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13

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Abstract

An insurance company has a large number N of potential customers characterized by i.i.d. r.v.'s $A_1, \dots, A_N$ giving the arrival rates of claims. Customers are risk averse, and a customer accepts an offered premium p according to his A-value. The modeling further involves a discount rate $d > r$ of customers, where r is the risk-free interest rate. Based on calculations of the customers' present values of the alternative strategies of insuring and not insuring, the portfolio size $n(p)$ is derived, and also the rate of claims from the insured customers is given. Further, the value of p which is optimal for minimizing the ruin probability is derived in a diffusion approximation to the Cramér-Lundberg risk process with an added liability rate L of the company. The solution involves the Lambert W function. Similar discussion is given for extensions involving customers having only partial information on their A and stochastic discount rates.

Keywords: Certainty equivalent, Cramér-Lundberg model, diffusion approximation, discounting, inverse Gamma distribution, Lambert W function, present value, risk aversion, ruin probability

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1 Introduction

For controlling its business, an insurance company has at its disposal several decision variables. In particular, strategies for dividend payments and reinsurance arrangements have been much studied (cf., e.g., the bibliography in [10]). The present paper focuses on a further decision variable, the premium p charged, that is an obvious choice but has received less detailed analysis. The rough picture is that decreasing p may increase the portfolio size but also decreases the profit per customer. Thus, it is not clear what is the sign in the change in the profit when changing p, and the question is what is a reasonable trade-off. This of course depends on the optimality criterion used; we return to this point later. In any case, a key first step is to identify the form of the portfolio size $n(p)$ as function of the premium p offered, and precise quantitative investigations of this seem to be few, if any, in the insurance mathematics literature (but note Højgaard [8] who takes the route of taking an unspecified function of the safety loading, and thereby the premium, as control). Providing explicit forms of $n(p)$ in a variety of models is one of the main contributions of this paper.

For a more precise formulation, consider the classical Cramér-Lundberg surplus process

$$R(t) = r_0 + ct - \sum_{i=1}^{M} Z_i,$$

where $R(t)$ is the surplus at time t, r_0 is the initial surplus, c is the rate of premium income per period, $M = M_\lambda$ is a Poisson process of intensity λ governing the arrival of claims, and $Z_1, Z_2, \ldots$ are the claim sizes (assumed i.i.d. and independent of M). In the diffusion approximation

$$\mathrm{d}x(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$$

where x is the reserve, we have

$$\mu = c - \lambda\overline{z}\,, \quad \sigma^2 = \lambda\overline{z^2}\,, \tag{1.1}$$

where $\overline{z} = \mathbb{E}Z$, $\overline{z^2} = \mathbb{E}Z^2$.

We assume that the market consists of N potential customers (agents) who may insure against their losses, and of whom the insurance company attracts $n = n(p)$ as customers, $0 \le n(p) \le N$, when the offered premium rate is p per customer. The compound Poisson process describing losses to the insurance company is the sum of n processes, one for each customer, and the gross premium c is collected from all n customers. We then have

$$c = n(p)p\,, \quad \lambda = \lambda(p) = n(p)\alpha(p)\,,$$

where $\alpha(p)$ is the arrival rate of the typical customer's claims at premium level p. We will see well motivated examples where indeed α depends on p; in contrast, for simplicity we make the assumption that the distribution of the claim size Z does not depend on p.

In the diffusion approximation,

$$\mu = n(p)p - n(p)\alpha(p)\overline{z} = n(p)\big(p - \alpha(p)\overline{z}\big)\,.$$

We expect that lowering p will increase $n(p)$. Hence, the effect could be to either increase or decrease μ. Letting p be large one expects a substantial loss of customers and hence of income. In most situations we meet, this is indeed the case. However, if $n(p)$ decreases slowly as $p \to \infty$, the overall premium inflow $pn(p)$ could potentially increase, and we will in fact see examples of this. In summary, the crux is the functional dependence of $n(p)$ (and possibly $\alpha(p)$) on p in the whole range $0 < p < \infty$.

A simplified approach is to postulate $\alpha(p) \equiv \alpha$ to be independent of p and $n(p)$ to have some specific *ad hoc* form, for example to be exponentially decreasing. We survey this line of attack briefly in Section 4. However, the main contribution of this paper (as we see it) is to present some ideas inspired from economics as well as classical Bayesian insurance modeling which provide some suggestions on less *ad hoc* forms of $n(p)$ and $\alpha(p)$. Doing so, we take the view that the customers have only one potential insurer, i.e., we neglect the market effects. The decision on whether to insure or not is then based on certain comparisons between the wealth resulting from one or the other decision and studied in more detail in Section 2; the modeling involves a discount factor d of the customers.

Roughly, in the naive setting either all customers insure or none according to whether $p < p_0$ or $p > p_0$ for a certain threshold value p_0 given in (2.2) at the end of Section 2. In more realistic settings, a customer is characterized by a random value A of his α or/and a random value D of his d, and the decision on whether to insure then involves comparisons of these r.v.'s to certain thresholds. When using such randomized values of $\alpha, d, (\alpha, d)$, it is throughout assumed that r.v.'s corresponding to different customers are i.i.d.; a classical example is A being Gamma in car insurance, cf. Bichsel [2], Bühlmann & Gisler [3] and Denuit *et al.* [5]. Examples of such modeling and associated optimization calculations are the main topic of Sections 3 and 5–8 occupying the rest of the paper.

Now consider control problems. Standard settings are optimization of dividend payout (before ruin), reinsurance arrangements and ruin probability minimization. In a controlled setting (assuming feedback control), μ and σ^2 will depend on the parameters $c, \lambda, \overline{z}, \overline{z^2}$ in (1.1) as well on the current level x of the reserve, the control strategy and possibly further parameters. As an example, we will focus here on finding the minimum $\psi^*(x)$ of the ruin probability $\psi(x)$ in the diffusion approximation, where $x = x(0)$. In the formulation given so far, this is a trivial problem, since by taking $p = \infty$ one expects that no customers will insure so that the reserve becomes freezed at its current value x corresponding to $\psi^*(x) = 0$. Accordingly, we will introduce a fixed rate $L > 0$ of liability payments. Then the diffusion approximation becomes

$$\mu = n(p)p - L - n(p)\alpha(p)\overline{z} = n(p)\big(p - \alpha(p)\overline{z}\big) - L\,, \quad \sigma^2(p) = n(p)\alpha(p)\overline{z^2}\,. \quad (1.2)$$

2 Customer's problem

The problem faced by the potential customer is to decide whether or not to insure. While insuring, the customer pays premium at the constant rate p. For simplicity, we consider a customer with infinite life length, discount rate d, and access to a riskless invest opportunity at interest rate $r > 0$. Since the decision problem is identical in every period, the customer's choice does not change over time. It is given by $\min(V_0, V_1)$ where V_0 is the cost of not insuring at time 0 and V_1 is the cost of insuring. Our modeling of this choice takes these costs as expected discounted present values, with an adjustment for risk aversion on the part of the potential customer. The standard transversality condition $d > r$ ensuring finite asset valuation is maintained throughout, i.e., present values are assessed using a discount rate exceeding the growth rate (cf. Gordon [6]).

As the funds used to pay the premium could alternatively (in case of not insuring) be invested, the total cost of insuring is identical to the value to the customer of this investment process. A possible approach to this valuation could be to introduce a rate of consumption out of running wealth and consider expected discounted utility of consumption, or fix a finite horizon and consider expected discounted terminal wealth. With risk aversion modelled using a concave utility of consumption function, as is common in economics and finance (including cases with both continuous and jump components, Øksendahl & Sulem [11]), this would introduce dependence on initial wealth. A finite horizon would introduce nonstationarity. Assume instead that the present (time $t = 0$) value to the customer of the deterministic flow at rate p is $\int_0^\infty \mathrm{e}^{-dt}\mathrm{d}w_t$, where w_t is the wealth generated by time t from investing this flow along with interest on accumulated wealth, i.e.,

$$\mathrm{d}w_t = (p + rw_t)\ \mathrm{d}t\,.$$

The solution to this differential equation with initial wealth w_0 is

$$w_t = \mathrm{e}^{rt}\left(w_0 + \frac{p}{r}\right) - \frac{p}{r}\,.$$

The present value to the customer of the lost cash flow (if he decides to insure and thus pay the premium) is therefore

$$V_1 = \int_0^\infty \mathrm{e}^{-dt}\mathrm{d}w_t = \int_0^\infty \mathrm{e}^{-dt} r\mathrm{e}^{rt}\left(w_0 + \frac{p}{r}\right)\mathrm{d}t = \frac{rw_0 + p}{d - r}\,.$$

As a special case, if $p = 0$ and only w_0 and interest on this is invested, the value is $rw_0/(d - r)$. This may be more or less than w_0, depending on the interest and discount rates.

The customer compares V_1 to the expected present value of the cost incurred over the inifinite horizon of not insuring, V_0. Write T_i for the time of arrival of the ith claim (with the convention $T_0 = 0$), Z_i for the random size of the claim, and $\overline{z} = \mathbb{E}Z_i$. Again, the funds used to pay the uninsured claims could alternatively be invested, and the total subjective cost of not insuring is given by the value to the customer of the investment process where Z_i is invested at T_i. Ignoring risk aversion, the above expression for V_1 at $p = 0$ would suggest that investing Z_i at T_i generates a present value of $rZ_i/(d-r)$ to the customer at time T_i, with expected present value at $t=0$ given by $r\overline{z}/(d-r)\cdot\mathbb{E}\mathrm{e}^{-dT_i}$. To capture the effect of risk aversion simply, assume instead that the customer's subjective expected cost of a claim is given by its certainty equivalent $\widehat{z}$ using an increasing and convex cost function φ. Specifically, $\widehat{z}$ is defined by $\varphi(\widehat{z})=\mathbb{E}\varphi(Z_i)$, and by convexity $\mathbb{E}\varphi(Z_i)>\varphi(\overline{z})$, so that $\varphi(\widehat{z})>\varphi(\overline{z})$. Since φ is increasing we get $\widehat{z}>\overline{z}$. The certainty equivalent is a convenient way to express the risk premium given by $\widehat{z}-\overline{z}>0$, see Pratt [9]. Replacing $\mathbb{E}Z_i=\overline{z}$ in the expected present value by $\widehat{z}$ and starting again with initial wealth w_0, it follows that

$$V_0=\frac{rw_0}{d-r}+\frac{r\widehat{z}}{d-r}\mathbb{E}\sum_{i=1}^{\infty}\mathrm{e}^{-dT_i}\ . \tag{2.1}$$

This is evaluated simply:

Lemma 1. *The present value of expected losses from the unpaid claims if not insuring is*

$$V_0=\frac{rw_0}{d-r}+\frac{r\widehat{z}\alpha}{(d-r)\,d}\ .$$

Proof. Writing $M_\infty=\mathbb{E}\sum_{i=1}^{\infty}\mathrm{e}^{-dT_i}$ and $M_1=\mathbb{E}\mathrm{e}^{-dT_1}$, we clearly have $M_\infty=M_1+\mathbb{E}\sum_{i=2}^{\infty}\mathrm{e}^{-dT_i}=M_1+\mathbb{E}\sum_{i=2}^{\infty}\mathrm{e}^{-dT_{i-1}}\mathrm{e}^{-d(T_i-T_{i-1})}$. Because the waiting times T_i-T_{i-1} between arrivals are i.i.d. exponential with arrival rate α, we can write $M_\infty=M_1+M_\infty M_1$, i.e., $M_\infty=M_1/(1-M_1)$. Further, since T_1 is $\exp(\alpha)$ we have $M_1=\int_0^\infty \mathrm{e}^{-dt}\alpha\mathrm{e}^{-\alpha t}\,\mathrm{d}t=\alpha/(\alpha+d)$, and so $M_\infty=\alpha/d$. □

When choosing to insure or not the customer compares the expected present values of the costs of paying the premium respectively the uncovered claims.

Corollary 2. *A customer insures iff $V_1<V_0$, i.e., he insures iff the offered premium p satisfies*

$$p<\frac{r\alpha\widehat{z}}{d}\ . \tag{2.2}$$

Thus, the customer may be willing to pay more for the insurance than the net premium $\alpha\overline{z}$, i.e., the expected net losses per period, provided

$$\frac{d\overline{z}}{r\widehat{z}}<1\,. \tag{2.3}$$

It seems reasonable to assume that d is only moderately larger than r whereas $\widehat{z}$ is substantially larger than $\overline{z}$, and so *the assumption* (2.3) *is maintained throughout in the following.*

3 Portfolio characteristics with stochastic arrival rates

A classical and practically highly relevant point of view in insurance mathematics is that the portfolio is non-homogeneous in the sense that the value A of the arrival rate α of claims of a customer is a r.v. with i.i.d. values over the portfolio. We will exemplify this via the classical example of car insurance where A is assumed to have a Gamma(s, β) distribution with density $\beta^s x^{s-1} e^{-\beta x}/\Gamma(s)$. Since in some cases ([2]) there is empirical evidence that s is close to 1, we assume in the following that $s = 1$, so that A has an exponential(β) distribution with density $\beta e^{-\beta x}$and tail $\overline{F}_A(x) = e^{-\beta x}$. See, e.g., Bichsel [2], Bühlmann & Gisler [3] and Denuit *et al.* [5] for more discussion and detail.

The condition (2.2) for the customer insuring now takes the form

$$A > \frac{pd}{r\widehat{z}}.$$

Now we have

$$\mathbb{P}\left(A > \frac{pd}{r\widehat{z}}\right) = \overline{F}_A\left(\frac{pd}{r\widehat{z}}\right) = e^{-\beta pd/r\widehat{z}}.$$

The number of insured individuals is this fraction out of the total population of N individuals:

Theorem 3. *In the model with exponential stochastic arrival rate A, the portfolio size is given by $n(p) = N e^{-\beta pd/r\widehat{z}}$.*

Theorem 3 gives the demand curve of the insurance company. Strictly speaking, this assumes a continuum of customers and so it is only the mean demand curve, and in principle a distribution of demand centered at $n(p)$ could be considered. However, we will assume that N is so large that this variation is negligible.

With stochastic arrival rates, the calculation of μ, σ^2 must be modified. The insurance company faces an arrival rate of claims that appropriately aggregates the arrival rates of those potential customers who choose to insure. More precisely, these arrival rates have the distribution of A given $A > pd/r\widehat{z}$. By the memoryless property of the exponential distribution, $\mathbb{E}[A \mid A > a] = a + 1/\beta$ if $a \geq 0$. It follows that under the the large population assumption,

$$\lambda = \lambda(p) = \mathbb{P}\left(A > \frac{pd}{r\widehat{z}}\right) \mathbb{E}\left(A \,\middle|\, A > \frac{pd}{r\widehat{z}}\right) N$$
$$= e^{-\beta pd/r\widehat{z}}[1/\beta + pd/r\widehat{z}]N = n(p)\big[1/\beta + pd/r\widehat{z}\big].$$

This shows that, again, the arrival rate of claims to the company takes the form $\lambda(p) = n(p)\alpha(p)$, but as noted in the introduction, both factors may depend on the premium charged. The functional form of $\alpha(p)$ reflects adverse selection: The higher the premium charged, the less desirable (to the company) the average customer.

In the diffusion approximation, we have

$$\mu(p) = pn(p) - L - \lambda(p)\overline{z} = n(p)\big(p(1 - d\overline{z}/r\widehat{z}) - \overline{z}/\beta\big) - L$$
$$= N e^{-\beta pd/r\widehat{z}}\big(p(1 - d\overline{z}/r\widehat{z}) - \overline{z}/\beta\big) - L, \tag{3.1}$$
$$\sigma^2(p) = n(p)[1/\beta + pd/r\widehat{z}]\overline{z^2} = N e^{-\beta pd/r\widehat{z}}[1/\beta + pd/r\widehat{z}]\overline{z^2}. \tag{3.2}$$

4 Ruin probability minimization in a simplified setting

As noted by Hipp & Taksar [7] in a more general setting, minimizing the ruin probability amounts to maximizing μ/σ^2 for each x. In our case, μ and σ^2 do not depend on x, so that the optimal parameter choice is global in x. The conclusion that minimizing the ruin probability amounts to maximizing μ/σ^2 is also immediate from the observation that the controlled process is a Brownian motion and so the ruin probability is $\psi(x) = \mathrm{e}^{-2\mu x/\sigma^2}$ when $x = x(0)$ is the initial reserve in the diffusion approximation.

Assume that $\alpha(p) \equiv \alpha$. Then

$$\frac{\mu}{\sigma^2} = \frac{n(p)(p-\alpha\overline{z}) - L}{n(p)\alpha\overline{z^2}} = \frac{p-\alpha\overline{z}}{\alpha\overline{z^2}} - \frac{L}{n(p)\alpha\overline{z^2}}$$

(note the independence of the level $R(t) = x$). For the purpose of maximization, we consider the derivatives

$$\frac{\partial(\mu/\sigma^2)}{\partial p} = \frac{1}{\alpha\overline{z^2}} + \frac{n'(p)L}{n(p)^2\alpha\overline{z^2}}, \qquad \frac{\partial^2(\mu/\sigma^2)}{\partial p^2} = \frac{L}{\alpha\overline{z^2}}\left(\frac{n''(p)n(p) - 2n'(p)^2}{n(p)^3}\right).$$

The first order condition for finding an interior extremum $\breve{p}$ of μ/σ^2 is that $\breve{p}$ solves

$$n'(\breve{p})L + n(\breve{p})^2 = 0. \tag{4.1}$$

The second order condition for the solution of (4.1) to be a proper optimum is that

$$\frac{n''(\breve{p})n(\breve{p})}{n'(\breve{p})^2} < 2 \quad \text{or equivalently} \quad \frac{\mathrm{d}}{\mathrm{d}p}\log n(p) > \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}\log n'(p) \tag{4.2}$$

at $p = \breve{p}$. I.e., the elasticity of the demand curve should be sufficiently high, more than half the elasticity of marginal demand n'.

We may also assume that the maximization has to be performed on an interval $[p_-, \infty)$ with $p_- \geq 0$; the most obvious possibilities are $p_- = 0$ and $p_- = \alpha\overline{z}$. The latter applies if the company does not want to charge less than the net premium $\alpha\overline{z}$, although this constraint need not be binding given the objective of minimization of ruin probability.

Clearly, without liabilities (the case $L = 0$), the optimum is $n = 0$, which may be achieved by setting the premium prohibitively high. For $L > 0$, the solution depends on the precise relation between p and n, as shown by the following examples.

Example 4. Assume $n(p) = K\mathrm{e}^{-bp}$. The first order condition (4.1) becomes

$$-bKL\mathrm{e}^{-b\breve{p}} + K^2\mathrm{e}^{-2b\breve{p}} = 0 \tag{4.3}$$

with solution

$$\breve{p} = \frac{c}{b} \quad \text{where} \quad c = \log K - \log b - \log L\,.$$

The l.h.s. in (4.2) collapses to unity identically, so that the second order condition is automatic. It follows that μ/σ^2 is maximized and the ruin probability minimized at $p^* = \breve{p}$ if $\breve{p} > p_-$ and $p^* = p_-$ otherwise.

Example 5. Assume $n(p) = K(1+bp)^{-\tau}$. Then

$$\alpha\overline{z^2}\frac{\mathrm{d}}{\mathrm{d}p}\frac{\mu(p)}{\sigma^2(p)} = 1 - \frac{L\tau b}{K}(1+bp)^{\tau-1}\,. \tag{4.4}$$

Equating (4.4) to 0 gives the first order condition with solution

$$\breve{p} = \frac{1}{b}\Big[\Big(\frac{K}{L\tau b}\Big)^{1/(\tau-1)} - 1\Big]\,.$$

At $p = 0$ we have $\mu = -K\alpha\overline{z} - L < 0$, giving ruin probability 1.

We have four cases:

(i) $\tau < 1$, $L\tau b/K < 1$. Here $\breve{p} < 0$ and (4.4) is positive for all $p > 0$, so that μ/σ^2 is maximized and the ruin probability minimized at $p^* = \infty$, with minimum $\psi^*(x) = 0$ since $\mu(p/\sigma^2 \to \infty$ by (4.4).

(ii) $\tau < 1$, $L\tau b/K > 1$. Here $\breve{p} > 0$ and (4.4) increases from a negative value to 1, so that $\breve{p}$maximizes, not minimizes, the ruin probability. Instead, μ/σ^2 is maximized and the ruin probability minimized at $p^* = \infty$, with minimum $\psi^*(x) = 0$.

(iii) $\tau > 1$, $L\tau b/K < 1$. Here $\breve{p} > 0$ and (4.4) decreases from a positive value to $-\infty$, so that $p^* = \breve{p}$ is the minimizer of the ruin probability if $\breve{p} > p_-$ and $p^* = p_-$ otherwise. We have $\psi^*(x) < 1$ or $\psi^*(x) = 1$ according as $\mu(p^*) > 0$ or $\mu(p^*) \le 0$.

(iv) $\tau > 1$, $L\tau b/K > 1$. Here $\breve{p} < 0$ and (4.4) is negative for all $p > 0$, so that μ/σ^2 is maximized and the ruin probability minimized at $p^* = 0$ if $p_- = 0$ and $p^* = p_-$ otherwise, giving $\psi^*(x) = 1$ for all x.

Example 6. Assume $n(p) = K_1[K_2 - p]^+$. The first order condition (4.1) becomes

$$\big[-K_1L + K_1^2[K_2 - \breve{p}]^2\big]\mathbf{1}_{\{\breve{p}<K_2\}} = 0$$

with solution

$$\breve{p} = K_2 - \sqrt{\frac{L}{K_1}} < K_2\,.$$

The l.h.s. in (4.2) vanishes identically, so that the second order condition is again automatic. It follows that μ/σ^2 is maximized and the ruin probability minimized at $p^* = \breve{p}$ if $\breve{p} > p_-$ and $p^* = p_-$ otherwise.

Example 5 includes cases where it pays for the company to accept a smaller portfolio size, at a higher profit per customer. We will encounter aspects of Example 4 in Section 5 and of (the richer) Example 5 in Section 6, though the set-ups are sufficiently different that a direct comparison is not possible.

5 Ruin probability minimization with stochastic arrival rates

Consider the stochastic arrival rate model of Section 3.

Proposition 1. *Let $x = x(0)$ be the initial reserve in the diffusion approximation, $\psi(x)$ the ruin probability and*

$$p^{\#} = \frac{(r\widehat{z})^2}{\beta(r\widehat{z} - d\overline{z})}\,.$$

Then $\psi(x) = 1$ for all $x, p > 0$ if $\mu(p^{\#}) \le 0$. If $\mu(p^{\#}) > 0$, then $\psi(x) < 1$ for all $x > 0$ and all p in some bounded open interval $I \subset (0, \infty)$ containing $p^{\#}$.

Proof. Differentiating (3.1), we get

$$\mu'(p) = N\mathrm{e}^{-\beta pd/r\widehat{z}}\left[-\frac{\beta d}{r\widehat{z}}\big(p(1 - d\overline{z}/r\widehat{z}) - \overline{z}/\beta\big) + 1 - d\overline{z}/r\widehat{z}\right]\,.$$

Therefore $\mu'(p) = 0$ for

$$p = \frac{1}{1 - d\overline{z}/r\widehat{z}}\left[\frac{r\widehat{z}}{\beta d}(1 - d\overline{z}/r\widehat{z}) + \frac{\overline{z}}{\beta}\right] = \frac{1}{1 - d\overline{z}/r\widehat{z}}\frac{r\widehat{z}}{\beta d} = p^{\#}\,,$$

and thus by (2.3), $\mu'(p) > 0$ for $p < p^{\#}$, $\mu'(p^{\#}) = 0$, and $\mu'(p) < 0$ for $p > p^{\#}$. Also clearly $\mu(0) < 0$ and $\mu(\infty) = -L < 0$. Therefore $\mu(p)$ attains a unique maximum at $p = p^{\#}$. Thus, if $\mu(p^{\#}) \le 0$, also $\mu(p) \le 0$ for all p and $\psi(x) = 1$ for all $x, p > 0$. If conversely $\mu(p^{\#}) > 0$, then $\mu(p) > 0$ for all p in some bounded open interval containing $p^{\#}$ and hence $\psi(x) = \mathrm{e}^{-2x\mu/\sigma^2} < 1$ for all $x > 0$. □

The case $\mu(p^{\#}) > 0$ may arise, certainly if N is sufficiently large compared to L: indeed, if p is so large that $p(1 - d\overline{z}/r\widehat{z}) > \overline{z}/\beta$, then $\mu(p) > 0$ for all large N and so also $\mu(p^{\#}) > 0$.

It remains to maximize $\mu(p)/\sigma^2(p)$ (and hence minimize the ruin probability $\psi(x)$) in the case $\mu(p^{\#}) > 0$. We start by remarking that the ratio of the $-\lambda(p)\overline{z}$ term in $\mu(p)$ and $\sigma^2(p) = \lambda(p)\overline{z^2}$ does not depend on p, hence vanishes by differentiation, and this explains that (maybe surprisingly) the optimizers in the following do not depend on $\widehat{z}$, only on $\widehat{z}$.

Theorem 7. *Assume $\mu(p^{\#}) > 0$. Then $\mu(p)/\sigma^2(p)$ is maximized, and hence $\psi(x)$ minimized, for $p = p^*$ where*

$$p^* = \frac{r\widehat{z}}{\beta d}W\left(\frac{Nr\widehat{z}}{L\beta d}\right) \in I\,, \tag{5.1}$$

where W is the Lambert W function. Further, $\psi^*(x) < 1$ for all $x > 0$.

[For W, see the Appendix.]

Proof. The first order condition $0 = \big(\mu(p)/\sigma^2(p)\big)'$ means (multiply by $\overline{z^2}$)

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p}{1/\beta + pd/r\widehat{z}} - \overline{z}/\beta - \frac{L}{N}\frac{1}{\mathrm{e}^{-\beta pd/r\widehat{z}}[1/\beta + pd/r\widehat{z}]}\right) \\
&= \beta\frac{(1+\beta pd/r\widehat{z})\cdot 1 - p\cdot\beta d/r\widehat{z}}{(1+\beta pd/r\widehat{z})^2} \\
&\quad + \frac{\beta L}{N}\frac{-\mathrm{e}^{-\beta pd/r\widehat{z}}\beta d/r\widehat{z}(1+\beta pd/r\widehat{z}) + \mathrm{e}^{-\beta pd/r\widehat{z}}\beta d/r\widehat{z}}{\mathrm{e}^{-2\beta pd/r\widehat{z}}(1+\beta pd/rz)^2}\,.
\end{aligned}
$$

Multiplying by $(1+\beta pd/r\overline{z})^2$, this becomes

$$
\begin{aligned}
0 &= \beta + \frac{\beta L}{N}\big(-\beta d/r\widehat{z}(1+\beta pd/r\widehat{z}) + \beta d/r\widehat{z}\big)\mathrm{e}^{\beta pd/r\widehat{z}}\,, \\
1 &= \frac{L}{N}\left(\frac{\beta d}{r\widehat{z}}\right)^2 p\mathrm{e}^{\beta pd/r\widehat{z}}\,, \quad \frac{Nr\widehat{z}}{L\beta d} = \beta pd/r\widehat{z}\mathrm{e}^{\beta pd/r\widehat{z}}\,, \\
\beta pd/r\widehat{z} &= W(Nr\widehat{z}/L\beta d)
\end{aligned}
$$

so that indeed the solution of $0 = \big(\mu(p)/\sigma^2(p)\big)'$ is given by the r.h.s. of (5.1).

Using the properties of W given in the Appendix and the shape of $\mu(p)$ discussed above also shows that there is a unique maximum of $\mu(p)/\sigma^2(p)$ at p^* given by (7.1), and that necessarily $p^* \in I$. □

In particular, p^* is strictly increasing in $N, r, \widehat{z}$, does not depend on $\overline{z}$, and is strictly decreasing in β, d, L. Further:

Corollary 8. *In the setting of Theorem 7, the portfolio size at the optimum is*

$$n(p^*) = N\mathrm{e}^{-W(Nr\widehat{z}/L\beta d)}\,.$$

This follows by combining Theorems 1 and 2. Similarly, p^* may be inserted in the foregoing expressions to find the optimum arrival rate of claims, drift and diffusion of the reserve, and the minimized ruin probability.

6 Customers with partial information

The discussion above assumes complete information on behalf of the customer in the sense that he knows his rate of claim occurences. In reality, this is of course not the case. We consider here the model where the customer's belief is that his rate is AS, where S is a r.v. independent of A; the S-values for different customers are assumed i.i.d. How S is centered compared to 1 expresses the optimism/pessimism of customers in assessing their true rate: a distribution left skewed compared to 1 is optimistic, a right skewed one is pessimistic.

When deciding on whether to insure or not, the customer will make his decision based upon AS rather than A. With the exponential assumption on A, we have

$$\mathbb{P}(AS > x) = \mathbb{E}\mathrm{e}^{-\beta x/S}\,.$$

An appealing possibility is to take S as inverse Gamma. That is, $1/S$ is Gamma(τ,δ) (say), i.e. with density $\delta^\tau x^{\tau-1}\mathrm{e}^{-\delta x}/\Gamma(\delta)$, so that

$$\mathbb{P}(AS > x) = \mathbb{E}\mathrm{e}^{-\beta x/S} = \Big(\frac{\delta}{\delta+\beta x}\Big)^{\tau} = \Big(\frac{1}{1+\beta x/\delta}\Big)^{\tau},$$

i.e., AS is Pareto. Using again the memoryless property of the exponential distribution, we further get

$$\alpha(p) = \mathbb{E}\big[A \,\big|\, AS > pd/r\widehat{z}\big] = \mathbb{E}\Big[\frac{1}{\beta} + \frac{pd}{r\widehat{z}S}\Big] = \frac{1}{\beta} + \frac{\tau pd}{\delta r\widehat{z}}.$$

Thus:

Theorem 9. *In the model with exponential stochastic arrival rate A and an inverse Gamma multiplicative subjective uncertainty factor S, the portfolio size is given by*

$$n(p) = N\,\mathbb{P}\big(AS > pd/r\widehat{z}\big) = \frac{N}{\big(1+\beta pd/\delta r\widehat{z}\big)^{\tau}}. \tag{6.1}$$

Further, the overall arrival rate of claims is

$$\lambda(p) = n(p)\alpha(p) = \frac{N}{\big(1+\beta pd/\delta r\widehat{z}\big)^{\tau}}\left(\frac{1}{\beta} + \frac{\tau pd}{\delta r\widehat{z}}\right). \tag{6.2}$$

It follows that in the diffusion approximation, we have

$$\begin{aligned}
\mu &= pn(p) - \lambda(p)\overline{z} - L = \frac{N}{(1+\beta pd/\delta r\widehat{z})^{\tau}}\Big(p - \Big[\frac{1}{\beta} + \frac{\tau pd}{\delta r\widehat{z}}\Big]\overline{z}\Big) - L\\
&= Nq^{-\tau}(p - m\overline{z}) - L,\\
\sigma^2 &= \frac{N}{(1+\beta pd/\delta r\widehat{z})^{\tau}}\left[\frac{1}{\beta} + \frac{\tau pd}{\delta r\widehat{z}}\right]\overline{z^2} = Nq^{-\tau}m\,\overline{z^2},
\end{aligned}$$

where

$$q = 1 + \frac{\beta d}{\delta r\widehat{z}}p, \quad p = \frac{\delta r\widehat{z}}{\beta d}(q-1), \quad m = \frac{1}{\beta} + \frac{\tau d}{\delta r\widehat{z}}p = \frac{1}{\beta} + \frac{\tau}{\beta}(q-1).$$

Consider the properties of the drift $\mu(p)$. As for boundary behavior, we get $\mu \sim -N\overline{z}/\beta - L$ as $p \downarrow 0$. At the other extreme,

$$\mu \approx \begin{cases} Np^{1-\tau}\Big(\dfrac{\delta r\widehat{z}}{\beta d}\Big)^{\tau}\Big(1 - \dfrac{\tau d\overline{z}}{\delta r\widehat{z}}\Big) \to \infty & \tau < 1,\ \dfrac{\tau d\overline{z}}{\delta r\widehat{z}} < 1,\\[2ex] Np^{1-\tau}\Big(\dfrac{\delta r\widehat{z}}{\beta d}\Big)^{\tau}\Big(1 - \dfrac{\tau d\overline{z}}{\delta r\widehat{z}}\Big) \to -\infty & \tau < 1,\ \dfrac{\tau d\overline{z}}{\delta r\widehat{z}} > 1,\\[2ex] -L & \tau > 1 \end{cases}$$

as $p \uparrow \infty$. The interpretation is that if $\tau < 1$, then $n(p) \to 0$ so slowly that the average premium inflow $pn(p)$ goes to ∞ as $p \uparrow \infty$ (at rate $p^{1-\tau}$). If furthermore customers are so pessimistic in their subjective assessment of their A that $\tau d\overline{z}/\delta r\widehat{z} =$

$\mathbb{E}S^{-1}\cdot d\overline{z}/r\widehat{z} < 1$, they are willing to pay enough premium to make the overall gain go to infinity (even with ever fewer customers) so that μ is maximized at $p = \infty$, while if $\mathbb{E}S^{-1}\cdot d\overline{z}/r\widehat{z} > 1$, they are so optimistic that there is a loss by taking p large (recall from (2.3) that $d\overline{z}/r\widehat{z} < 1$). Conversely, if $\tau > 1$ then $pn(p) \to 0$ and all that matters in the limit $p \uparrow \infty$ is the fixed liability payment L.

Finally consider ruin probability minimization. As $p \downarrow 0$, $\overline{z^2}\mu/\sigma^2 \sim -\overline{z} - L\beta/N < 0$. As $p \uparrow \infty$,

$$\sigma^2 \sim Np^{1-\tau}(\delta r\widehat{z}/\beta d)^\tau(\tau d/\delta r\widehat{z})\overline{z^2}\,,$$

and thus

$$\overline{z^2}\frac{\mu}{\sigma^2} \sim \begin{cases} \dfrac{\delta r\widehat{z}}{\tau d}\Big(1-\dfrac{\tau d\overline{z}}{\delta r\widehat{z}}\Big) > 0 & \tau < 1,\ \dfrac{\tau d\overline{z}}{\delta r\widehat{z}} < 1\,, \\ \dfrac{\delta r\widehat{z}}{\tau d}\Big(1-\dfrac{\tau d\overline{z}}{\delta r\widehat{z}}\Big) < 0 & \tau < 1,\ \dfrac{\tau d\overline{z}}{\delta r\widehat{z}} > 1\,, \\ -\infty & \tau > 1\,. \end{cases}$$

We further get

$$\begin{aligned} \overline{z^2}\frac{\mu(p)}{\sigma^2(p)} &= \frac{p}{m} - \overline{z} - \frac{L}{N}\frac{q^\tau}{m}\,, \\ \overline{z^2}\frac{\mathrm{d}}{\mathrm{d}p}\frac{\mu(p)}{\sigma^2(p)} &= \frac{m-\tau pd/\delta r\widehat{z}}{m^2} - \frac{L}{N}\frac{\tau q^{\tau-1}m\beta d/\delta r\widehat{z} - q^\tau\tau d/\delta r\widehat{z}}{m^2} \\ &= \frac{1}{\beta m^2} - \frac{Lq^{\tau-1}\tau d}{Nm^2\delta r\widehat{z}}(\beta m - q) \\ &= \frac{1}{\beta m^2} - \frac{L\tau(\tau-1)d}{N\delta r\widehat{z}m^2}q^{\tau-1}(q-1) \end{aligned}$$

Note that this is expression is positive if $\tau < 1$ and always positive at $p = 0$ (corresponding to $q = 1$). Putting things together and noting the sign behavior of $(\mu(p)/\sigma^2(p))'$, we get:

Theorem 10. *In the model with exponential stochastic arrival rate A and an inverse Gamma multiplicative subjective uncertainty factor S, we have:*

(i) *If $\tau < 1$, then $\mu(p)/\sigma^2(p)$ is monotonically increasing in p. If $\tau d\overline{z}/\delta r\widehat{z} < 1$, then $\mu(p)/\sigma^2(p)$ is maximized at $p = \infty$ and hence the ruin probability $\psi(x)$ minimized, with*

$$\psi^*(x) = \exp\Big\{-2x\frac{\delta r\widehat{z}}{\tau d}\Big(1-\frac{\tau d\overline{z}}{\delta r\widehat{z}}\Big)\Big\} < 1.$$

If $\tau d\overline{z}/\delta r\widehat{z} > 1$, then $\mu(p)/\sigma^2(p)$ is negative for all p and hence $\psi(x) = 1$.

(ii) *If $\tau > 1$, then $\mu(p)/\sigma^2(p)$ has a unique maximum in $(0,\infty)$, say at $\breve{p}$, where $\breve{p} = (\breve{q}-1)\delta r\widehat{z}/(\beta d)$ with $\breve{q}$ the unique solution in $(1,\infty)$ of*

$$q^\tau - q^{\tau-1} = \frac{N\delta r\widehat{z}}{L\tau(\tau-1)\beta d}\,. \tag{6.3}$$

If $\mu(\breve{p}) \le 0$, then $\psi(x) = 1$ for all x, whereas if $\mu(\breve{p}) > 0$, then the ruin probability is minimized for $p = p^* = \breve{p}$, with

$$\psi^*(x) = \exp\Big\{-2x\Big(\frac{p}{m} - \overline{z} - \frac{L}{N}\frac{q^\tau}{m}\Big)/\overline{z^2}\Big\} < 1,$$

where $p = \breve{p}$, $q = \breve{q}$, $m = \dfrac{1}{\beta} + \dfrac{\tau}{\beta}(\breve{q} - 1)$.

7 Stochastic discount rates

An alternative to the model where each customer has a separate arrival rate is that the discount rate d varies over the population but $A \equiv \alpha$ is fixed. That is, for each customer the discount rate is the outcome of a r.v. D, such that the D's for different customers are i.i.d.

A case that is easily analysized is that where $1/D$ is exponentially(β) distributed. Indeed, since the fundamental inequality (2.2) $p < r\alpha\widehat{z}/d$ takes the form $p < r\alpha\widehat{z}/D$, it has the same form as in Section 3 (where it read $p < rA\widehat{z}/d$), only with A replaced by $1/D$ and d by $1/\alpha$. In Section 2, inequality (2.2) was derived explicitly from the potential customer's cost minimization problem under the condition $d > r$, and insurance above the net premium in addition required condition (2.3). With stochastic discount rates, both conditions may be violated with some probability across the population. One possibility would be to consider a distribution for D with support concentrated in $(r, r\widehat{z}/\overline{z})$. For simplicity, we maintain $1/D \sim \exp(\beta)$ and regard the decision criterion $p < r\alpha\widehat{z}/D$ as a behavioral assumption, not necessarily attached to the analysis in Section 2, and similarly for condition (2.3). With this interpretation of the modified model, the analysis of Section 3 carries through without changes, and we get:

Corollary 11. *Assume that the stochastic discount rate is a r.v. D and that $A \equiv \alpha$ is fixed. Then $\mu(p)/\sigma^2(p)$ is maximized for $p = p^*$ where*

$$p^* = \frac{\alpha r\widehat{z}}{\beta} W\Big(\frac{N\alpha r\widehat{z}}{L\beta}\Big) . \tag{7.1}$$

If $\mu(p^*) > 0$, then $\psi(x) < 1$ for all x and all p in a bounded open interval containing p^*, and p^* is the unique minimizer of the ruin probability for all x. The optimum portfolio size in this case is $n(p^*) = N\mathrm{e}^{-W(N\alpha r\widehat{z}/L\beta)}$. If $\mu(p^*) \le 0$, then $\psi(x) = 1$ for all $x, p > 0$.

8 Stochastic arrival and discount rates

The models with stochastic arrival and discount rates may be combined, by assuming both the arrival rate $\alpha = A$ and the discount rate $d = D$ to be r.v.'s. The customer's criterion for insuring at premium p becomes

$$\frac{rA\widehat{z}}{D} > p \,,$$

and under appropriate conditions, there is an easy reduction to the partial information case. Indeed, assuming independence and A to be exponential(β), D to be gamma(τ,δ), we are back to the setting of Section 6 and get:

Corollary 12. *Assume the arrival rate $\alpha = A$ and the discount rate $d = D$ to be independent r.v.'s with A exponential(β), D gamma(τ,δ). Then:*

(i) *the portfolio size is given by*

$$n(p) = N\,\mathbb{P}\bigl(A/D > p/r\widehat{z}\bigr) = \frac{N}{\bigl(1+\beta p/\delta r\widehat{z}\bigr)^{\tau}}\,. \tag{8.1}$$

Further, the overall arrival rate of claims is

$$\lambda(p) = n(p) = \frac{N}{\bigl(1+\beta p/\delta r\widehat{z}\bigr)^{\tau}}\Bigl(\frac{1}{\beta}+\frac{\tau p}{\delta r\widehat{z}}\Bigr)\,. \tag{8.2}$$

(ii) *If $\tau < 1$, then $\mu(p)/\sigma^2(p)$ is monotonically increasing in p. If $\tau\overline{z}/\delta r\widehat{z} < 1$, then $\mu(p)/\sigma^2(p)$ is maximized at $p=\infty$ and hence the ruin probability $\psi(x)$ minimized, with*

$$\psi^*(x) = \exp\Bigl\{-2x\frac{\delta r\widehat{z}}{\tau}\Bigl(1-\frac{\tau\overline{z}}{\delta r\widehat{z}}\Bigr)\Bigr\} < 1.$$

If $\tau\overline{z}/\delta r\widehat{z} > 1$, then $\mu(p)/\sigma^2(p)$ is negative for all p and hence $\psi(x)=1$.

(iii) *If $\tau > 1$, then $\mu(p)/\sigma^2(p)$ has a unique maximum in $(0,\infty)$, say at $\breve{p}$, where $\breve{p} = (\breve{q}-1)\delta r\widehat{z}/\beta$ with $\breve{q}$ the unique solution in $(1,\infty)$ of*

$$q^{\tau} - q^{\tau-1} = \frac{N\delta r\widehat{z}}{L\tau(\tau-1)\beta}\,. \tag{8.3}$$

If $\mu(\breve{p}) \le 0$, then $\psi(x)=1$ for all x, whereas if $\mu(\breve{p}) > 0$, then the ruin probability is minimized for $p = p^* = \breve{p}$, with

$$\psi^*(x) = \exp\Bigl\{-2x\Bigl(\frac{p}{m}-\overline{z}-\frac{L}{N}\frac{q^{\tau}}{m}\Bigr)/\overline{z^2}\Bigr\} < 1,$$

where $p=\breve{p}$, $q=\breve{q}$, $m = \dfrac{1}{\beta}+\dfrac{\tau}{\beta}(\breve{q}-1)$.

Appendix: The Lambert W function

The Lambert W function (e.g. Corless *et al.* [4]) is defined as solution of $x = we^{w}$, i.e. as the function implicitly given by $x = W(x)e^{W(x)}$. We are only concerned with the case of x being real and positive, and in this case W is strictly increasing and a bijection $(0,\infty)\to(0,\infty)$.

Computationally, one may note that W is available in Maple and Matlab, and that an alternative expression is given by the series expansion

$$W(x) = \sum_{n=1}^{\infty}\frac{(-n)^n}{n!}x^n\,.$$

References

[1] S. Asmussen & H. Albrecher (2010) *Ruin Probabilities* (2nd ed.). World Scientific.

[2] F. Bichsel (1964) Erfahrungstariffierung in der Motorfahrzeug-Haftphlicht-Versicherung. *Mitt. Verein. Schweiz. Versich. Math.*, 119–130.

[3] H. Bühlmann & A. Gisler (2005) *A Course in Credibility Theory and its Applications.* Springer-Verlag.

[4] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey & D. E. Knuth (1996). On the Lambert W function. *Advances in Computational Mathematics* **5**, 329–359.

[5] M. Denuit, X. Maréchal, S. Pitrebois & J.-F. Walhin (2007) *Actuarial Modeling of Claim Counts. Risk Classification, Credibility and Bonus-Malus Systems.* Wiley.

[6] M.J. Gordon (1959) Dividends, earnings and stock prices, *Review of Economics and Statistics* **41** , 99–105.

[7] C. Hipp & M. Taksar (2010) Optimal non-proportional reinsurance control, *Insurance: Mathematics and Economics* **47**, 246–254.

[8] B. Højgaard (2002) Optimal dynamic premium control in non-life insurance. Maximizing dividend pay-outs. *Scand. Actuarial J.* **2002**, 225–245.

[9] J.W. Pratt (1964) Risk aversion in the small and in the large, *Econometrica* **32**, 122–136.

[10] H. Schmidli (2008) *Stochastic Control in Insurance.* Springer-Verlag.

[11] B. Øksendahl & A. Sulem (2005) *Applied Stochastic Control of Jump Diffusions.* Springer-Verlag.