

$$A_{n,x}^\sharp = \{\xi_n^\sharp > x + na\} \quad \text{and} \quad D_x^\sharp = \bigcup_{n\geq 1} A_{n,x}^\sharp.$$

Assume there exists a function $e(x) \uparrow \infty$ such that, for any $t > 0$, there exists a limit

$$\lim_{x\to\infty} \frac{\mathbb{P}(D^\sharp_{x+te(x)})}{\mathbb{P}(D_x^\sharp)} = \overline{G}(t) \tag{3.11}$$

with $\lim_{t\to\infty} \overline{G}(t) = 0$. Remark that condition (3.11) is nothing else than condition (1.3) since $\mathbb{P}(D_x^\sharp) \sim \sum_{n\geq 1} \mathbb{P}(A_{n,x}^\sharp) \sim \frac{1}{a}\overline{F}^I(x)$.

On the event $D_x^\sharp$, introduce random variable

$$\tau^\sharp \equiv \tau^\sharp(x) = \min\{n \geq 1 : \mathbf{1}(A_{n,x}^\sharp = 1)\}.$$

Then the following result holds:

Lemma 3.3. *Assume that the distribution F^I is subexponential and that* (1.3) *holds. Then the conditional distribution of $\tau^\sharp/e(x)$ given $D_x^\sharp$ converges to the distribution G (say, of the random variable W).*

Proof. Indeed,

$$\begin{aligned}\mathbb{P}(a\tau^\sharp/e(x) > t \,|\, \tau^\sharp < \infty) &= \mathbb{P}(\tau^\sharp > \frac{t}{a}e(x) \,|\, \tau^\sharp < \infty) \\ &\sim \frac{\sum_{n>\frac{t}{a}e(x)} \mathbb{P}(\xi_n^\sharp > x + na)}{\mathbb{P}(D_x^\sharp)} \sim \frac{\mathbb{P}(D^\sharp_{x+te(x)})}{\mathbb{P}(D_x^\sharp)} \to \overline{G}(t).\end{aligned}$$

□

We now return to the modulated random walk. On the event $\{M^{rw} > x\}$, we similarly introduce random variable

$$\tau^{rw} = \tau^{rw}(x) = \min\{n \geq 1 : Z_n > x\}.$$

Recall from Proposition 3.2 that

$$\{M^{rw} > x\} \sim D_x = \bigcup_{n\geq 1} A_{n,x}.$$

Then, by Lemma 3.3, we obtain:

Lemma 3.4. *Under the assumptions of Theorem* 3.1 *with* $C > 0$ *and* (1.3)*, the conditional distribution of* $a\tau^{rw}/e(x)$*, conditioned on* $\{M^{rw} > x\}$*, converges to the distribution* G*.*

Indeed, the equivalence

$$\mathbb{P}(a\tau^\sharp > te(x) \,|\, \tau^\sharp < \infty) \sim \mathbb{P}(a\tau^{rw} > te(x) \,|\, \tau^{rw} < \infty)$$

holds since we may represent conditional probabilities as ratios of probabilities where the both numerators and the both denominators are pairwise asymptotically proportional, with the same coefficient C.

Further, by Proposition 3.2 and Lemma 3.4, one may deduce the following result.

Theorem 3.5. *Assume* (1.3) *to hold. Then, under the conditions of Theorem* 3.1 *and the assumption* $C > 0$*, the distribution of*

$$\left(\frac{a\tau^{rw}}{e(x)}, \frac{Z_{\tau^{rw}-1}}{e(x)}, \max_{0\leq m\leq \tau^{rw}-1} \frac{|Z_m + ma|}{\tau^{rw}}, \frac{Z_{\tau^{rw}} - x}{e(x)}\right), \tag{3.12}$$

conditioned on $\{M^{rw} > x\}$*, converges to the distribution of* $(W, -W, 0, W')$ *where* W *and* W' *have the same distribution* G *and, for any positive* u *and* v*,*

$$\mathbb{P}(W > u, W' > v) = \mathbb{P}(W > u + v). \tag{3.13}$$

This result is a complete analogue of Theorem 1.1 from Asmussen & Klüppelberg (1996) which was obtained in the case of an ordinary random walk.

Proof. We have already proved the convergence of the first component in (3.12). From that and from (3.7), one may conclude that

$$\mathbb{P}\big(|Z_m + ma| \leq m\varepsilon_m + h(x) \qquad \forall m < \tau^{rw} \,|\, \tau^{rw} < \infty\big) \to 1.$$

Then the convergence of the second and third components in (3.12) follows if we take $h(x) \to \infty$ such that $h(x) = \mathrm{o}\big(e(x)\big)$.

It remains to show the convergence of the last component in (3.12). This follows from

$$\big\{Z_{\tau^{rw}} - x > ve(x)\big\} \sim \bigcup_{n\geq 1}\{\xi_n - x - na > ve(x)\},$$

$$\begin{aligned}\mathbb{P}\big(Z_{\tau^w} - x > ave(x)\big) &\sim \mathbb{P}\Big(\bigcup_{n\geq 1}\{\xi_n - x - na > ve(x)\}\Big)\\ &\sim \sum_{n\geq 1}\mathbb{P}\big(\xi_n - x - na > ve(x)\big) = \sum_{n\geq 1}\mathbb{P}\big(\xi_n > x + ve(x) + na\big) \sim \mathbb{P}(D_{x+ve(x)})\end{aligned}$$

(here we assume that $Z_{\tau^{rw}} = -\infty$ if $\tau^{rw} = \infty$). Similarly, equality (3.13) follows since

$$\begin{aligned}\big\{a\tau^{rw} > ue(x), Z_{\tau^{rw}} - x > ve(x)\big\} &\sim \bigcup_{n>ue(x)} \{\xi_n > x + na, \xi_n - na - x > ve(x)\} \\ &= \bigcup_{n>ue(x)} \{\xi_n > x + na + ve(x)\},\end{aligned}$$

and then

$$\begin{aligned}&\mathbb{P}\big(a\tau^{rw} > ue(x), Z_{\tau^{rw}} - x > ve(x)\big) \sim \sum_{n>ue(x)} \mathbb{P}\big(\xi_n > x + na + ve(x)\big) \\ &\quad \sim C \sum_{n>ue(x)} \mathbb{P}\big(\xi_n^\sharp > x + na + ve(x)\big) = C \sum_{n\ge 1} \mathbb{P}\big(\xi_n^\sharp > x + na + (v+u)e(x)\big) \\ &\quad \sim C\mathbb{P}(D^\sharp_{x+(u+v)e(x)}) \sim \mathbb{P}(D_{x+(u+v)e(x)}).\end{aligned}$$

□

4 Continuous-time modulated regenerative processes

We consider now a continuous-time process $Z(t)$ introduced in Section 1 and assume that, more generally, it is a *regenerative process which is modulated by a discrete-time regenerative process* Y. This means that (compare with the previous Section!)

(i) conditionally on Y, the random elements $V_{k+1} = \{Z(t) - Z(T_k), 0 \le t \le R_{k+1}\}$ are independent;

(ii) for any n,

$$\mathbb{P}(V_n \in \cdot \,|\, Y) = \mathbb{P}(V_n \in \cdot \,|\, Y_n) \quad \text{a.s.} \tag{4.1}$$

Let further, as in Theorem 1.1,

$$\xi_k = Z(T_{k+1}) - Z(T_k)\,, \qquad \xi_k^* = \sup_{T_k \le t < T_{k+1}} Z(t) - Z(T_k)$$

and assume conditions of Theorem 3.1 and (1.3) to hold. Then the statements of Theorems 3.1 and 3.5 hold too.

Note that $M \equiv \sup_{t\ge 0} Z(t)$ may be also represented as $M = \sup_{n\ge 0}(\xi_1 + \cdots + \xi_n + \xi^*_{n+1})$. Then we have the following result:

Theorem 4.1. *Assume that conditions of Theorem* 3.1 *and* (1.3) *hold, and that* $C > 0$ *in Theorem* 3.1. *Assume further that, for all* $y \in \mathcal{Y}$,

$$\mathbb{P}(\xi_n^* > x \,|\, Y = y) \sim \overline{F}_y(x) \quad a.s.$$

and that

$$\mathbb{P}(\xi_n^* > x) \le c\overline{F}(x),$$

for some $c \geq 1$ and all x. Then, as $x \to \infty$,

$$\{M > x\} \sim \{M^{rw} > x\} \sim \bigcap_{n\geq 1} K_{n,x} \cap A_{n,x} \tag{4.2}$$

and, for $\widehat{\tau}^{rw} \equiv \widehat{\tau}^{rw}(x) = \min\{n \geq 1 : Z_{n-1} + \xi_n^* > x\}$,

$$\mathbb{P}(\tau^{rw} = \widehat{\tau}^{rw} \mid M > x) \to 1, \quad x \to \infty \tag{4.3}$$

and

$$\mathbb{P}(\tau^{rw} = \widehat{\tau}^{rw} \mid M^{rw} > x) \to 1, \quad x \to \infty. \tag{4.4}$$

Therefore the statement of Theorem 3.5 continues to hold if one replaces in (3.12) *τ^{rw} by $\widehat{\tau}^{rw}$ and then $Z_{\tau^{rw}}$ by $Z_{\widehat{\tau}^{rw}-1} + \xi^*_{\widehat{\tau}^{rw}}$.*

The proof of Theorem 4.1 follows from routine minor modification of calculations from the previous Section.

5 Proof of Theorem 1.3

Now we assume that the process $Z(t)$ is regenerative. This means that Y is a constant and, as a corollary, that conditions (3.1)-(3.4) and (C1)-(C4) are redundant. Also, the ξ_n are i.i.d. in this case and, therefore, we may take $\tau^\sharp = \min\{n : \xi_n > x + na\}$.

Let $\mathbb{P}^{(x)}$ denote the conditional probability given $\tau < \infty$, write $T_n = R_1 + \cdots + R_n$ and recall the definition of $\widehat{\tau}^{rw}$ from Theorem 4.1 Note that since the events $\tau < \infty$ and $\widehat{\tau}^{rw} < \infty$ coincide, and are equivalent to each of the events $\tau^{\mathrm{rw}} < \infty$ and $\tau^\sharp < \infty$ (see Lemma 3.4), we may use either of the four in conditioning arguments.

The proof of Theorem 1.3 is a straightforward combination of Theorem 3.5, Theorem 4.1 and of the following two lemmas. Both use the fact, implicit in [8] and also a consequence of (4.3) and (4.4) of Theorem 4.1, that

$$\mathbb{P}^{(x)}\big(\tau \in [T_{\tau^{\mathrm{rw}}-1}, T_{\tau^{\mathrm{rw}}})\big) \to 1\,, \qquad x \to \infty \tag{5.1}$$

since asymptotically

$$\{\tau \in [T_{\tau^{\mathrm{rw}}-1}, T_{\tau^{\mathrm{rw}}})\} \subseteq \{\tau^{rw} = \widehat{\tau}^{rw}\}.$$

Lemma 5.1. *$T_{\tau^{\mathrm{rw}}-1}/e(x) \to \mu W/a$ in $\mathbb{P}^{(x)}$-distribution.*

Proof. We use the representation

$$\frac{T_{\tau^{\mathrm{rw}}-1}}{e(x)} = \frac{T_{\tau^{\mathrm{rw}}-1}}{\tau^{\mathrm{rw}}} \cdot \frac{\tau^{\mathrm{rw}}}{e(x)}. \tag{5.2}$$

Choose $N = N(x) \to \infty$ from Proposition 3.2. The first fraction in the LHS of (5.2) converges to μ in $\mathbb{P}^{(x)}$ probability since, by the independence of $A_{n,x}$ and T_{n-1} and by the SLLN,

$$\begin{aligned}\big\{\big|T_{\tau^{\mathrm{rw}}-1}/\tau^{\mathrm{rw}} - \mu\big| \leq \varepsilon,\, \tau^{\mathrm{rw}} < \infty\big\} &\sim \bigcup_{n\geq 1}\big\{\big|T_{n-1}/n - \mu\big| \leq \varepsilon\big\} \cap A_{n,x} \\ &\sim \bigcup_{n\geq N}\big\{\big|T_{n-1}/n - \mu\big| \leq \varepsilon\big\} \cap A_{n,x} \sim \bigcup_{n\geq N} A_{n,x} \sim \bigcup_{n\geq 1} A_{n,x} \sim \{\tau^{\mathrm{rw}} < \infty\}.\end{aligned}$$

Then the second fraction converges to W/a by Theorem 3.5, and the result follows. $\square$

Recall that $\tau = \sum_1^{\widehat{\tau}^{rw}-1} R_i + t_{\widehat{\tau}^{rw}}$.

Lemma 5.2. *Under the conditions of Theorem* 1.3, $t_{\widehat{\tau}^{rw}}/e(x) \to$ *in* $\mathbb{P}^{(x)}$*-probability.*

Proof. By Theorem 3.5 and Theorem 4.1, for any $\delta > 0$, one can choose $K > 0$ such that

$$\mathbb{P}\big(\widehat{\tau}^{rw}/e(x) > K\big) \le \delta/2,$$

for all x large enough. Then, for any $y > 0$,

$$\mathbb{P}^{(x)}\big(t_{\widehat{\tau}^{rw}}/e(x) > y\big) \le \big(1 + \mathrm{o}(1)\big) \frac{\sum_{n \le Ke(x)} \mathbb{P}\big(t_n > ye(x), \xi_n > x + na\big)}{\overline{F}^I(x)/a} + \delta$$

If (1.4) holds, then also $\limsup_{x\to\infty} e\big(x+ke(x)\big)/e(x) < \infty$, for any $k > 0$. Therefore, the latter sum in the numerator is equivalent to

$$\sum_{n \le Ke(x)} \mathbb{P}\big(t_1 > ye(x + na) \,\big|\, \xi_1 > x + na\big)\mathbb{P}(\xi_1 > x + na) = \mathrm{o}(1)\overline{F}^I(x).$$

Then we complete the proof by letting first $x \to \infty$ and then $\delta \to 0$. $\square$

Combining this with the statements of Theorem 4.1 and Lemma 5.1 completes the proof of Theorem 1.3.

6 Examples

Example 6.1. In the setting of Section 3, we may assume that $Z(t)$ is a right-continuous piecewise constant process with $Z(n) = Z_n$. We shall show here that, under a natural extra assumption, Theorem 1.3 holds for this model as well. Note that because of the result of [17], we need not verify the conditions of Theorem 1.1 (which may be messy); all that is needed is to establish (1.5).

Assume in addition that the distribution of the cycle length, R, has a lighter tail than $\overline{F}(x)$, in the following strong sense: there exists constant $c > 1$ such that

$$\mathbb{P}(cR > x) = \mathrm{o}\big(\overline{F}(x)\big), \quad x \to \infty. \tag{6.1}$$

Let ξ be the increment over the cycle, $\xi = \sum_1^R \xi_i$. For (1.5) to hold, it suffices to show that, for any $y > 0$,

$$\mathbb{P}(R > yx,\, \xi > x) = \mathrm{o}\big(\overline{F}(x)\big)\,, \qquad x \to \infty, \tag{6.2}$$

where F is the reference distribution. For any fixed x_0 and for $x \ge x_0$, as $x \to \infty$,

$$\begin{aligned} \mathbb{P}(R > yx,\, \xi > x) &= \mathbb{E}(\mathbb{P}(\xi > x \,|\, R, Y_0, \dots, Y_R)\mathbf{1}(Y > yx)) \\ &\le \sum_{k \ge xy} \overline{F^k}(x)\mathbb{P}(R = k) \qquad\qquad \text{(by property (C2))} \\ &\le \sum_{k \ge x_0 y} \overline{F^k}(x)\mathbb{P}(R = k) \sim \overline{F}(x)\mathbb{E}[R; R > x_0 y] \end{aligned}$$

where, under assumption (6.1), the last equivalence follows from [14], Theorem 1. By letting $x_0 \to \infty$, we obtain (6.2). $\square$

Example 6.2. The Björk-Grandell model ([11]) is a regenerative risk process, such that in addition to the cycle length R also the rate Λ of claims arrivals within a cycle is random. All claims are i.i.d. with distribution H, with mean m, and independent of (R, Λ), and there is a constant rate 1 of premium inflow. The infinite horizon ruin probabilities are discussed in [11] for the light-tailed case and in [8] for the heavy-tailed case. As noted in [8], heavy tails of ξ may occur in at least three ways: (i) F is heavy-tailed; (ii) Λ is heavy-tailed; (iii) R is heavy-tailed for sufficiently large values of Λ. Under some (not necessarily minimal) assumptions, we shall give arguments to identify the limiting conditional behavior of τ.

For the following estimates, one may keep in mind that

$$\xi = X_1 + \cdots + X_{M_{R\Lambda}} - R \tag{6.3}$$

with M an independent Poisson process at unit rate. For the tail asymptotics of ξ, the $-R$ term may often be neglected (see [4] for some preliminary discussion and [1] for a more complete picture). Also, with light-tailed claims one may frequently approximate $X_1 + \cdots + X_{M_{R\Lambda}}$ by $mR\Lambda$; the relevant large deviations arguments are given in detail in [8] and will not be repeated here.

Consider first case (i) with R, Λ both light-tailed. Using (6.3) and an independence result from [4], it is standard that

$$\mathbb{P}(\xi > x) \sim \mathbb{E}(R\Lambda)\overline{F}(x)\,.$$

By a classical inequality due to Kesten, to each $\delta > 0$ there is a $C_\delta < \infty$ such that $\mathbb{P}(X_1 + \cdots + X_n > x) \le C_\delta \mathrm{e}^{n\delta}\overline{F}(x)$ for all n. With

$$p = \mathbb{P}(M_{R\Lambda} = 1) > 0\,, \qquad q = p\mathbb{P}(X_1 > x + R) \sim p\overline{F}(x)\,,$$

we get

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{sR} \,|\, \xi > x] &= \frac{\mathbb{E}[\mathrm{e}^{sR};\, \xi > x]}{\mathbb{P}(\xi > x)} \le \frac{1}{q}\mathbb{E}\big[\mathrm{e}^{sR};\, X_1 + \cdots + X_{M_{R\Lambda}} > x\big] \\ &\le \frac{1}{q}\mathbb{E}\big[\mathrm{e}^{sR} C_\delta \mathrm{e}^{\delta M_{R\Lambda}}\overline{F}(x)\big] \sim \frac{C_\delta}{p}\mathbb{E}\big[\mathrm{e}^{sR}\mathrm{e}^{R\Lambda(\mathrm{e}^\delta - 1)}\big]\,.\end{aligned}$$

Taking s, δ small enough, this expression is finite, and its independence of x together with $e(x) \to \infty$ then gives (1.5) and the conclusion of Theorem 1.3.

Consider next case (ii) with F light-tailed and (R, Λ) satisfying $\mathbb{P}(\Lambda > x) \sim x^{-\alpha}$ with $\alpha > 1$ and $\mathbb{E}R^{\alpha'} < \infty$ for some $\alpha' > \alpha$. Then by Breiman's theorem ([12], [13], [15]), $\mathbb{P}(mR\Lambda > x) \sim cx^{-\alpha}$ where $c = m^\alpha \mathbb{E}R^\alpha$. By a large deviations argument,

$$\mathbb{P}(M_{R\Lambda} > x) = \mathbb{P}(R\Lambda > x) + \mathrm{O}(\mathrm{e}^{-\varepsilon_1 x})$$

for some $\varepsilon_1 > 0$. A further large deviations given in [6] then shows that

$$\big\{X_1 + \cdots + X_{M_{R\Lambda}} > x\big\}\Delta\{mR\Lambda > x\} = A(x)$$

where $\mathbb{P}A(x) = \mathrm{O}(\mathrm{e}^{-\varepsilon x})$ for some $\varepsilon > 0$. In particular $X_1 + \cdots + X_{M_{R\Lambda}}$ has asymptotic tail $cx^{-\alpha}$. Hence so has $\xi = X_1 + \cdots + X_{M_{R\Lambda}} - R$ (see [4]; note that this is non-trivial due to dependence). Let $\alpha < \alpha'' < \alpha'$ and let R^* be a r.v. with distribution

$$\mathbb{P}(R^* \in \mathrm{d}t) = \mathbb{E}\big[R^{\alpha''}; R \in \mathrm{d}t\big]/\mathbb{E}(R^{\alpha''})\,.$$

Then

$$\begin{aligned}\mathbb{E}[R^{\alpha''}\,|\,\xi > x] &\sim \frac{\mathbb{E}[R^{\alpha''};\,\xi > x]}{cx^{-\alpha}} \le \frac{1}{cx^{-\alpha}}\mathbb{E}\big[R^{\alpha''};\,X_1+\cdots+X_{M_{R\Lambda}} > x\big]\\ &= \frac{1}{cx^{-\alpha}}\mathbb{E}\big[R^{\alpha''};mR\Lambda > x\big] + \mathrm{O}(\mathrm{e}^{-\varepsilon x})\\ &= \frac{1}{cx^{-\alpha}}\mathbb{E}R^{\alpha''}\mathbb{P}(R^*\Lambda > x/m) + \mathrm{O}(\mathrm{e}^{-\varepsilon x}).\end{aligned}$$

Another application of Breiman's theorem justified by the choice of α'' shows that $R^*\Lambda$ has a distribution tail asymptotically proportional to $x^{-\alpha}$. Hence $\mathbb{E}[R^{\alpha''}\,|\,\xi > x]$ stays bounded as $x \to \infty$, and arguing as above gives (1.5) and the conclusion of Theorem 1.3.

In contrast, the behavior in case (iii) is different, see Section 7. □

Example 6.3. Let Z be a two-stage fluid model, where a cycle R is composed of two stages such that the first has deterministic length a_1 and the second a random length R_2 with a subexponential distribution F with mean $a_2 < a_1$. In stage 1, Z decreases deterministically at rate 1 and in stage 2, Z increases deterministically at rate 1 (thus $-a = a_2 - a_1 < 0$). Clearly, $\xi > x$ occurs if and only $R_2 > x + a_1$. Thus $\mathbb{P}(\xi > x) = \overline{F}(x + a_1) \sim \overline{F}(x)$ and given $\xi > x$, R is at least x. Since $e(x) = \mathrm{O}(x)$ in all examples, condition (1.5) can not hold and more precisely, given $\xi > x$, R is of order $x + e(x)$. Therefore τ is of order $e(x)$ in the regularly varying case but with a larger multiplier than μW, and of order $x \gg e(x)$ for other subexponential distributions.

Note that this example shows that the regenerative setting is more flexible than the Markov additive one: if one considers the discrete time analogue, the increments in each Markov stage are bounded and there is thus no version of condition (C2) of Section 3 with F heavy-tailed. On the other hand, conditions may be easier to verify in the Markov additive setting. □

7 Different growth rates

The following result is straightforward given Lemma 5.1 and the proof of Theorem 1.3. For simplicity (to avoid distinction between x and $e(x)$) we state it only for the regularly varying case where the r.v. W in Theorem 1.2 is Pareto:

Corollary 7.1. *Assume that F in* (1.2) *is regularly varying and that instead of Condition* (1.5) *we have*

$$\mathbb{P}\big(t_1/e^*(x) > y\,\big|\,\xi > x\big) \to \mathbb{P}(W^* > y) \qquad \text{for all } y, \tag{7.1}$$

some function $e^*(x)$ with $\liminf e^*(x)/x > 0$ and some r.v. W^*.
(i) *If $e^*(x) \sim dx$ for some d, then*

$$\frac{\tau}{x} \to W\mu/a + d(1+W)W^* \qquad \text{in } \mathbb{P}^{(x)}\text{-distribution;}$$

(ii) *if $e^*(x)/x \to \infty$, then*

$$\frac{\tau}{e^*\big(x(1+W)\big)} \to dW^* \qquad \text{in } \mathbb{P}^{(x)}\text{-distribution}$$

with W, W^* independent in both (i) *and* (ii)*, and W independent of τ in* (ii)*. In particular, if $e^*(x) \sim dx^\beta$ with $\beta > 1$, then $\tau/x^\beta \to d(1+W)^\beta W^*$ in $\mathbb{P}^{(x)}$-distribution.*

Proof. The asymptotic $\mathbb{P}^{(x)}$-distribution of τ is the same as the symptotic distribution of $\sum_1^{\widehat{\tau}^{\mathrm{rw}}-1} R_i + t_{\widehat{\tau}^{\mathrm{rw}}}$. Here $\sum_1^{\widehat{\tau}^{\mathrm{rw}}-1} R_i/x \to \mu W/a$ in $\mathbb{P}^{(x)}$-distribution (Lemma 5.1). More generally,

$$\frac{1}{x}\Bigl(\sum_1^{\widehat{\tau}^{\mathrm{rw}}-1} R_i, \sum_1^{\widehat{\tau}^{\mathrm{rw}}-1} \xi_i\Bigr) \to (\mu W/a, W)\,.$$

Given $W = w$, $\xi_{\widehat{\tau}^{\mathrm{rw}}}$ will asymptotically have to exceed $xw + x$, implying that $t_{\widehat{\tau}^{\mathrm{rw}}}/e^*\bigl(x(w+1)\bigr) \to W^*$ and the conclusion of (i) since the limit W^* does not depend of w. For (ii), just note that in this case $\sum_1^{\widehat{\tau}^{\mathrm{rw}}-1} R_i$ may be neglected. □

We next first give an example of $e^*(x) \sim dx$ and thereafter some discussion of what may happen if $e^*(x)/x \to \infty$.

Example 7.2. We return to the Björk-Grandell model in case (iii). Here one expects that given $\Lambda = \lambda$, the surplus process $\sum_1^{N(t)} U_i - t$ can be approximated by $\lambda m t - t$, and this is confirmed by the large deviations bounds in [8]. Therefore the behavior should be like a fluid model with heavy-tailed on periods, so that the exceedance time of x within a cycle should be of order x and accordingly makes a genuine contribution to τ.

We next verify this statement and and make it more precise, assuming as in [8] that claims are light-tailed and independent of (R, Λ), that for some $\lambda_0 > 1/\mu$

$$\begin{aligned} \mathbb{P}\bigl(R > t \,\big|\, \Lambda = \lambda\bigr) &= \overline{F}(t)\,, \qquad \lambda > \lambda_0\,,\\ \mathbb{P}\bigl(R > t \,\big|\, \Lambda = \lambda\bigr) &\le \overline{G}(t)\,, \qquad \lambda \le \lambda_0, \end{aligned}$$

for some regularly varying F with $\overline{F}(t) = L(t)/t^\alpha$ (L slowly varying), some G satisfying $\overline{G}(t) = \mathrm{o}\bigl(\overline{F}(t)\bigr)$, and the following regularity condition:

$$\sup_{x \ge x_0} \frac{L(x/y)}{L(x)} \le g(y) \tag{7.2}$$

for all $y > 0$, some $x_0 > 0$ and some function $g(y)$ with $\mathbb{E}\bigl[\Lambda^\alpha g(\Lambda)\bigr] < \infty$.

It is then shown in [8] that the conditions of Theorem 1.1 are satisfied and that

$$\psi(x) \sim c_1 \overline{F}(x) \tag{7.3}$$

where

$$c_1 = \frac{c}{(\alpha - 1)\bigl[\mathbb{E}R - m\mathbb{E}(\Lambda R)\bigr]}\,, \qquad c = \mathbb{E}\bigl[(\Lambda m - 1)^\alpha; \Lambda > \lambda_0\bigr]\,.$$

This depends on the estimate

$$\mathbb{P}(\xi > x) \sim c\overline{F}(x)\,.$$

As preparation for the study of the ruin time, we first recall the proof of (7.2). That the event $\xi > x$ occurs is by the LD arguments equivalent to $R(\Lambda m - 1) > x$, and so

$$\mathbb{P}(\xi > x) \sim \int_{\lambda_0}^{\infty} f_\Lambda(\lambda)\overline{F}\big(x/(\lambda m-1)\big)\,\mathrm{d}\lambda \tag{7.4}$$

$$= \int_{\lambda_0}^{\infty} f_\Lambda(\lambda)(\lambda m-1)^{\alpha}\frac{L\big(x/(\lambda m-1)\big)}{x^{\alpha}}\,\mathrm{d}\lambda \tag{7.5}$$

$$\sim \int_{\lambda_0}^{\infty} f_\Lambda(\lambda)(\lambda m-1)^{\alpha}\frac{L(x)}{x^{\alpha}}\,\mathrm{d}\lambda = c\overline{F}(x)\,, \tag{7.6}$$

where the last $\sim$ follows by dominated convergence justified by (7.2). If $\xi > x, \tau \le xt$ is to occur, we need in additional $x/(\lambda m - 1) \le xt$, and so by the same dominated convergence argument

$$\mathbb{P}(\xi > x, \tau \le xt) \sim \int_{\lambda_0 \vee (1/t+1)/m}^{\infty} f_\Lambda(\lambda)\overline{F}\big(x/(\lambda m-1)\big)\,\mathrm{d}\lambda \tag{7.7}$$

$$\sim \int_{\lambda_0 \vee (1/t+1)/m}^{\infty} f_\Lambda(\lambda)(\lambda m-1)^{\alpha}\frac{L(x)}{x^{\alpha}}\,\mathrm{d}\lambda = c\overline{F}(x)W^*(t)\,, \tag{7.8}$$

where W^* is the distribution with c.d.f.

$$\mathbb{P}(W^* \le t) = \begin{cases} \dfrac{1}{c}\displaystyle\int_{(1/t+1)/m}^{\infty} f_\Lambda(\lambda)(\Lambda m-1)^{\alpha}\,\mathrm{d}\lambda & t \le 1/(\lambda_0 m - 1) \\ 1 & t > 1/(\lambda_0 m-1)\,. \end{cases}$$

From Corollary 7.1 we therefore conclude that $\tau(x)/x \to W^*(1+W)$ in $\mathbb{P}^{(x)}$-distribution. □

We proceed to discussing when τ may grow at larger rates than $e(x)$ and how fast the rate may be. If $t_x = \mathbb{E}[R \,|\, \xi = x] \to \infty$ faster than $e(x)$, one expects R given $\xi > x$ (and hence often τ) to grow at a faster rate than $e(x)$. At a first sight, one could conjecture that any rate is possible. This is, however, not possible because of the requirement $\mathbb{E}R < \infty$. Suppose, for example, that F is a discrete subexponential distribution with point probabilities $f_x = \mathbb{P}(\xi = x) \sim c_1/x^{\alpha+1}$. Assuming $t_x \sim c_2 x^{\beta}$, we then get

$$\infty > \mathbb{E}R = \sum_0^{\infty} t_x f_x \approx \sum_0^{\infty} c_2 x^{\beta}\, c_1/x^{\alpha+1}\,,$$

implying $\beta < \alpha$. The following result gives the more precise upper bound $c/\overline{F}(x)$ and is more satisfying by being in terms of the growth rate of τ rather than expected values:

Theorem 7.3. *Let $F(x) = \mathbb{P}(\xi \le x)$ be a discrete subexponential distribution with point probabilities $f_0, f_1, \ldots$ and h a function with $h(x)/e(x) \to \infty$. Assume that $\mathbb{P}\big(R > \varepsilon h(x) \,\big|\, \xi > x\big) \ge \delta$ for some $\varepsilon, \delta > 0$ and all large x. Then $h(x) \le c/\overline{F}(x)$ for some constant c.*

Proof. Define $t(x)$ as above and let

$$k(x) = \mathbb{E}[R \,|\, \xi > x] = \frac{1}{\overline{F}(x)}\big(t_{x+1}f_{x+1} + t_{x+2}f_{x+2} + \cdots\big) \tag{7.9}$$

Multiplying by $\overline{F}(x)$ and subtracting the resulting equation with x replaced by $x+1$, it follows that

$$t_{x+1} = \frac{1}{f_{x+1}}\big(k(x)\overline{F}(x) - k(x+1)\overline{F}(x+1)\big)\,.$$

This expression needs to be positive which gives $k(x+1)/k(x) < \overline{F}(x)/\overline{F}(x+1)$ and, multiplying from $x = 1$ to $y-1$,

$$k(y) < k(1)\overline{F}(1)\frac{1}{\overline{F}(y)}\,. \tag{7.10}$$

However, clearly $k(x) = \mathbb{E}[R \,|\, \xi > x] \geq \varepsilon\delta h(x)$, from which we conclude

$$h(x) \leq \frac{k(x)}{\varepsilon\delta} < \frac{k(1)\overline{F}(1)}{\varepsilon\delta}\,\frac{1}{\overline{F}(x)}$$ □

That the upper bound of order $1/\overline{F}(x)$ is attainable follow from the following example:

Example 7.4. Let F be a discrete subexponential distribution with point probabilities $f_0 > 0, f_1, f_2, \ldots$, A discrete-time regenerative process Z is constructed as follows. At the start of a cycle, a r.v. X with distribution F is drawn. If $X = 0$, one takes $R = 1, \xi = -b$. If $X = x > 0$, one takes $R = h(x)$ for some suitable $h(x) \uparrow \infty$, $Z_0 = \ldots Z_{h(x)-2} = 0$, $\xi = Z_{h(x)-1} = x$ (one then needs to choose b such that $\mathbb{E}\xi < 0$).

The question is now whether all rates are attainable. To discuss this point, let $h(x) \uparrow \infty$. By Theorem 7.3, h must satisfy $h(x) = \mathrm{O}\big(1/\overline{F}(x)\big)$. Conversely, the construction works if (7.10) holds and gives a risk process such that τ grows at rate at least $h(x)$.

For example, if F is regularly varying with index $\alpha > 1$, this allows for growth rates $h(x)$ of order x^β with $1 < \beta < \alpha$, whereas $e(x)$ only is of order x.

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