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Abstract

In this paper we present some new asymptotic results for high frequency statistics of Brownian semi-stationary ($\mathcal{BSS}$) processes. More precisely, we will show that singularities in the weight function, which is one of the ingredients of a $\mathcal{BSS}$ process, may lead to non-standard limits of the realised quadratic variation. In this case the limiting process is a convex combination of shifted integrals of the intermittency function. Furthermore, we will demonstrate the corresponding stable central limit theorem. Finally, we apply the probabilistic theory to study the asymptotic properties of the realised ratio statistics, which estimates the smoothness parameter of a $\mathcal{BSS}$ process.

Keywords: Brownian semi-stationary processes, high frequency data, limit theorems, stable convergence

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1 Introduction

In the last years Brownian semi-stationary processes and their tempo-spatial extensions, *ambit fields*, have been widely studied in the literature. This class of models has been originally proposed by Barndorff-Nielsen and Schmiegel [8] in the context of turbulence modeling. In their general form, Brownian semi-stationary processes without drift are defined as

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s), \qquad t \in \mathbb{R}$$

where μ is a constant, W is a Brownian measure on $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a deterministic weight function with $g(t) = 0$ for $t \leq 0$, and σ is a càdlàg processes. If σ is stationary and independent of W, then $(X_t)_{t \in \mathbb{R}}$ is stationary, which explains the name Brownian semi-stationary process. In the framework of turbulence modeling, $(X_t)_{t \in \mathbb{R}}$ denotes the velocity of a turbulent flow in the direction of the mean field measured at a

fixed point in space. The stochastic process $(\sigma_t)_{t\in\mathbb{R}}$ embodies the *intermittency* of the dynamics of X. We refer to [8, 9, 10, 11] for application of Brownian semi-stationary processes and ambit fields to turbulence modeling, and to [2, 6] for further applications in mathematical finance and biology.

Recently, probabilistic properties of high frequency statistics of $\mathcal{BSS}$ processes have been investigated in several papers. We refer to a series of articles [4, 5, 13], which studies the asymptotic behaviour of (multi)power variation of $\mathcal{BSS}$ models. Typically, the weight function g considered in the aforementioned work has the form

$$g(x) = x^\alpha f(x), \qquad \alpha \in (-1/2, 0) \cup (0, 1/2),$$

where f is a sufficiently smooth function slowly varying at 0 and with rapid decay at infinity. This type of weight functions satisfies $g \in \mathbb{L}^2(\mathbb{R})$, but $g' \notin \mathbb{L}^2(\mathbb{R})$ since g' is not square integrable near 0; in other words, the latter property means that 0 is the only *singularity point* of the weight function g. As a consequence, the process X is not a semimartingale. Moreover, its local behaviour corresponds to the one of a fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$.

Understanding the limit theory for $\mathcal{BSS}$ processes requires an analysis of the following probability measure. For any $A \in \mathcal{B}(\mathbb{R})$, we define

$$\pi_n(A) := \frac{\int_A \{g(x + \Delta_n) - g(x)\}^2 dx}{\int_{\mathbb{R}} \{g(x + \Delta_n) - g(x)\}^2 dx}. \tag{1.1}$$

In the setting of weight functions as above, we deduce that $\pi_n \xrightarrow{d} \delta_0$ as $\Delta_n \to 0$, where δ_0 denotes the Dirac measure at 0 (cf. [4]). In this case the limit of the power variation of a $\mathcal{BSS}$ process is given as

$$\Delta_n \tau_n^{-p} \sum_{i=1}^{[t/\Delta_n]} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p \xrightarrow{\text{u.c.p.}} m_p \int_0^t |\sigma_s|^p ds, \qquad \text{as } \Delta_n \to 0, \tag{1.2}$$

where $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$, τ_n is a certain normalizing sequence and $\xrightarrow{\text{u.c.p.}}$ stands for convergence in probability uniformly on compact sets. In [4, 5] the asymptotic mixed normality of (multi)power variation is proved and the paper [13] studies the application of the limit theory to estimation of the *smoothness parameter* α. We remark that the asymptotic results are quite similar to the theory of power variations of continuous Itô semimartingales (cf. [7, 15] among many others), although the methodologies of proofs are completely different.

The aim of this paper is to demonstrate that other type of limits for power variations may appear when the weight function g exhibits further singularity points. More precisely, we will prove that

$$\Delta_n \tau_n^{-2} \sum_{i=1}^{[t/\Delta_n]} (X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 \xrightarrow{\text{u.c.p.}} \int_0^\infty \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds \right) \pi(d\theta), \qquad \text{as } \Delta_n \to 0, \tag{1.3}$$

where π is a finite probability measure on $[0, \infty)$ whose support is a subset of all singularity points of g. Consequently, the limit theory for $\mathcal{BSS}$ processes is richer

than the corresponding theory for continuous Itô semimartingales. Furthermore, we will show the associated stable central limit theorem including the setting of higher order differences. We remark that this type of limits has already appeared in [8]. The authors proved convergence in probability for the realised quadratic variation under the independence assumption between σ and W, and under further conditions on certain measures associated with g, which identify π. However, it remained quite unclear when a given weight function g satisfies the proposed set of conditions. The main goal of our paper is to show that singularity points of g, i.e. all points around which g' is not square integrable, determine the support and the weights of π. We remark that in physics multiple singularity points of g lead to non-homogeneous turbulent flows. Moreover, we will study the effect of this new class of weight functions g on smoothness parameter estimation. More precisely, we will present the asymptotic behaviour of a realised ratio statistic that compares the realised quadratic variation at two different frequencies. Applying the limit theory on a short enough time interval, we will derive the central limit theorem for the smoothness parameter of the model, which turns out to be the minimal power associated with singularity points of g. This shows some robustness of the realised ratio statistics investigated in [4, 5] with respect to misspecification of the kernel g.

The paper is organised as follows. Section 2 presents the main framework and a set of assumptions. In Section 3 we demonstrate the complete asymptotic theory for the realised quadratic variation of $\mathcal{BSS}$ processes, including the law of large numbers and the associated stable central limit theorem. In Section 4 we apply the probabilistic results to determine the asymptotic behaviour of a realised ratio statistic, which is an estimator of the smoothness parameter of X. Finally, all proofs are collected in Section 5.

2 The setting

2.1 Model

We start with a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ on which our processes are defined. We consider a $\mathcal{BSS}$ process $(X_t)_{t\in\mathbb{R}}$ (without drift) given as

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(ds), \qquad t \in \mathbb{R, \tag{2.1}$$

where W is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted white noise on $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a deterministic weight function satisfying $g(t) = 0$ for $t \leq 0$ and $g \in \mathbb{L}^2(\mathbb{R})$. The intermittency process σ is assumed to be an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted càdlàg process. We recall that $(\mathcal{F}_t)$-adapted white noise W is zero-mean Gaussian random measure on $\{A \in \mathcal{B}(\mathbb{R}) : \lambda(A) < \infty\}$, where λ denotes the Lebesgue measure, whose covariance kernel is given by

$$\mathbb{E}[W(A)W(B)] = \lambda(A \cap B).$$

The finiteness of the process X is guaranteed by the condition

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 ds < \infty \qquad \text{almost surely,} \tag{2.2}$$

for any $t \in \mathbb{R}$, which we assume from now on. The underlying observations of the $\mathcal{BSS}$ process X are

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{\Delta_n[t/\Delta_n]}$$

with $\Delta_n \to 0$ and t fixed. In other words, we are in the framework of infill asymptotics. Our realised quadratic variation statistics will be based upon higher order increments of X computed at different frequencies. For any $k \in \mathbb{N}$ and $v = 1, 2$, the k-th order difference $\Delta_{i,k}^{n,v} X$ at frequency $v\Delta_n$ and at stage $i \geq vk$ is defined by

$$\Delta_{i,k}^{n,v} X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-vj)\Delta_n}. \tag{2.3}$$

The quantity $\Delta_{i,k}^{n,v} X$ is a particular example of a k-th order filter applied to the process X. When $v = 1$ we usually write $\Delta_{i,k}^n X$ instead of $\Delta_{i,k}^{n,1} X$. For instance,

$$\Delta_{i,1}^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n} \quad \text{and} \quad \Delta_{i,2}^n X = X_{i\Delta_n} - 2X_{(i-1)\Delta_n} + X_{(i-2)\Delta_n}.$$

The realised quadratic variation statistic based upon $\Delta_{i,k}^{n,v} X$ is defined as

$$QV(X, k, v\Delta_n)_t := \sum_{i=vk}^{[t/\Delta_n]} (\Delta_{i,k}^{n,v} X)^2 \tag{2.4}$$

As in [4, 5], the *Gaussian core* G is crucial for understanding the fine structure of X. The process $G = (G_t)_{t \in \mathbb{R}}$ is a zero-mean stationary Gaussian process given by

$$G_t := \int_{-\infty}^{t} g(t-s)W(ds), \qquad t \in \mathbb{R}. \tag{2.5}$$

We remark that $|G_t| < \infty$ since $g \in \mathbb{L}^2(\mathbb{R})$. A straightforward computation shows that the correlation kernel r of G has the form

$$r(t) = \frac{\int_0^\infty g(u)g(u+t)du}{\|g\|_{\mathbb{L}^2(\mathbb{R})}^2}, \qquad t \geq 0.$$

Another important quantity for the asymptotic theory is the variogram R, i.e.

$$R(t) := \mathbb{E}[(G_{t+s} - G_s)^2] = 2\|g\|_{\mathbb{L}^2(\mathbb{R})}^2 (1 - r(t)), \quad \tau_k(v\Delta_n) := \sqrt{\mathbb{E}[(\Delta_{i,k}^{n,v} G)^2]}. \tag{2.6}$$

The quantity $\tau_k(v\Delta_n)$ will appear as a proper scaling in the law of large numbers for the statistic $QV(X, k, v\Delta_n)$.

2.2 Main assumptions

As we mentioned in the introduction, understanding the asymptotic behaviour of the probability measure

$$\pi_n(A) = \frac{\int_A \{g(x + \Delta_n) - g(x)\}^2 dx}{\int_{\mathbb{R}} \{g(x + \Delta_n) - g(x)\}^2 dx}, \qquad A \in \mathcal{B}(\mathbb{R}),$$

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is absolutely crucial for determining the limit theory for the realised quadratic variation $QV(X, 1, v\Delta_n)$ (for $QV(X, k, v\Delta_n)$, $k \geq 2$, there exists an analogous probability measure). Indeed, the condition

$$\pi_n \xrightarrow{d} \pi,$$

where π is a certain probability measure on $\mathbb{R}$, is necessary (but not sufficient) to obtain a non-standard law of large numbers at (1.3). In [4, 5, 13] it has been dealt with the case $\pi = \delta_0$, and hence we obtained a rather standard convergence as in (1.2). However, due to a moving average structure of the process X, even trivial weight functions g may lead to $\pi \neq \delta_0$ as the next simple example shows.

Example 2.1. Let us consider the function $g(x) = 1_{[0,1]}(x)$. A simple computation shows that

$$\pi_n(A) = \frac{\{\lambda(A \cap [-\Delta_n, 0]) + \lambda(A \cap [1 - \Delta_n, 1])\}}{2\Delta_n},$$

and consequently $\pi_n \xrightarrow{d} \pi = \frac{1}{2}(\delta_0 + \delta_1)$. Indeed, the convergence in (1.3) with $\tau_n^2 = \tau_1(\Delta_n)^2 = 2\Delta_n$ can be shown in a straightforward manner. For our weight function g, we deduce that

$$X_t = Y_t - Y_{t-1} \qquad \text{with } Y_t = \int_{-1}^t \sigma_s W(ds),$$

for $t \geq -1$. Noticing that Y is a martingale, we easily conclude

$$\Delta_n \tau_n^{-2} \sum_{i=1}^{[t/\Delta_n]} (X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 \xrightarrow{\text{u.c.p.}} \frac{1}{2} \left(\int_0^t \sigma_s^2 ds + \int_{-1}^{t-1} \sigma_s^2 ds \right), \qquad \text{as } \Delta_n \to 0,$$

which confirms (1.3). This example demonstrates that if $g(x) = \sum_{i=1}^l a_i 1_{[\theta_i^{(1)}, \theta_i^{(2)}]}$ with $0 \leq \theta_1^{(1)} < \theta_1^{(2)} < \theta_2^{(1)} < \theta_2^{(2)} < \cdots < \theta_l^{(2)} < \infty$ then

$$\text{supp}(\pi) = \{\theta_1^{(1)}, \theta_1^{(2)}, \ldots, \theta_l^{(1)}, \theta_l^{(2)}\} \quad \text{and} \quad \pi(\{\theta_i^{(1)}\}) = \pi(\{\theta_i^{(2)}\}) = \frac{a_i^2}{2 \sum_{i=1}^l a_i^2},$$

and (1.3) holds. $\qquad \square$

Barndorff-Nielsen and Schmiegel [8] provide conditions on certain rather complex measures associated with g (including $\pi_n \xrightarrow{d} \pi$), which are sufficient for proving law of large numbers of the type (1.3) under the independence assumption between σ and W. However, it is not a priori clear when a given weight function g satisfies those conditions. Furthermore, conditions ensuring the associated central limit theorem are expected to be even more complex.

In this paper we follow a different route. We present an explicit large class of weight functions g, which leads to the law of large numbers of (1.3), such that the limiting probability measure π is easily identified. Moreover, the associated central limit theorems are obtained (the limit theory does not require independence of σ and W). The crucial message of this paper is that singularity points of g defined below determine the support and the weights of π.

Let $0 = \theta_0 < \theta_1 < \cdots < \theta_l < \infty$ be a set of given points and $\alpha_0, \ldots, \alpha_l \in (-1/2, 0) \cup (0, 1/2)$. For any function $h \in C^m(\mathbb{R})$, $h^{(m)}$ denotes the m-th derivative of h. Recall that $k \geq 1$ stands for the order of the filter defined in (2.3). We introduce the following set of assumptions.

(A) For $\delta < \frac{1}{2} \min_{1 \leq i \leq l}(\theta_i - \theta_{i-1})$ it holds that

    (i) $g(x) = x^{\alpha_0} f_0(x)$ for $x \in (0, \delta)$.

    (ii) $g(x) = |x - \theta_i|^{\alpha_i} f_i(x)$ for $x \in (\theta_i - \delta, \theta_i) \cup (\theta_i, \theta_i + \delta)$, $i = 1, \ldots, l$.

    (iii) $g(\theta_i) = 0$, $f_i \in C^k((\theta_i - \delta, \theta_i + \delta))$ and $f_i(\theta_i) \neq 0$ for $i = 0, \ldots, l$.

    (iv) $g \in C^k(\mathbb{R} \setminus \{\theta_0, \ldots, \theta_l\})$ and $g^{(k)} \in \mathbb{L}^2\left(\mathbb{R} \setminus \cup_{i=0}^{l}(\theta_i - \delta, \theta_i + \delta)\right)$.

    (v) For any $t > 0$

$$F_t = \int_{\theta_l+1}^{\infty} g^{(k)}(s)^2 \sigma_{t-s}^2 ds < \infty. \tag{2.7}$$

We also set

$$\alpha := \min\{\alpha_0, \ldots, \alpha_l\}, \qquad \mathcal{A} := \{0 \leq i \leq l : \alpha_i = \alpha\}. \tag{2.8}$$

Let us give some remarks on this set of conditions.

Remark 2.2. The points $\theta_0, \ldots, \theta_l$ are singularities of g in the sense that $g^{(k)}$ is not square integrable around these points, because $\alpha_0, \ldots, \alpha_l \in (-1/2, 0) \cup (0, 1/2)$ and conditions (A)(i)–(iii) hold. Condition (A)(iv) indicates that g exhibits no further singularities. The papers [4, 5, 13] deal with the framework of a single singularity at 0. $\qquad\square$

Remark 2.3. The parameter $\alpha \in (-1/2, 0) \cup (0, 1/2)$ defined at (2.8) determines the smoothness coefficient of the $\mathcal{BSS}$ process X. In some sense, the coefficients α_i with $i \in \mathcal{A}$ will dominate when proving the limit theory for $QV(X, k, v\Delta_n)_t$. In particular, we will prove that $\mathrm{supp}(\pi) = \{\theta_i\}_{i \in \mathcal{A}}$. $\qquad\square$

Remark 2.4. The weight function considered in Example 2.1 obviously does not satisfy the assumption (A). Indeed, in the framework of Example 2.1 the limit theory for $QV(X, k, v\Delta_n)_t$ relies on semimartingale methods (cf. [7]) as X is a difference of two martingales (although X is not a semimartingale). In the case of Assumption (A) we are in the framework of fractional processes. More precisely, the small scale behaviour of the Gaussian core G of X is close to the small scale bahaviour of a fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$. In this situation the limit theory for $QV(X, k, v\Delta_n)_t$ relies on Malliavin calculus and Bernstein's blocking technique. $\qquad\square$

Remark 2.5. In papers [4, 5, 13], where $l = 0$ holds, the function f_0 is assumed to be slowly varying at $\theta_0 = 0$. In this setting more assumptions are required to establish the limit theory than mere Condition (A). In our paper we impose a bit stronger assumptions on functions f_j, $j = 0, \ldots, l$, in order to avoid a longer set of further conditions.

    Note that condition (A)(ii) implies a symmetric behaviour of the function g around the points θ_j, $j = 1, \ldots, l$. Instead we could have assumed different power behaviour left and right from θ_j. Although certain constants in the limit theorems would change in this case, the asymptotic theory remains essentially the same. $\qquad\square$

3 Limit theorems

3.1 Law of large numbers

For any number $k \geq 1$ and $v = 1, 2$, we introduce a k-th order filter associated with g via

$$\Delta_k^{n,v} g(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} g(x - vj\Delta_n), \qquad x \in \mathbb{R}. \tag{3.1}$$

There is a straightforward relationship between the scaling quantity $\tau_k(v\Delta_n)$ defined at (2.6) and the function $\Delta_k^{n,v} g$, namely

$$\tau_k(v\Delta_n)^2 = \|\Delta_k^{n,v} g\|_{\mathbb{L}^2(\mathbb{R})}^2.$$

Now, we define the corresponding measures associated with $\Delta_k^{n,v} g$:

$$\pi_{n,k}^v(A) := \frac{\int_A (\Delta_k^{n,v} g(x))^2 dx}{\|\Delta_k^{n,v} g\|_{\mathbb{L}^2(\mathbb{R})}^2}, \qquad A \in \mathcal{B}(\mathbb{R}). \tag{3.2}$$

In order to identify the limit of $\pi_{n,k}^v$, we define the following functions

$$h_0(x) := f_0(\theta_0) \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x - j)_+^{\alpha_0}, \tag{3.3}$$

$$h_i(x) := f_i(\theta_i) \sum_{j=0}^{k} (-1)^j \binom{k}{j} |x - j|^{\alpha_i}, \qquad i = 1, \ldots, l,$$

where $x_+ := \max\{x, 0\}$. At this stage we suppress the dependence of functions h_i on the index k. Our first result presents the limiting measure π_k, which will appear in the law of large numbers.

Proposition 3.1. *Assume that Condition (A) holds. Then we deduce that*

$$\pi_{n,k}^v \xrightarrow{d} \pi_k,$$

for any $k \geq 1$ and $v = 1, 2$, where the probability measure π_k is given as

$$\mathrm{supp}(\pi_k) = \{\theta_i\}_{i \in \mathcal{A}}, \quad \pi_k(\theta_i) = \frac{\|h_i\|_{\mathbb{L}^2(\mathbb{R})}^2 \mathbb{1}_{i \in \mathcal{A}}}{\sum_{j=0}^{l} \|h_j\|_{\mathbb{L}^2(\mathbb{R})}^2 \mathbb{1}_{j \in \mathcal{A}}}. \tag{3.4}$$

Recalling the definition of the set $\mathcal{A}$ at (2.8), Proposition 3.1 says that only singularities corresponding to the minimal indexes α_i (i.e. indexes with $\alpha_i = \alpha$) contribute to the limit. We remark that the norms $\|h_i\|_{\mathbb{L}^2(\mathbb{R})}$ are indeed finite, since for $|x|$ large enough

$$|h_i(x)|^2 \leq C|x|^{2(\alpha_i - k)} \quad \text{and} \quad 2(\alpha_i - k) < -1,$$

for any $k \geq 1$ and $\alpha_i \in (-1/2, 0) \cup (0, 1/2)$ due to Taylor expansion. Our next result is the law of large numbers for the statistic $QV(X, k, v\Delta_n)$.

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Theorem 3.2. *Assume that condition (A) holds. Then*

$$\frac{\Delta_n}{\tau_k(v\Delta_n)^2} QV(X, k, v\Delta_n)_t \xRightarrow{u.c.p.} QV(X, k)_t := \int_0^\infty \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds \right) \pi_k(d\theta), \quad (3.5)$$

where the probability measure π_k is introduced in (3.4).

3.2 Central limit theorem

Now, we will present a stable central limit theorem associated with convergence in (3.5). Let us shortly recall the notion of stable convergence, which is originally due to Rényi [20]. We say that a sequence of processes Y^n converges stably in law to a process Y, where Y is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in the space $\mathbb{D}([0, T])$ equipped with the uniform topology $(Y^n \xrightarrow{d_{st}} Y)$ if and only if

$$\lim_{n \to \infty} \mathbb{E}[f(Y^n)Z] = \mathbb{E}'[f(Y)Z]$$

for any bounded and continuous function $f : \mathbb{D}([0, T]) \to \mathbb{R}$ and any bounded $\mathcal{F}$-measurable random variable Z. We refer to [1, 20] for a detailed study of stable convergence. Note that stable convergence is a stronger mode of convergence than weak convergence, but it is weaker than u.c.p. convergence.

The stable central limit theorem associated with convergence in (3.5) is different compared to the corresponding result in the case of a single singularity (cf. [4, 5]). In particular, as we will see below, the limiting process is not an $\mathcal{F}$-conditional martingale on every interval $[0, T]$, but just for small enough T. For the purpose of statistical inference we present a joint central limit theorem for the pair $(QV(X, k, \Delta_n), QV(X, k, 2\Delta_n))$.

Theorem 3.3. *Assume that condition (A) holds and the intermittency process σ is Hölder continuous of order $\gamma > 1/2$. If $k = 1$ we further assume that $\alpha_j \in (-\frac{1}{2}, 0)$ for all $0 \le j \le l$. Then, under condition*

$$\alpha_i - \alpha > 1/4 \qquad for \ all \ i \notin \mathcal{A}, \tag{3.6}$$

we obtain the stable convergence

$$\Delta_n^{-1/2} \left(\frac{\Delta_n}{\tau_k(\Delta_n)^2} QV(X, k, \Delta_n)_t - QV(X, k)_t, \right.$$

$$\left. \frac{\Delta_n}{\tau_k(2\Delta_n)^2} QV(X, k, 2\Delta_n)_t - QV(X, k)_t \right)^\star \xrightarrow{d_{st}} L_t = \int_0^t \mathcal{V}_s^{1/2} dB_s \quad (3.7)$$

on $\mathbb{D}^2([0, \min_{1 \le j \le l}(\theta_j - \theta_{j-1})])$ equipped with the uniform topology, where B is a two-dimensional Brownian motion, independent of $\mathcal{F}$, defined on an extension of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $x^\star$ denotes the transpose of x. The matrix $\mathcal{V}_s$ is given by

$$\mathcal{V}_s = \left(\int_0^\infty \sigma_{s-\theta}^2 \pi_k(d\theta) \right)^2 \Lambda_k, \tag{3.8}$$

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where the 2×2 matrix $\Lambda_k = (\lambda_{ij}^k)_{1 \leq i,j \leq 2}$ is defined by

$$\lambda_{11}^k = \lim_{n \to \infty} \Delta_n^{-1} \operatorname{var}\left(\frac{\Delta_n}{\hat{\tau}_k(\Delta_n)^2} QV(B^H, k, \Delta_n)_1 \right),$$

$$\lambda_{22}^k = \lim_{n \to \infty} \Delta_n^{-1} \operatorname{var}\left(\frac{\Delta_n}{\hat{\tau}_k(2\Delta_n)^2} QV(B^H, k, 2\Delta_n)_1 \right) \qquad (3.9)$$

$$\lambda_{12}^k = \lim_{n \to \infty} \Delta_n^{-1} \operatorname{cov}\left(\frac{\Delta_n}{\hat{\tau}_k(\Delta_n)^2} QV(B^H, k, \Delta_n)_1, \frac{\Delta_n}{\hat{\tau}_k(2\Delta_n)^2} QV(B^H, k, 2\Delta_n)_1 \right)$$

with B^H being a fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$ and $\hat{\tau}_k(v\Delta_n)^2 := \mathbb{E}[(\Delta_{i,k}^{n,v} B^H)^2]$.

The Hölder condition is a standard requirement for the validity of the blocking technique applied in the proofs (cf. [4, 5]). As we remarked earlier, the singularity points θ_i with $i \notin \mathcal{A}$ do not affect the law of large numbers in (3.5). However, they are responsible for a certain bias, which might explode in the central limit theorem. Assumption (3.6) guarantees that it does not happen.

The appearance of the fractional Brownian motion in the definition of the matrix Λ_k is explained by the fact that the local behaviour of the Gaussian core G is close to the local behaviour of B^H with $H = \alpha + 1/2$. In the terminology of the theory of Gaussian fields it means that B^H is a *tangent process* of G. In particular, the correlation structure of increments of G converges to the correlation structure of increments of B^H.

Remark 3.4. The limiting process L is an $\mathcal{F}$-conditional Gaussian martingale on the interval $[0, \min_{1 \leq j \leq l}(\theta_j - \theta_{j-1})]$. Outside of this interval the $\mathcal{F}$-conditional martingale property gets lost. One may still show a stable central limit theorem with an $\mathcal{F}$-conditional Gaussian process as the limit, but only when $\theta_j - \theta_{j-1} \in \mathbb{N}$ for all j, since otherwise the covariance structure of the original statistic does not converge. We dispense with the exact presentation of this case. $\qquad \square$

Remark 3.5. The limits in (3.9) are indeed finite and can be computed explicitly. To see this, let us define the fractional Brownian noise of order k and scale $v = 1, 2$ via

$$\Delta_{i,k}^v B^H := \sum_{j=0}^{k} (-1)^j \binom{k}{i} B_{i-vj}^H, \qquad (3.10)$$

and set

$$\rho_k^{v_1,v_2}(j) := \operatorname{corr}(\Delta_{i,k}^{v_1} B^H, \Delta_{i+j,k}^{v_2} B^H) \qquad (3.11)$$

(Recall that B^H has stationary increments.) Using the covariance kernel of the fractional Brownian motion one can compute the quantity $\rho_k^{v_1,v_2}(j)$ explicitly. For instance,

$$\rho_1^{1,1}(j) = \tfrac{1}{2}\big(|j+1|^{2H} - 2|j|^{2H} + |j-1|^{2H}\big), \qquad j \geq 1.$$

A straightforward computation shows that $|\rho_k^{v_1,v_2}(j)| = O(|j|^{2(H-k)})$ as $|j| \to \infty$. Hence, using H-self similarity of B^H and the formula $\mathbb{E}[(Y_1^2 - 1)(Y_2^2 - 1)] = 2\mathbb{E}[Y_1 Y_2]^2$

for jointly normal vector (Y_1, Y_2) with standard normal marginal distribution, we conclude that

$$\lambda_{v_1, v_2}^k = 2\Big(1 + \sum_{j \in \mathbb{Z}\setminus\{0\}} \rho_k^{v_1, v_2}(j)^2\Big),$$

where the latter series is finite for all $k \geq 2$ and also for $k = 1$ if $H = \alpha + 1/2 < 3/4$ holds. The condition $H < 3/4$ is well known in the framework of Breuer-Major central limit theorems for quadratic functionals (see [12]). This condition directly translates to $\alpha < 1/4$. However, we require an additional restriction $\alpha < 0$ when $k = 1$ in Theorem 3.3 due to a certain bias, which might affect the central limit theorem. □

Remark 3.6. Theorem 3.3 deals with realised quadratic variation only, since it is sufficient for the estimation of the smoothness parameter α as we will see below. However, we do think that the asymptotic theory can be extended to functionals of the type

$$V(X, h, k, v\Delta_n)_t := \sum_{i=vk}^{[t/\Delta_n]} h\left(\frac{\Delta_{i,k}^{n,v} X}{\tau_k(v\Delta_n)}\right),$$

where $h \in C^1(\mathbb{R})$ is an even function. The main step of the proof is the approximation

$$\Delta_{i,k}^{n,v} X \approx \sum_{j=0}^{l} \sigma_{i\Delta_n - \theta_j} \Delta_{i,k}^{n,v} G^{(j)},$$

where $\Delta_{i,k}^{n,v} G^{(j)}$, $j = 0, \ldots, l$, are certain Gaussian random variables. Using Bernstein's blocking technique, which amounts in freezing the intermittency process σ in the beginning of sub-blocks, the asymptotic behaviour of the statistic $V(X, h, k, v\Delta_n)$ is determined by the functional

$$Q(z, \widetilde{h}, k, v\Delta_n)_t := \sum_{i=vk}^{[t/\Delta_n]} \widetilde{h}\left(z_0 \frac{\Delta_{i,k}^{n,v} G^{(0)}}{\tau_k(v\Delta_n)}, \ldots, z_l \frac{\Delta_{i,k}^{n,v} G^{(l)}}{\tau_k(v\Delta_n)}\right), \qquad z \in \mathbb{R}^{l+1},$$

where $\widetilde{h} \in C^1(\mathbb{R}^{l+1})$. The central limit theorem for a standardized version of $Q(\cdot, \widetilde{h}, k, v\Delta_n)$ relies on the stable convergence of finite dimensional distributions and tightness. The convergence of finite dimensional distributions is a classical setting of Breuer-Major central limit theorem. It can be shown via method of moments or using more modern methods of Malliavin calculus (see [18, 19] among others). We remark that in the case $h(x) = x^p$, where p is an even number, we do not need to consider the process $Q(\cdot, \widetilde{h}, k, v\Delta_n)$ and the proof becomes simpler due to binomial formula. □

4 The ratio statistic

The smoothness parameter α defined at (2.8) describes the Hölder continuity index of X, i.e. X is Hölder continuous of any order smaller than $H = \alpha + 1/2$. In the context of turbulence modeling the parameter α is connected to the so called

Kolmogorov's 2/3-law (see [17]). It predicts that $\alpha \approx -1/6$ (or, in other words, $2(\alpha + 1/2) \approx 2/3$). From this perspective it is important to construct a consistent estimator of α to check if $\mathcal{BSS}$ models adequately describe the physical laws.

The next lemma is crucial for estimating α.

Lemma 4.1. *Assume that conditions (A) and (3.6) hold. When $k = 1$ we further assume that $\alpha_j \in (-1/2, 0)$ for all $0 \le j \le l$. Then we obtain*

$$\tau_k(v\Delta_n)^2 = (v\Delta_n)^{2\alpha+1} \sum_{j=1}^{l} \|h_j\|_{\mathbb{L}^2(\mathbb{R})}^2 1_{j\in\mathcal{A}} + o(\Delta_n^{2\alpha+3/2}), \tag{4.1}$$

where the functions h_j were defined in (3.3).

Now, Lemma 4.1 and Theorem 3.2 provide a direct way of estimating the scaling parameter α. Indeed, we observe that

$$S_n := \frac{QV(X, k, 2\Delta_n)_t}{QV(X, k, \Delta_n)_t} \xrightarrow{\mathbb{P}} 2^{2\alpha+1},$$

for any fixed $t > 0$. Thus, a consistent estimator of α is given via

$$\widehat{\alpha}_n = \frac{1}{2}\left(\log_2\left(\frac{QV(X, k, 2\Delta_n)_t}{QV(X, k, \Delta_n)_t}\right) - 1\right) \xrightarrow{\mathbb{P}} \alpha, \tag{4.2}$$

where $\log_2$ denotes the logarithm at basis 2. We remark that this is exactly the same estimator as proposed in [4, 5] for $\mathcal{BSS}$ processes with a single singularity at 0. A feasible central limit theorem for $\widehat{\alpha}_n$ is obtained as follows. Note that the applicability of the result below relies on an a priori knowledge of the lower bound of $\min_{1 \le j \le l}(\theta_j - \theta_{j-1})$.

Theorem 4.2. *Assume that the conditions of Theorem 3.3 hold and choose $t < \min_{1 \le j \le l}(\theta_j - \theta_{j-1})$.*

(i) Define

$$QQ(X, k, v\Delta_n)_t := \sum_{i=vk}^{[t/\Delta_n]} (\Delta_{i,k}^{n,v} X)^4. \tag{4.3}$$

Then we obtain that

$$\frac{\Delta_n}{\tau_k(v\Delta_n)^4} QQ(X, k, v\Delta_n)_t \xrightarrow{u.c.p.} 3 \int_0^t \left(\int_0^\infty \sigma_{s-\theta}^2 \, \pi_k(d\theta)\right)^2 ds$$

(ii) Furthermore, we have for any fixed $t > 0$

$$\frac{2\log(2)QV(X, k, \Delta_n)_t(\widehat{\alpha}_n - \alpha)}{\sqrt{\frac{1}{3}QQ(X, k, \Delta_n)_t(-1, 1)\Lambda_k^n(-1, 1)^\star}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{4.4}$$

where $\log$ denotes the logarithm at basis e and the matrix Λ_k^n is defined as Λ_k in (3.9), where the unknown parameter α is replaced by $\widehat{\alpha}_n$ (recall that due to Remark 3.5 the matrix Λ_k is a function of α).

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Proof. Here we demonstrate the proof of part (ii), while part (i) will be proved in Section 5. First of all, we remark that under the condition $t < \min_{1 \le j \le l}(\theta_j - \theta_{j-1})$ we may apply the result of Theorem 3.3. Define the functions $r(x,y) = 2^{2\alpha+1}\frac{y}{x}$ and $h(x) = \frac{1}{2}(\log_2(x) - 1)$. Lemma 4.1 implies that

$$\frac{\tau_k(2\Delta_n)^2}{\tau_k(\Delta_n)^2} = 2^{2\alpha+1} + o(\Delta_n^{1/2}).$$

Hence,

$$S_n = r\left(\frac{\Delta_n}{\tau_k(\Delta_n)^2} QV(X,k,\Delta_n)_t, \frac{\Delta_n}{\tau_k(2\Delta_n)^2} QV(X,k,2\Delta_n)_t\right) + o_{\mathbb{P}}(\Delta_n^{1/2}).$$

Putting things together we conclude that

$$\Delta_n^{-1/2}(\widehat{\alpha}_n - \alpha)$$
$$= h \circ r\left(\frac{\Delta_n}{\tau_k(\Delta_n)^2} QV(X,k,\Delta_n)_t, \frac{\Delta_n}{\tau_k(2\Delta_n)^2} QV(X,k,2\Delta_n)_t\right) + o_{\mathbb{P}}(1).$$

Applying Theorem 3.3 and delta method for stable convergence we deduce that

$$\Delta_n^{-1/2}(\widehat{\alpha}_n - \alpha) \xrightarrow{d_{st}} \mathcal{MN}(0, V^2),$$

where $\mathcal{MN}(0, V^2)$ denotes a mixed normal variable with mean 0 and conditional variance V^2 defined by

$$V^2 := \frac{(-1,1)\Lambda_k(-1,1)^{\star} \int_0^t \left(\int_0^\infty \sigma_{s-\theta}^2\, \pi_k(d\theta)\right)^2 ds}{\left(2\log(2)QV(X,k)_t\right)^2},$$

where the matrix Λ_k is defined by (3.9). Notice that Λ_k is a continuous function of α due to Remark 3.5. Hence,

$$\Lambda_k^n \xrightarrow{\mathbb{P}} \Lambda_k.$$

The two other random quantities involved in the definition of V^2 can be directly estimated via part (i) of Theorem 4.2 and Theorem 3.2. Consequently, the properties of stable convergence imply part (ii) of Theorem 4.2. $\qquad\square$

Note that the standardized statistic in (4.4) is feasible as it does not require the knowledge of the weight function g. We remark that (4.4) coincides with the statistic presented in [13, Proposition 4.2] in the framework of a single singularity at 0. This demonstrates that the test statistic in (4.4) is robust to model misspecification within the setting of Assumption (A) and condition (3.6). In the context of turbulence modeling this is a very important property.

5 Proofs

5.1 Proof of Proposition 3.1 and Lemma 4.1

We first prove Lemma 4.1 as its proof essentially implies Proposition 3.1. Throughout this section all positive constants are denoted by C, or C_p if they depend on an external parameter p, although they may change from line to line.

Proof of Lemma 4.1. We assume without loss of generality that $l = 2$, $\alpha_0 = \alpha_1 = \alpha$ and $\alpha_2 - \alpha > 1/4$ (since condition (3.6) was assumed). Moreover, let $v = 1$. Recall the identity

$$\tau_k(\Delta_n)^2 = \|\Delta_k^{n,1} g\|_{\mathbb{L}^2(\mathbb{R})}^2$$

(cf. Section 3.1). We consider the decomposition

$$\|\Delta_k^{n,1} g\|_{\mathbb{L}^2(\mathbb{R})}^2 = \sum_{j=0}^{2} \int_{\theta_j - \delta}^{\theta_j + \delta} \Delta_k^{n,1} g(x)^2 dx + \int_{\delta}^{\theta_1 - \delta} \Delta_k^{n,1} g(x)^2 dx + \int_{\theta_1 + \delta}^{\theta_2 - \delta} \Delta_k^{n,1} g(x)^2 dx$$
$$+ \int_{\theta_2 + \delta}^{\infty} \Delta_k^{n,1} g(x)^2 dx. \tag{5.1}$$

We will now show that

$$\int_{\theta_j - \delta}^{\theta_j + \delta} \Delta_k^{n,1} g(x)^2 dx = \Delta_n^{2\alpha+1} \|h_j\|_{\mathbb{L}^2(\mathbb{R})} + o(\Delta_n^{2\alpha+3/2}), \qquad j = 0, 1, \tag{5.2}$$

and all other terms in the decomposition are $o(\Delta_n^{2\alpha+3/2})$ under the assumptions of Lemma 4.1. We start with the negligibility of the three last terms in (5.1). Using Taylor expansion of order k and integrability condition (A)(iv), we immediately conclude that

$$\int_{\delta}^{\theta_1 - \delta} \Delta_k^{n,1} g(x)^2 dx = O(\Delta_n^{2k}), \quad \int_{\theta_1 + \delta}^{\theta_2 - \delta} \Delta_k^{n,1} g(x)^2 dx = O(\Delta_n^{2k}),$$
$$\int_{\theta_2 + \delta}^{\infty} \Delta_k^{n,1} g(x)^2 dx = O(\Delta_n^{2k}),$$

so all these terms are $o(\Delta_n^{2\alpha+3/2})$ under assumptions of Lemma 4.1. Now, we show (5.2) for $j = 0$; the case $j = 1$ works similarly. Proving this statement for $j = 0$ essentially means that we can replace $f_0(x)$ involved in the integral by the constant $f_0(0)$. Let $\varepsilon > 0$ be small enough with $\varepsilon >> \Delta_n$. Using again Taylor expansion of order k and integrability condition (A)(iv), we conclude that (recall that $g(x) = 0$ for $x \leq 0$)

$$\int_{-\delta}^{\delta} \Delta_k^{n,1} g(x)^2 dx = \int_0^{\varepsilon} \Delta_k^{n,1} g(x)^2 dx + O(\Delta_n^{2k} \varepsilon^{2(\alpha-k)+1}).$$

When we replace the function f_0 that appears in the latter integral by a constant $f_0(0)$, we deduce by substitution $x = \Delta_n y$

$$f_0(0)^2 \int_0^{\varepsilon} (\Delta_k^{n,1}(x^{\alpha}))^2 dx = \Delta_n^{2\alpha+1} \int_0^{\varepsilon/\Delta_n} h_0(y)^2 dy$$

$$= \Delta_n^{2\alpha+1} \int_0^{\infty} h_0(y)^2 dy + O(\Delta_n^{2k} \varepsilon^{2(\alpha-k)+1}),$$

since $|h_0(x)|^2 \leq C|x|^{2(\alpha-k)}$ for large x and $2(\alpha - k) < -1$. Note that the dominating

term is exactly the one given in (5.2). Now, let us evaluate the difference

$$\int_0^\varepsilon \Delta_k^{n,1} g(x)^2 dx - f_0(0)^2 \int_0^\varepsilon (\Delta_k^{n,1}(x^\alpha))^2 dx$$

$$= \int_0^{k\Delta_n} \Delta_k^{n,1} g(x)^2 - f_0(0)^2 (\Delta_k^{n,1}(x^\alpha))^2 dx$$

$$+ \int_{k\Delta_n}^\varepsilon \Delta_k^{n,1} g(x)^2 - f_0(0)^2 (\Delta_k^{n,1}(x^\alpha))^2 dx.$$

Using differentiability of f_0 we immediately conclude that

$$\int_0^{k\Delta_n} |\Delta_k^{n,1} g(x)^2 - f_0(0)^2 (\Delta_k^{n,1}(x^\alpha))^2| dx = O(\Delta_n^{2\alpha+2}).$$

The other integral has to be treated differently. In the following we present the computations only for $k = 1, 2$ (in fact, the case $k \geq 3$ is easier to treat). We start with $k = 1$. Using binomial rule, differentiability of f_0 and substitution, we conclude that

$$\left| \int_{\Delta_n}^\varepsilon \Delta_1^{n,1} g(x)^2 - f_0(0)^2 (\Delta_1^{n,1}(x^\alpha))^2 dx \right| \leq C\Delta_n^{2\alpha+2} \int_1^{\varepsilon/\Delta_n} |h_0(y)| y^{\alpha+1} dy$$

Using the inequality $|h_0(x)| \leq C|x|^{\alpha-1}$ for large x, we deduce that

$$\Delta_n^{2\alpha+2} \int_k^{\varepsilon/\Delta_n} |h_0(y)| y^{\alpha+1} dy \leq C\Delta_n \varepsilon^{2\alpha+1}$$

Setting $\varepsilon = \Delta_n^{1/2}$, we deduce that all involved small order terms are $o(\Delta_n^{2\alpha+3/2})$ when $\alpha < 0$. Now, we consider the case $k = 2$. Since f_0 is twice continuously differentiable, we may apply the Taylor expansion to

$$f_0(x + m\Delta_n) = f_0(0) + mf_0'(0)\Delta_n + \tfrac{1}{2} m^2 \Delta_n^2 f_0''(x_m),$$

where $x_m \in (0, x + m\Delta_n)$ and $m = 0, 1, 2$. Using the above Taylor expansion and the binomial formula, we deduce that

$$\left| \int_{k\Delta_n}^\varepsilon \Delta_k^{n,1} g(x)^2 - f_0(0)^2 (\Delta_k^{n,1}(x^\alpha))^2 dx \right| \leq C(q_1(n, \varepsilon) + q_2(n, \varepsilon) + q_3(n, \varepsilon)),$$

where

$$q_1(n, \varepsilon) = \int_{k\Delta_n}^\varepsilon h_0(x)^2 x dx,$$

$$q_2(n, \varepsilon) = \Delta_n \int_{k\Delta_n}^\varepsilon |h_0(x)||(x + 2\Delta_n)^\alpha - (x + \Delta_n)^\alpha| dx,$$

$$q_3(n, \varepsilon) = \int_{k\Delta_n}^\varepsilon |h_0(x)| x^{\alpha+2} dx.$$

Applying the substitution $x = \Delta_n y$, we get

$$q_1(n, \varepsilon) = O(\Delta_n^{2\alpha+2}), \quad q_2(n, \varepsilon) = O(\Delta_n^{2\alpha+2}), \quad q_3(n, \varepsilon) = O(\Delta_n^2 \varepsilon^{2\alpha+1}).$$

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Setting now $\varepsilon = \Delta_n^{1/2}$, we conclude that all second order terms are $o(\Delta_n^{2\alpha+3/2})$.

Finally, let us treat the case $j = 2$. Since

$$\int_{\theta_2-\delta}^{\theta_2+\delta} \Delta_k^{n,1} g(x)^2 dx = O(\Delta_n^{2\alpha_2+1})$$

as shown above and $\alpha_2 - \alpha > 1/4$, we see that this term is $o(\Delta_n^{2\alpha+3/2})$. Consequently, we obtain the assertion of Lemma 4.1. $\qquad\square$

Proof of Proposition 3.1. The assertion of Proposition 3.1 now easily follows from the proof of Lemma 4.1. First of all, it implies that

$$\|\Delta_k^{n,v} g\|_{\mathbb{L}^2(\mathbb{R})}^2 = (v\Delta_n)^{2\alpha+1} \sum_{j=0}^{l} \|h_j\|_{\mathbb{L}^2(\mathbb{R})}^2 1_{j\in\mathcal{A}} + o(\Delta_n^{2\alpha+1}),$$

even without condition (3.6). On the other hand the proof of Lemma 4.1 also implies that

$$\int_{\theta_j-\varepsilon}^{\theta_j+\varepsilon} \Delta_k^{n,v} g(x)^2 dx = (v\Delta_n)^{2\alpha+1} \|h_j\|_{\mathbb{L}^2(\mathbb{R})}^2 + o(\Delta_n^{2\alpha+1}) \qquad \text{if } j \in \mathcal{A},$$

for any $\varepsilon < \min_{1\leq i\leq l}(\theta_i - \theta_{i-1})$, and

$$\int_a^b \Delta_k^{n,v} g(x)^2 dx = o(\Delta_n^{2\alpha+1})$$

if the interval $[a, b]$ does not contain any θ_j with $j \in \mathcal{A}$. This completes the proof of Proposition 3.1. $\qquad\square$

5.2   Some preliminaries

Before we prove the main results of the paper, we start with some preliminaries. We remark that the intermittency process σ is cádlág, hence σ_- is locally bounded. Since our Theorems 3.2, 3.3 and 4.2 are stable under localization (cf. [7]), we may and will assume that σ is bounded on compact intervals.

Recalling the notation of (3.1) we introduce the following Gaussian random variables

$$\Delta_{i,k}^{n,v} G^{(j)} := \int_{i\Delta_n-\theta_j-\delta}^{i\Delta_n-\theta_j+\delta} \Delta_k^{n,v} g(i\Delta_n - s) W(ds), \qquad j = 0, \dots, l, \qquad (5.3)$$

where the constant $\delta > 0$ was defined in (A). Notice that the above Gaussian variables are independent for different j's when computed at the same stage $i\Delta_n$. One of the key steps of our proofs is to show the approximation

$$\Delta_{i,k}^{n,v} X \approx \sum_{j=0}^{l} \sigma_{(i-vk)\Delta_n-\theta_j} \Delta_{i,k}^{n,v} G^{(j)}. \qquad (5.4)$$

(cf. Sections 5.3 and 5.4.) The ideas behind the proofs of Theorems 3.2, 3.3 and 4.2 follow a similar structure as presented in [4, 5], although the situation is more complex due to multiple singularities of the weight function g. First of all, we will

use a blocking technique, which amounts in considering a subdivision of the interval $[0, t]$ into equidistant sub-blocks and freezing the intermittency process σ within each sub-block. In a second step, we will prove joint limit theorems over the sub-blocks applying Malliavin calculus and properties of stable convergence.

We start with the limit theory for the Gaussian variables $\Delta_{i,k}^{n,v} G^{(j)}$, which has been essentially treated in [4]. Define

$$\tau_{k,j}(v\Delta_n)^2 := \mathbb{E}[(\Delta_{i,k}^{n,v} G^{(j)})^2], \quad r_{k,n}^{v_1,v_2,j_1,j_2}(q) := \mathrm{corr}\big(\Delta_{1,k}^{n,v_1} G^{(j_1)}, \Delta_{1+q,k}^{n,v_2} G^{(j_2)}\big). \quad (5.5)$$

We consider the statistics

$$QV(k, v\Delta_n)_t^{j_1,j_2} := \Delta_n \sum_{i=vk}^{[t/\Delta_n]} \frac{\Delta_{i,k}^{n,v} G^{(j_1)} \Delta_{i,k}^{n,v} G^{(j_2)}}{\tau_{k,j_1}(v\Delta_n)\tau_{k,j_2}(v\Delta_n)}, \qquad 0 \le j_1, j_2 \le l, \quad (5.6)$$

and set

$$\rho_k^{v_1,v_2,j}(q) := \mathrm{corr}\big(\Delta_{1,k}^{v_1} B^{H_j}, \Delta_{1+q,k}^{v_2} B^{H_j}\big), \quad (5.7)$$

where B^{H_j} is a fractional Brownian motion with Hurst parameter $H_j = \alpha_j + 1/2$ and the quantity $\Delta_{i,k}^v B^{H_j}$ is defined in (3.10). The next result is essentially a combination of [4, Theorems 1 and 2] and [5, Section 2].

Theorem 5.1. *Assume that Condition (A) holds.*

(i) *We have the convergence*

$$QV(k, v\Delta_n)_t^{j_1,j_2} \xrightarrow{u.c.p.} QV(k)_t^{j_1,j_2} := \delta_{j_1,j_2} t, \qquad j_1, j_2 \in \mathcal{A}, \quad (5.8)$$

where $\delta_{j_1,j_2} = 1$ when $j_1 = j_2$ and 0 otherwise.

(ii) *When $k = 1$ we further assume that $\alpha_j < 0$ for all $0 \le j \le l$. Then*

$$\Delta_n^{-1/2}\big(QV(k, v\Delta_n)^{j_1,j_2} - QV(k)^{j_1,j_2}\big)_{j_1,j_2 \in \mathcal{A}, j_1 \le j_2}^{v=1,2} \xrightarrow{d_{st}} V = (V_{k,v}^{j_1,j_2})_{j_1,j_2 \in \mathcal{A}, j_1 \le j_2}^{v=1,2}, \quad (5.9)$$

on $\mathbb{D}^{|\mathcal{A}|(|\mathcal{A}|+1)}([0, \min_{1 \le j \le l}(\theta_j - \theta_{j-1})])$, where V is a Gaussian martingale, defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space and independent of $\mathcal{F}$. The covariance structure is given as

$$\mathbb{E}'[V_{k,v}^{j_1,j_2}(t) V_{k,v'}^{j_1',j_2'}(s)] = 0 \qquad when\ (j_1, j_2) \ne (j_1', j_2'),$$

$$\mathbb{E}'[V_{k,v}^{j,j}(t) V_{k,v'}^{j,j}(s)] = 2\min\{t, s\}\Big(1 + \sum_{q \in \mathbb{Z}\setminus\{0\}} \rho_k^{v,v',j}(q)^2\Big),$$

$$\mathbb{E}'[V_{k,v}^{j_1,j_2}(t) V_{k,v'}^{j_1,j_2}(s)] = \min\{t, s\}\Big(1 + \sum_{q \in \mathbb{Z}\setminus\{0\}} \rho_k^{v,v',j_1}(q)\rho_k^{v,v',j_2}(q)\Big) \quad for\ j_1 \ne j_2.$$

We remark that for $j, j_1, j_2 \in \mathcal{A}$ we immediately conclude that

$$\mathbb{E}'[V_{k,v}^{j,j}(1) V_{k,v'}^{j,j}(1)] = \lambda_{v,v'}^k, \quad (5.10)$$

$$\mathbb{E}'[V_{k,v}^{j_1,j_2}(1) V_{k,v'}^{j_1,j_2}(1)] = \tfrac{1}{2}\lambda_{v,v'}^k \qquad for\ j_1 \ne j_2,$$

where $\lambda_{v,v'}^k$ is defined by (3.9).

We divide the proof into several steps.

Step 1. We will not work with the random variables $\Delta_{i,k}^{n,v}G^{(j)}$ directly, but with their approximations. For any $0 \leq j \leq l$, we define a new Gaussian process

$$\widetilde{G}_t^{(j)} := \int_{\mathbb{R}} \widetilde{g}^{(j)}(t-s) W(ds),$$

where the function $\widetilde{g}^{(j)}$ $(j \geq 1)$ is given via

$$\widetilde{g}^{(j)}(x) = \widetilde{f}_j(x)|x - \theta_j|^{\alpha_j},$$

with $\widetilde{f}_j = f_j$ on $x \in (\theta_j - \delta/2, \theta_j + \delta/2)$, $\widetilde{f}_j = 0$ outside of the interval $(\theta_j - \delta, \theta_j + \delta)$, and $\widetilde{f}_j \in C^k(\mathbb{R})$ (the function $\widetilde{f}_0$ is defined similarly). Then, if we consider the k-th order increments $\Delta_{i,k}^{n,v}\widetilde{G}^{(j)}$ of $\widetilde{G}$ at frequency $v\Delta_n$, we readily deduce that

$$\mathbb{E}[(\Delta_{i,k}^{n,v}\widetilde{G}^{(j)} - \Delta_{i,k}^{n,v}G^{(j)})^2] \leq C\Delta_n^{2k} \tag{5.11}$$

due to assumption (A). Now, let us define the statistics

$$\widetilde{QV}(k, v\Delta_n)_t^{j_1,j_2} := \Delta_n \sum_{i=vk}^{[t/\Delta_n]} \frac{\Delta_{i,k}^{n,v}\widetilde{G}^{(j_1)} \Delta_{i,k}^{n,v}\widetilde{G}^{(j_2)}}{\widetilde{\tau}_{k,j_1}(v\Delta_n)\widetilde{\tau}_{k,j_2}(v\Delta_n)}, \quad \widetilde{\tau}_{k,j}(v\Delta_n)^2 := \mathbb{E}[(\Delta_{i,k}^{n,v}\widetilde{G}^{(j)})^2], \tag{5.12}$$

for $0 \leq j_1, j_2 \leq l$. Then, due to (5.11), $\widetilde{\tau}_{k,j}(v\Delta_n)/\tau_{k,j}(v\Delta_n) \to 1$ and

$$\widetilde{QV}(k, v\Delta_n)_t^{j_1,j_2} - QV(k, v\Delta_n)_t^{j_1,j_2} \stackrel{\text{u.c.p.}}{\Longrightarrow} 0, \tag{5.13}$$

and also

$$\Delta_n^{-1/2}\left(\widetilde{QV}(k, v\Delta_n)_t^{j_1,j_2} - QV(k, v\Delta_n)_t^{j_1,j_2}\right) \stackrel{\text{u.c.p.}}{\Longrightarrow} 0 \tag{5.14}$$

under the assumption of Theorem 5.1(ii), which is due to Cauchy-Schwarz inequality. Thus, it suffices to prove the asymptotic theory for the statistics $\widetilde{QV}(k, v\Delta_n)_t^{j_1,j_2}$. $\quad\square$

Step 2. In this step we analyze the correlation structure of the increments $\Delta_{i,k}^{n,v}\widetilde{G}^{(j)}$. We define

$$\widetilde{r}_{k,n}^{v_1,v_2,j_1,j_2}(q) := \text{corr}\left(\Delta_{1,k}^{n,v_1}\widetilde{G}^{(j_1)}, \Delta_{1+q,k}^{n,v_2}\widetilde{G}^{(j_2)}\right). \tag{5.15}$$

The next proposition describes the asymptotic behaviour of the correlation function $\widetilde{r}_{k,n}^{v_1,v_2,j_1,j_2}(q)$.

Proposition 5.2. *Assume that Condition (A) holds. Then we obtain that*

$$\widetilde{r}_{k,n}^{v_1,v_2,j,j}(q) \to \rho_k^{v_1,v_2,j}(q), \tag{5.16}$$

where $\rho_k^{v_1,v_2,j}(q)$ is defined at (5.7). Furthermore, for any $\epsilon > 0$ there exists $C > 0$ such that

$$|\widetilde{r}_{k,n}^{v_1,v_2,j,j}(q)| \leq C|q|^{2H_j-2k-\epsilon}. \tag{5.17}$$

Likewise, if $j_1, j_2 \in \mathcal{A}$, $j_1 > j_2$ then for any $\epsilon > 0$ there exists $C > 0$ such that

$$|\widetilde{r}_{k,n}^{v_1,v_2,j_1,j_2}(q)| \leq C|q + \Delta_n^{-1}(\theta_{j_1} - \theta_{j_2})|^{2\alpha+1-2k-\epsilon}. \tag{5.18}$$

Proof. We start with the proof of (5.16) and (5.17). Without loss of generality we prove it only for the case $k = 1$, $v_1 = v_2 = 1$ and $j = 0$; the rest follows by similar arguments. For simplicity we set $\widetilde{r}_n(q) := \widetilde{r}_{1,n}^{1,1,0,0}(q)$, $\rho(q) := \rho_1^{1,1,0}(q)$ and $\Delta_i^n \widetilde{G}^{(0)} := \Delta_{i,1}^{n,1} \widetilde{G}^{(0)}$. Observe that

$$\text{cov}\big(\Delta_1^n \widetilde{G}^{(0)}, \Delta_{1+q}^n \widetilde{G}^{(0)}\big) = \int_{(1+q)\Delta_n - \delta}^{\Delta_n + \delta} \{\widetilde{g}^{(0)}(\Delta_n - s) - \widetilde{g}^{(0)}(-s)\}$$
$$\times \{\widetilde{g}^{(0)}((1+q)\Delta_n - s) - \widetilde{g}^{(0)}(q\Delta_n - s)\} ds.$$

Now, recalling that $\widetilde{g}^{(0)}(x) = x^{\alpha_0} f_0(x)$ for $x \in (0, \delta/2)$, we conclude as in the proof of Lemma 4.1

$$\Delta_n^{-(2\alpha_0 + 1)} \text{cov}\big(\Delta_1^n \widetilde{G}^{(0)}, \Delta_{1+q}^n \widetilde{G}^{(0)}\big)$$
$$\to f_0(0)^2 \int_{\mathbb{R}} \{(1-s)_+^{\alpha_0} - (-s)_+^{\alpha_0}\}\{(q+1-s)_+^{\alpha_0} - (q-s)_+^{\alpha_0}\} ds. \quad (5.19)$$

The latter limit is, up to a factor $f_0(0)^2$, the covariance function of a (non-standard) fractional Brownian noise $(\widetilde{B}_i^{\alpha_0 + 1/2} - \widetilde{B}_{i-1}^{\alpha_0 + 1/2})_{i \geq 1}$, where $\widetilde{B}^{\alpha_0 + 1/2}$ is defined as

$$\widetilde{B}_t^{\alpha_0 + 1/2} := \int_{\mathbb{R}} \{(t-s)_+^{\alpha_0} - (-s)_+^{\alpha_0}\} W(ds).$$

Thus, using again Lemma 4.1, we deduce that

$$\widetilde{r}_n(q) \to \rho(q) \qquad \text{as } n \to \infty,$$

which completes the proof of (5.16).

Now, we define the function

$$\widetilde{R}_u := \int_{-\delta}^{\delta} (\widetilde{g}^{(0)}(-s) - \widetilde{g}^{(0)}(-u-s))^2 ds$$

and note that $\widetilde{R}_{\Delta_n} = \widetilde{\tau}_{1,0}(\Delta_n)^2$. According to [3, Lemma 1] and conditions (A1)–(A3) therein, it is sufficient to show that

$$\widetilde{R}_u = u^{2\alpha_0 + 1} Z_u, \qquad u > 0,$$

where $Z \in C^2(0, \infty)$ and $\lim_{u \to 0} Z_u \neq 0$, to conclude (5.17). Observe that for $u < \delta$

$$\widetilde{R}_u = \int_0^u s^{2\alpha_0} \widetilde{f}_0^2(s) ds + \int_u^\delta (s^{\alpha_0} \widetilde{f}_0(s) - (s-u)^{\alpha_0} \widetilde{f}_0^2(s-u))^2 ds$$

$$= u^{2\alpha_0 + 1} \left(\int_0^1 x^{2\alpha_0} \widetilde{f}_0^2(ux) dx + \int_1^{\delta/u} (x^{\alpha_0} \widetilde{f}_0(ux) - (x-1)^{\alpha_0} \widetilde{f}_0^2(ux - u))^2 dx \right)$$

$$= u^{2\alpha_0 + 1} Z_u.$$

Now, $Z \in C^2(0, \infty)$ since $\widetilde{f}_0 \in C^2(0, \delta)$; see condition (A)(iii). Furthermore,

$$\lim_{u \to 0} Z_u = f_0^2(0) \left(\int_0^1 x^{2\alpha_0} dx + \int_1^\infty (x^{\alpha_0} - (x-1)^{\alpha_0})^2 dx \right),$$

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where the limit is finite since $\alpha_0 < 1/2$ and strictly positive because $f_0(0) \neq 0$ (see again (A)(iii)). Thus, (5.17) follows.

Finally, we need to prove (5.18). We assume without loss of generality that $k = 1$, $v_1 = v_2 = 1$ and $\theta_{j_1} > \theta_{j_2} > 0$. For $q < 0$ such that $(1+q)\Delta_n - \theta_{j_2} - \delta \in [\Delta_n - \theta_{j_1} - \delta, \Delta_n - \theta_{j_1} - \delta)$, we obtain that

$$
\begin{aligned}
\mathrm{cov}&\big(\Delta_1^n \widetilde{G}^{(j_1)}, \Delta_{1+q}^n \widetilde{G}^{(j_2)}\big) \\
&= \int_{(1+q)\Delta_n - \theta_{j_2} - \delta}^{\Delta_n - \theta_{j_1} + \delta} \{\widetilde{g}^{(j_1)}(\Delta_n - s) - \widetilde{g}^{(j_1)}(-s)\} \\
&\qquad\qquad \times \{\widetilde{g}^{(j_2)}((1+q)\Delta_n - s) - \widetilde{g}^{(j_2)}(q\Delta_n - s)\}ds \\
&= \int_{(1+q)\Delta_n - \theta_{j_2} - \delta}^{\Delta_n - \theta_{j_1} + \delta} \{\widetilde{f}_{j_1}(\Delta_n - s)|\Delta_n - s - \theta_{j_1}|^\alpha - \widetilde{f}_{j_1}(-s)|s + \theta_{j_1}|^\alpha\} \\
&\qquad\qquad \times \{\widetilde{f}_{j_2}((1+q)\Delta_n - s)|(1+q)\Delta_n - s - \theta_{j_2}|^\alpha \\
&\qquad\qquad\qquad - \widetilde{f}_{j_2}(q\Delta_n - s)|q\Delta_n - s - \theta_{j_2}|^\alpha\}ds,
\end{aligned}
$$

where we recall that $\alpha_{j_1} = \alpha_{j_2} = \alpha$. In the next step we compare this expression with the following covariance

$$
\begin{aligned}
\mathrm{cov}&\big(\Delta_1^n \widetilde{G}^{(j_1)}, \Delta_{1+\bar{q}}^n \widetilde{G}^{(j_1)}\big) \\
&= \int_{(1+\bar{q})\Delta_n - \theta_{j_1} - \delta}^{\Delta_n - \theta_{j_1} + \delta} \{\widetilde{f}_{j_1}(\Delta_n - s)|\Delta_n - s - \theta_{j_1}|^\alpha - \widetilde{f}_{j_1}(-s)|s + \theta_{j_1}|^\alpha\} \\
&\qquad\qquad \times \{\widetilde{f}_{j_1}((1+\bar{q})\Delta_n - s)|(1+\bar{q})\Delta_n - s - \theta_{j_1}|^\alpha \\
&\qquad\qquad\qquad - \widetilde{f}_{j_1}(\bar{q}\Delta_n - s)|\bar{q}\Delta_n - s - \theta_{j_1}|^\alpha\}ds.
\end{aligned}
$$

Now, by setting $\bar{q} = [q + \Delta_n^{-1}(\theta_{j_1} - \theta_{j_2})]$ and recalling that all functions $\widetilde{f}_j$ satisfy the same assumption (A)(iii), and keeping in mind Lemma 4.1, we conclude that

$$
|r_n^{j_1, j_2}(q)| \leq C|r_n^{j_1, j_1}(\bar{q})|,
$$

which implies (5.18) by applying (5.17). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Step 3. Due to Step 1 it suffices to prove Theorem 5.1 for the statistics $\widetilde{QV}(k, v\Delta_n)_t^{j_1, j_2}$. We start with part (i). Assertions (5.16) and (5.17) immediately imply the convergence (5.8) by [4, Theorem 1] (or, more precisely, by its multivariate extension).

Part (ii) essentially follows from [4, Theorem 2]. First, we observe that our multivariate statistic is a functional of a Gaussian process. In this case it is sufficient to prove asymptotic normality for each component and to identify the covariance structure (this is due to the results of [19]). The asymptotic normality follows from the square summability of the bound in (5.17), i.e.

$$
\sum_{q=1}^{\infty} q^{2(2\alpha + 1 - 2k + \epsilon)} < \infty
$$

for $\epsilon > 0$ small enough (if $k = 1$ we require that $\alpha < 1/4$), which is a sufficient condition for asymptotic normality of each component due to [4, Theorem 2]. Furthermore, the convergence in (5.16) easily identifies the covariance structure of each component (see again [4, Theorem 2]), hence the last two identities of Theorem 5.1.

Now, let us prove the asymptotic independence of the involved components. As before we assume without loss of generality that $k = 1$, $v_1 = v_2 = 1$. We define

$$\widetilde{V}_n^j(t) = \Delta_n^{-1/2}\big(\widetilde{QV}(1, \Delta_n)_t^{j,j} - QV(1)_t^{j,j}\big)$$

and show that $\mathbb{E}[\widetilde{V}_n^{j_1}(t)\widetilde{V}_n^{j_2}(s)] \to 0$ for $j_1, j_2 \in \mathcal{A}$ with $j_1 \neq j_2$ (the asymptotic independence of all other components is shown in exactly the same manner). Recall that $|t - s| < \min_{1 \leq j \leq l}(\theta_j - \theta_{j-1})$. We deduce that

$$\mathbb{E}[\widetilde{V}_n^{j_1}(t)\widetilde{V}_n^{j_2}(s)] = 2\Delta_n \sum_{i_1=1}^{[t/\Delta_n]} \sum_{i_2=1}^{[s/\Delta_n]} |\widetilde{r}_n^{j_1,j_2}(i_2 - i_1)|^2$$

Now, for the sake of demonstration, we consider the case $t = s = \min_{1 \leq j \leq l}(\theta_j - \theta_{j-1}) = \theta_{j_1} - \theta_{j_2}$ (so $\theta_{j_1} > \theta_{j_2}$); in fact, the situation $t, s < \theta_{j_1} - \theta_{j_2}$ is easier to treat. Then the estimate (5.18) gives

$$\big|\mathbb{E}[\widetilde{V}_n^{j_1}(t)\widetilde{V}_n^{j_2}(s)]\big| \leq C\Delta_n \sum_{i=-[t/\Delta_n]+1}^{[t/\Delta_n]-1} |i + \Delta_n^{-1}(\theta_{j_1} - \theta_{j_2})|^{2(2\alpha-1+\epsilon)}([t/\Delta_n] - |i|)$$

Let $w \in (0, 1)$. We conclude that

$$\Delta_n \sum_{i=-w[t/\Delta_n]+1}^{[t/\Delta_n]-1} |i + \Delta_n^{-1}(\theta_{j_1} - \theta_{j_2})|^{2(2\alpha-1+\epsilon)}([t/\Delta_n] - |i|) \leq C(1-w)\Delta_n^{-4\alpha+1-2\epsilon}.$$

On the other hand we have that

$$\Delta_n \sum_{i=-[t/\Delta_n]+1}^{-w[t/\Delta_n]} |i + \Delta_n^{-1}(\theta_{j_1} - \theta_{j_2})|^{2(2\alpha-1+\epsilon)}([t/\Delta_n] - |i|) \leq C(1-w),$$

since $t = \theta_{j_1} - \theta_{j_2}$, $\alpha < 1/4$ and $\epsilon > 0$ can be chosen arbitrarily small. Hence, letting first $\Delta_n \to 0$ and then $w \to 1$ we obtain the desired convergence $\mathbb{E}[\widetilde{V}_n^{j_1}(t)\widetilde{V}_n^{j_2}(s)] \to 0$. $\qquad\square$

Finally, let us note that due to Theorem 5.1(i) we have that

$$\Delta_n \tau_k(\Delta_n)^{-2} \sum_{i=vk}^{[t/\Delta_n]} (\Delta_{i,k}^{n,v} G^{(j)})^2 = o_{\mathbb{P}}(\Delta_n^{1/2}) \qquad \text{for all } j \notin \mathcal{A},$$

due to Lemma 4.1 and the condition $\alpha_j - \alpha > 1/4$. On the other hand, when $j_1 \neq j_2$ and either $j_1 \notin \mathcal{A}$ or $j_2 \notin \mathcal{A}$, then we conclude that

$$\text{var}\big(QV(k, v\Delta_n)_t^{j_1,j_2}\big) = O_{\mathbb{P}}(\Delta_n)$$

under assumptions of Theorem 5.1 (the arguments are similar to the proof of Theorem 5.1). Thus, using again Lemma 4.1, we conclude that

$$\Delta_n \tau_k(\Delta_n)^{-2} \sum_{i=vk}^{[t/\Delta_n]} \Delta_{i,k}^{n,v} G^{(j_1)} \Delta_{i,k}^{n,v} G^{(j_2)} = o_{\mathbb{P}}(\Delta_n^{1/2}) \qquad (5.20)$$

whenever $j_1 \notin \mathcal{A}$ or $j_2 \notin \mathcal{A}$, under conditions of Theorem 3.3.

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5.3 Proof of Theorem 3.2 and Theorem 4.2(i)

Proof of Theorem 3.2. Below we apply a blocking technique, which means that we subdivide the interval $[0, t]$ into sub-blocks and freeze the intermittency process within each block. We remark that the statistic $QV(X, k, v\Delta_n)_t$ is increasing in t and the limiting process $QV(X, k)_t$ at (3.5) is continuous in t. For this reason it is sufficient to prove pointwise convergence

$$\frac{\Delta_n}{\tau_k(v\Delta_n)^2} QV(X, k, v\Delta_n)_t \xrightarrow{\mathbb{P}} QV(X, k)_t = \int_0^\infty \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds \right) \pi_k(d\theta)$$

for a fixed $t > 0$.

Now, we fix a natural number m and introduce the decomposition

$$\frac{\Delta_n}{\tau_k(v\Delta_n)^2} QV(X, k, v\Delta_n)_t - QV(X, k)_t = A_n + B_{n,m} + C_{n,m} + D_m, \qquad (5.21)$$

where

$$A_n := \frac{\Delta_n}{\tau_k(v\Delta_n)^2} \sum_{i=vk}^{[t/\Delta_n]} \left((\Delta_{i,k}^{n,v} X)^2 - \left(\sum_{j=0}^l \sigma_{(i-vk)\Delta_n - \theta_j} \Delta_{i,k}^{n,v} G^{(j)} \right)^2 \right)$$

$$B_{n,m} := \frac{\Delta_n}{\tau_k(v\Delta_n)^2} \left(\sum_{i=vk}^{[t/\Delta_n]} \left(\sum_{j=0}^l \sigma_{(i-vk)\Delta_n - \theta_j} \Delta_{i,k}^{n,v} G^{(j)} \right)^2 \right.$$
$$\left. - \sum_{r=1}^{[mt]} \sum_{i \in I_m(r)} \left(\sum_{j=0}^l \sigma_{(r-1)/m - \theta_j} \Delta_{i,k}^{n,v} G^{(j)} \right)^2 \right)$$

$$C_{n,m} := \frac{\Delta_n}{\tau_k(v\Delta_n)^2} \sum_{r=1}^{[mt]} \sum_{i \in I_m(r)} \left(\sum_{j=0}^l \sigma_{(r-1)/m - \theta_j} \Delta_{i,k}^{n,v} G^{(j)} \right)^2$$
$$- \frac{1}{m} \sum_{r=1}^{[mt]} \int_0^\infty \sigma_{(r-1)/m - \theta}^2 \pi_k(d\theta)$$

$$D_m := \frac{1}{m} \sum_{r=1}^{[mt]} \int_0^\infty \sigma_{(r-1)/m - \theta}^2 \pi_k(d\theta) - \int_0^\infty \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds \right) \pi_k(d\theta)$$

with

$$I_m(r) := \left\{ i \colon i\Delta_n \in \left(\frac{r-1}{m}, \frac{r}{m} \right] \right\}.$$

Let us give an interpretation to the introduced decomposition. The term A_n is the error associated with the crucial approximation introduced in (5.4). In a second step we divide the interval $[0, t]$ into $[mt]$ sub-blocks and freeze the intermittency σ in the beginning of each block; the associated error is represented by $B_{n,m}$. Within each sub-block we apply the law of large numbers to the Gaussian part. The error of this procedure is denoted by $C_{n,m}$. Finally, D_m represents the error of a Riemann sum approximation. Next we will prove that

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|A_n + B_{n,m} + C_{n,m} + D_m| > \epsilon) = 0,$$

for any $\epsilon > 0$. This will complete the proof of Theorem 3.2. □

The term A_n. The convergence $A_n \xrightarrow{\mathbb{P}} 0$ is shown exactly as in [4, Section 7.3]. Therein the proof is given for the case of a single singularity at 0. However, it directly extends to the case of multiple singularities. □

The term $B_{n,m}$. Observe that

$$|B_{n,m}| \leq \frac{\Delta_n}{\tau_k(v\Delta_n)^2} \sum_{r=1}^{[mt]} \sum_{j_1,j_2=0}^{l} \sup_{s \in (\frac{r-2}{m}, \frac{r}{m}]} |\sigma_{(r-1)/m-\theta_{j_1}} \sigma_{(r-1)/m-\theta_{j_2}} - \sigma_{s-\theta_{j_1}} \sigma_{s-\theta_{j_2}}|$$
$$\times \Big| \sum_{i \in I_m(r)} \Delta_{i,k}^{n,v} G^{(j_1)} \Delta_{i,k}^{n,v} G^{(j_2)} \Big| + R_{n,m}$$

with $\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|R_{n,m}| > \epsilon) = 0$. The dominating term converges in probability to

$$B_m := \frac{1}{m} \sum_{r=1}^{[mt]} \sum_{j \in \mathcal{A}} \pi(\theta_j) \sup_{s \in (\frac{r-2}{m}, \frac{r}{m}]} |\sigma_{(r-1)/m-\theta_j}^2 - \sigma_{s-\theta_j}^2|$$

as $n \to \infty$ due to Theorem 5.1(ii) and convergence $\tau_{k,j}(v\Delta_n)^2 / \tau_k(v\Delta_n)^2 \to \pi(\theta_j)$. Observing the estimation

$$|B_m| \leq \sum_{j \in \mathcal{A}} \pi(\theta_j) \int_0^t \sup_{s \in (\frac{[um]-1}{m}, \frac{[um]+1}{m}]} |\sigma_{\frac{[um]}{m}-\theta_j}^2 - \sigma_{s-\theta_j}^2| du,$$

we conclude that
$$B_m \xrightarrow{\mathbb{P}} 0 \qquad \text{as } m \to \infty,$$

by bounded convergence theorem, since σ is cádlág and bounded on compact intervals. Thus, the proof of this part is completed. □

The term $C_{n,m}$. According to Theorem 5.1(ii), we have

$$\Delta_n \sum_{i \in I_m(r)} \frac{\Delta_{i,k}^{n,v} G^{(j_1)} \Delta_{i,k}^{n,v} G^{(j_2)}}{\tau_{k,j_1}(v\Delta_n) \tau_{k,j_2}(v\Delta_n)} \xrightarrow{\mathbb{P}} \delta_{j_1,j_2} m^{-1} t$$

for any r. Furthermore, the proof of Proposition 3.1 shows that

$$\frac{\tau_{k,j}(v\Delta_n)^2}{\tau_k(v\Delta_n)^2} \to \pi(\theta_j).$$

Thus, for any fixed m, we conclude that

$$C_{n,m} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty,$$

which completes the proof of this part. □

The term D_m. Recall that the measure π is finite and the process σ is cádlág bounded. Hence, the convergence $D_m \xrightarrow{\mathbb{P}} 0$ as $m \to \infty$ follows by Lebesgue integrability. This completes the proof of Theorem 3.2. □

Proof of Theorem 4.2(i). The proof of this result follows along the same lines as the previous one. For the treatment of the terms $B_{n,m}$ and $C_{n,m}$ we use the convergence in probability

$$\Delta_n \sum_{i=vk}^{[t/\Delta_n]} \frac{(\Delta_{i,k}^{n,v} G^{(j_1)} \Delta_{i,k}^{n,v} G^{(j_2)})^2}{(\tau_{k,j_1}(v\Delta_n)\tau_{k,j_2}(v\Delta_n))^2} \overset{\text{u.c.p.}}{\Longrightarrow} t \qquad \text{when } j_1 \neq j_2,$$

$$\Delta_n \sum_{i=vk}^{[t/\Delta_n]} \frac{(\Delta_{i,k}^{n,v} G^{(j)})^4}{(\tau_{k,j}(v\Delta_n))^4} \overset{\text{u.c.p.}}{\Longrightarrow} 3t,$$

which follows from a general result of [4, Theorem 1]. The remaining proof of Theorem 3.2 applies directly to Theorem 4.2(i). $\qquad\square$

5.4   Proof of Theorem 3.3

Here we use a slightly different decomposition than in the proof of Theorem 3.2. Observe that

$$\Delta_n^{-1/2}\left(\frac{\Delta_n}{\tau_k(v\Delta_n)^2}QV(X,k,v\Delta_n)_t - QV(X,k)_t\right) = \widetilde{A}_n^v + \widetilde{B}_{n,m}^v + \widetilde{C}_{n,m}^v + \widetilde{D}_n^v,$$

$$(5.22)$$

where

$$\widetilde{A}_n^v := \frac{\Delta_n^{1/2}}{\tau_k(v\Delta_n)^2} \sum_{i=vk}^{[t/\Delta_n]} \left((\Delta_{i,k}^{n,v} X)^2 - \left(\sum_{j=0}^{l} \sigma_{(i-vk)\Delta_n - \theta_j} \Delta_{i,k}^{n,v} G^{(j)}\right)^2\right)$$

$$\widetilde{B}_{n,m}^v := \Delta_n^{1/2}\left(\frac{1}{\tau_k(v\Delta_n)^2} \sum_{i=vk}^{[t/\Delta_n]} \left(\sum_{j=0}^{l} \sigma_{(i-vk)\Delta_n - \theta_j} \Delta_{i,k}^{n,v} G^{(j)}\right)^2\right.$$

$$- \sum_{i=vk}^{[t/\Delta_n]} \int_0^\infty \sigma_{(i-vk)\Delta_n - \theta}^2 \pi_k(d\theta)\right)$$

$$- \Delta_n^{1/2}\left(\frac{1}{\tau_k(v\Delta_n)^2} \sum_{r=1}^{[mt]} \sum_{i\in I_m(r)} \left(\sum_{j=0}^{l} \sigma_{(r-1)/m - \theta_j} \Delta_{i,k}^{n,v} G^{(j)}\right)^2\right.$$

$$\left.- \frac{1}{m}\sum_{r=1}^{[mt]} \int_0^\infty \sigma_{(r-1)/m - \theta}^2 \pi_k(d\theta)\right)$$

$$\widetilde{C}_{n,m}^v := \Delta_n^{1/2}\left(\frac{1}{\tau_k(v\Delta_n)^2} \sum_{r=1}^{[mt]} \sum_{i\in I_m(r)} \left(\sum_{j=0}^{l} \sigma_{(r-1)/m - \theta_j} \Delta_{i,k}^{n,v} G^{(j)}\right)^2\right.$$

$$\left.- \frac{1}{m}\sum_{r=1}^{[mt]} \int_0^\infty \sigma_{(r-1)/m - \theta}^2 \pi_k(d\theta)\right)$$

$$\widetilde{D}_n^v := \Delta_n^{-1/2}\left(\Delta_n \sum_{i=vk}^{[t/\Delta_n]} \int_0^\infty \sigma_{(i-vk)\Delta_n - \theta}^2 \pi_k(d\theta) - \int_0^\infty \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds\right) \pi_k(d\theta)\right)$$

In the next step we will prove that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}(\|\widetilde{A}_n^v + \widetilde{B}_{n,m}^v + \widetilde{D}_n^v\|_\infty > \epsilon) = 0,$$

for any $\epsilon > 0$. The term $\widetilde{C}_{n,m}^v$ will give us the central limit theorem. More precisely, we will show that

$$(\widetilde{C}_{n,m}^1, \widetilde{C}_{n,m}^2) \xrightarrow{d_{st}} C_m \qquad \text{as } n \to \infty,$$

for some process C_m and $C_m \xrightarrow{d_{st}} L$, where the process L is defined at (3.7). This would complete the proof of Theorem 3.3.

Term $\widetilde{A}_n^v$. The convergence $\widetilde{A}_n^v \xRightarrow{\text{u.c.p.}} 0$ has been proved for $k = 1$ in [4, Section 7] and for $k = 2$ in [5, Section 5.2] (the latter proof easily extends to any $k \geq 2$). Although both results are only valid for the case of single singularity at 0, they extend to the case of multiple singularities exactly as in the proof of Theorem 3.2. $\qquad\square$

Term $\widetilde{B}_{n,m}^v$. The negligibility of the quantity $\widetilde{B}_{n,m}^v$ is proven by means of fractional calculus in a recent work [14, Section 4]. $\qquad\square$

Term $\widetilde{C}_{n,m}^v$. We recall first that

$$\frac{\tau_{k,j}(v\Delta_n)^2}{\tau_k(v\Delta_n)^2} = \pi(\theta_j) + o(\Delta_n^{1/2}) \quad j \in \mathcal{A}, \qquad \frac{\tau_{k,j}(v\Delta_n)^2}{\tau_k(v\Delta_n)^2} = o(\Delta_n^{1/2}) \quad j \notin \mathcal{A},$$

which follows from the proof of Proposition 3.1 and condition (3.6). Define the statistics

$$S_{n,m}^{v,j_1,j_2}(r) := \Delta_n^{1/2}\left(\frac{1}{\tau_k(v\Delta_n)^2}\sum_{i\in I_m(r)}\Delta_{i,k}^{n,v}G^{(j_1)}\Delta_{i,k}^{n,v}G^{(j_2)} - \delta_{j_1,j_2}\frac{\pi(\theta_{j_1})}{m}\right),$$

for $j_1, j_2 \in \mathcal{A}$. Applying Theorem 5.1(ii) and the properties of stable convergence, we conclude that

$$\left(\sigma_{(r-1)/m-\theta_j}, S_{n,m}^{v,j_1,j_2}(r)\right)_{j,j_1,j_2\in\mathcal{A}}^{r=1,\ldots,m, \ v=1,2}$$

$$\xrightarrow{d_{st}} \left(\sigma_{(r-1)/m-\theta_j}, \sqrt{\pi(\theta_{j_1})\pi(\theta_{j_2})}\left\{V_{k,v}^{j_1,j_2}\left(\frac{r}{m}\right) - V_{k,v}^{j_1,j_2}\left(\frac{r-1}{m}\right)\right\}\right)_{j,j_1,j_2\in\mathcal{A}}^{r=1,\ldots,m, \ v=1,2},$$

where the process V is defined at (5.9). Next, we observe that

$$\Delta_n^{1/2}\frac{1}{\tau_k(v\Delta_n)^2}\sum_{i\in I_m(r)}\Delta_{i,k}^{n,v}G^{(j_1)}\Delta_{i,k}^{n,v}G^{(j_2)} = o_{\mathbb{P}}(1)$$

when $j_1 \notin \mathcal{A}$ or $j_2 \notin \mathcal{A}$, which is due to (5.20). Hence, it holds that

$$\widetilde{C}_{n,m}^v = \sum_{r=1}^{[mt]}\sum_{j_1,j_2\in\mathcal{A}}\sigma_{(r-1)/m-\theta_{j_1}}\sigma_{(r-1)/m-\theta_{j_2}}S_{n,m}^{v,j_1,j_2}(r) + o_{\mathbb{P}}(1).$$

Now, applying the continuous mapping theorem for stable converge and recalling the identity (5.10), we deduce that

$$(\widetilde{C}_{n,m}^1, \widetilde{C}_{n,m}^2) \xrightarrow{d_{st}} \sum_{r=1}^{[mt]} \left(\int_0^\infty \sigma^2_{\frac{r-1}{m}-\theta} \, \pi_k(d\theta) \right) \Lambda_k^{1/2} (B_{\frac{r}{m}} - B_{\frac{r-1}{m}}) \qquad \text{as } n \to \infty,$$

where Λ_k and B are defined at (3.7). Finally,

$$\sum_{r=1}^{[mt]} \left(\int_0^\infty \sigma^2_{\frac{r-1}{m}-\theta} \, \pi_k(d\theta) \right) \Lambda_k^{1/2} (B_{\frac{r}{m}} - B_{\frac{r-1}{m}}) \xrightarrow{d_{st}} L_t \qquad \text{as } m \to \infty,$$

which completes this step. $\qquad\square$

Term $\widetilde{D}_n^v$. Since σ is Hölder continuous of order γ with $\gamma > 1/2$, we readily deduce that $\widetilde{D}_n^v \xrightarrow{\text{u.c.p.}} 0$. This completes the proof of Theorem 3.3. $\qquad\square$

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