THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Some Recent Developments in Ambit Stochastics

Ole E. Barndorff-Nielsen, Emil Hedevang,
Jürgen Schmiegel and Benedykt Szozda

15

Research Report

No. 03 | February 2015

Some Recent Developments in Ambit Stochastics

Ole E. Barndorff-Nielsen[1], Emil Hedevang[1], Jürgen Schmiegel[2] and Benedykt Szozda[1]

[1]Department of Mathematics, Aarhus University, {oebn,hedevang,szozda}@math.au.dk
[2]Department of Engineering, Aarhus University, schmiegl@eng.au.dk

Abstract

Some of the recent developments in the rapidly expanding field of Ambit Stochastics are here reviewed. After a brief recall of the framework of Ambit Stochastics three topics are considered: (i) Methods of modelling and inference for volatility/intermittency processes and fields (ii) Universal laws in turbulence and finance in relation to temporal processes (iii) Stochastic integration for time changed volatility modulated Levy-driven Volterra processes.

Keywords: ambit stochastics, extended subordination, finance, metatimes, non-semimartingales, stochastic integration, time-change, stochastic volatility/intermittency, turbulence, universality.

Contents

1 Introduction

Ambit Stochastics is a general framework for the modelling and study of dynamic processes in space-time. The present paper outlines some of the recent developments in the area, with particular reference to finance and the statistical theory of turbulence. Two recent papers ([9] and [39]) provide surveys that focus on other sides of Ambit Stochastics.

A key characteristic of the Ambit Stochastics framework, which distinguishes this from other approaches, is that beyond the most basic kind of random input it also specifically incorporates additional, often drastically changing, inputs referred to as volatility or intermittency.

Such "additional" random fluctuations generally vary, in time and/or in space, in regard to *intensity* (activity rate and duration) and *amplitude*. Typically the volatility/intermittency may be further classified into continuous and discrete (i.e. jumps) elements, and long and short term effects. In turbulence the key concept of energy dissipation is subsumed under that of volatility/intermittency.

The concept of (stochastic) *volatility/intermittency* is of major importance in many fields of science. Thus volatility/intermittency has a central role in *mathematical finance and financial econometrics*, in *turbulence*, in *rain and cloud studies* and other aspects of environmental science, in relation to *nanoscale emitters*, *magnetohydrodynamics*, and to *liquid mixtures of chemicals*, and last but not least in the *physics of fusion plasmas* (see e.g. the book [30] among many others).

As described here, volatility/intermittency is a *relative* concept, and its meaning depends on the particular setting under investigation. Once that meaning is clarified the question is how to assess the volatility/intermittency empirically and then to describe it in stochastic terms, for incorporation in a suitable probabilistic model. Important issues concern the modelling of propagating stochastic volatility/intermittency fields and the question of predictability of volatility/intermittency.

Section 2 briefly recalls some main aspects of Ambit Stochastics that are of relevance for the dicussions in the subsequent sections. The modelling of volatility/intermittency and energy dissipation is a main theme in Ambit Stochastics and several approaches to this are discussed in section 3. A leading principle in the development of Ambit Stochastics has been to take the cue from recognised stylised features – or universality traits – in various scientific areas, particularly turbulence, as the basis for model building; and in turn to seek new such traits using the models as tools. We discuss certain universal features observed in finance and turbulence and indicate ways to reproduces them in section 4. Integration with ambit processes/fields as integrators is discussed in section 5.

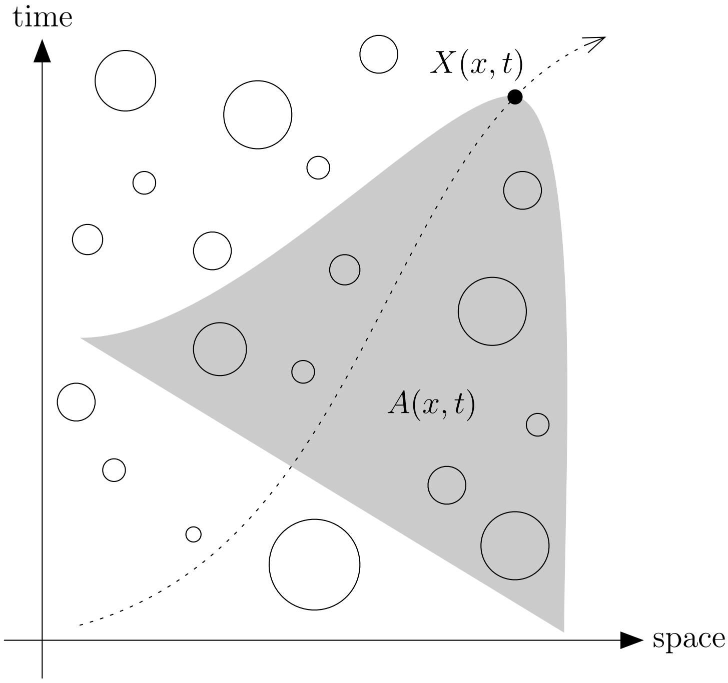

Figure 1: A spatio-temporal ambit field. The value $Y(x,t)$ of the field at the point marked by the black dot is defined through an integral over the corresponding ambit set $A(x,t)$ marked by the shaded region. The circles of varying sizes indicate the stochastic volatility/intermittency. By considering the field along the dotted path in space-time an ambit process is obtained.

2 Ambit Stochastics

In terms of mathematical formulae, in its original form [18] (cf. also [17]) an ambit field is specified by

$$Y(x,t) = \mu + \int_{A(x,t)} g(x,\xi,t,s)\sigma(\xi,s)\, L(d\xi\, ds) + Q(x,t) \tag{2.1}$$

where

$$Q(x,t) = \int_{D(x,t)} q(x,\xi,t,s)\chi(\xi,s)\, d\xi\, ds. \tag{2.2}$$

Here t denotes time while x gives the position in d-dimensional Euclidean space. Further, $A(x,t)$ and $D(x,t)$ are subsets of $\mathbb{R}^d \times \mathbb{R}$ and are termed *ambit sets*, g and q are deterministic weight functions, and L denotes a Lévy basis (i.e. an independently scattered and infinitely divisible random measure). Further, σ and χ are stochastic fields representing aspects of the volatility/intermittency. In Ambit Stochastics the models of the volatility/intermittency fields σ and χ are usually themselves specified as ambit fields. We shall refer to σ as the *amplitude volatility* component. Figure 1 shows a sketch of the concepts.

The development of Y along a curve in space-time is termed an ambit process. As will be exemplified below, ambit processes are not of the semimartingale type, even in the purely temporal case, i.e. where there is no spatial component x.

In a recent extension the structure (2.1) is generalised to

$$Y(x,t) = \mu + \int_{A(x,t)} g(x,\xi,t,s)\sigma(\xi,s)\, L_T(d\xi\, ds) + Q(x,t) \tag{2.3}$$

where Q is like (2.2) or the exponential thereof, and where T is a *metatime* expressing a further volatility/intermittency trait. The relatively new concept of metatime generalises that of subordination of stochastic processes by time change (as discussed for instance in [23]) to subordination of random measures by random measures. We shall not consider this concept and its applications further here but refer to the discussion given in [10].

Note however that in addition to modelling volatility/intermittency through the components σ, χ and T, in some cases this may be supplemented by probability mixing or Lévy mixing as discussed in [13].

It might be thought that ambit sets have no role in purely temporal modelling. However, examples of their use in such contexts will be discussed in section 3.

In many cases it is possible to choose specifications of the volatility/intermittency elements σ, χ and T such that these are infinitely divisible or even selfdecomposable, making the models especially tractable analytically. We recall that the importance of the concept of selfdecomposability rests primarily on the possibility to represent selfdecomposable variates as stochastic integrals with respect to Lévy bases, see [35].

So far, the main applications of ambit stochastics has been to turbulence and, to a lesser degree, to financial econometrics and to bioimaging. An important new area of applications is to particle transport in fluids.

The paper [26] develops a general theory for integrals

$$X(x,t) = \int_{\mathbb{R}^d \times \mathbb{R}} h(x,y,t,s)\, M(dx\, dx)$$

where h is a predictable stochastic function and M is a dispersive signed random measure. Central to this is that the authors establish a notion of characteristic triplet of M, extending that known in the purely temporal case. A major problem solved in that regard has been to merge the time and space aspects in a general and tractable fashion. Armed with that notion they determine the conditions for existence of the integral, analogous to those in [41] but considerably more complicated to derive and apply. An important property here is that now predictable integrands are allowed (in the purely temporal case this was done in [25]). Applications of the theory to Ambit Stochastics generally, and in particular to superposition of stochastic volatility models, is discussed.

Example 2.1. We can briefly indicate the character of some of the points on Ambit Stochastics made above by considering the following simple model

$$Y(t) = \int_{-\infty}^{t} g(t-s)\sigma(s)\, B_T(ds) + \int_{-\infty}^{t} g(t-s)\sigma(s)^2\, ds. \tag{2.4}$$

Here the setting is purely temporal and B_T is the time change of Brownian motion B by a chronometer T (that is, an increasing, cádlág and stochastically continuous process ranging from $-\infty$ to ∞), and the volatility/intermittency process σ is assumed stationary. Here σ and T represent respectively the amplitude and the intensity of the volatility/intermittency. If T has stationary increments then the process Y is stationary. Note that (2.4) can be seen as a stationary analogue of the BNS model introduced by Barndorff-Nielsen and Shephard in [15].

Note that in case T increases by jumps only, the infinitesimal of the process B_T cannot be reexpressed in the form $\chi(s)B(ds)$, as would be the case if T was of type $T_t = \int_0^t \psi(u)\, du$ with $\chi = \sqrt{\psi}$.

Further, for the exemplification we take g to be of the gamma type

$$g(s) = \frac{\lambda^\nu}{\Gamma(\nu)} s^{\nu-1} e^{-\lambda s} 1_{(0,\infty)}(s). \tag{2.5}$$

Subject to a weak condition on σ, the stochastic integral in (2.4) will exist if and only if $\nu > 1/2$ and then Y constitutes a stationary process in time. Moreover, Y is a semimartingale if and only if ν does not lie in one of the intervals $(1/2, 1)$ and $(1, 3/2)$. Note also that the sample path behaviour is drastically different between the two intervals where T is a pure-jump process, since, as $t \to 0$, $g(t)$ tends to ∞ when $\nu \in (1/2, 1)$ and to 0 when $\nu \in (1, 3/2)$. Further, the sample paths are purely discontinuous if $\nu \in (1/2, 1)$ but purely continuous (of Hölder index $H = \nu - 1/2$) when $\nu \in (1, 3/2)$.

The cases where $\nu \in (1/2, 1)$ have a particular bearing in the context of turbulence, the value $\nu = 5/6$ having a special role in relation to the Kolmogorov-Obukhov theory of statistical turbulence, cf. [4] and [36].

3 Modelling of volatility/intermittency/energy dissipation

A very general approach to specifying volatility/intermittency fields for inclusion in an ambit field, as in (2.1), is to take $\tau = \sigma^2$ as being given by a Lévy-driven Volterra field, either directly as

$$\tau(x, t) = \int_{\mathbb{R}^2 \times \mathbb{R}} f(x, \xi, t, s)\, L(d\xi, ds) \tag{3.1}$$

with f positive and L a Lévy basis (different from L in (2.1),) or in exponentiated form

$$\tau(x, t) = \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}} f(x, \xi, t, s)\, L(d\xi, ds) \right). \tag{3.2}$$

When the goal is to have stationary volatility/intermittency fields, such as in modelling homogeneous turbulence, that can be achieved by choosing L to be homogeneous and f of translation type. However, the potential in the specifications (3.1) and (3.2) is much wider, giving ample scope for modelling inhomogeneous fields, which are by far the most common, particularly in turbulence studies. Inhomogeneity can be expressed both by not having f of translation type and by taking the Lévy basis L inhomogeneous.

In the following we discuss two aspects of the volatility/intermittency modelling issue. Subsection 3.1 presents a general class of ambit processes, called trawl processes, where although they are in principal temporal the construction relies on two or higher dimensional ambit sets. Trawl processes have proved to be a useful tool for the modelling of volatility/intermittency and in particular for the modelling of the energy dissipation as outlined in subsection 3.2. In Subsection 3.4 we outline the applicability of selfdecomposability to the construction of volatility/intermittency fields.

3.1 Trawl processes

The simplest non-trivial kind of ambit field is perhaps the *trawl process*, first introduced in [3]. In a trawl process, the kernel function and the volatility field are constant and equal to 1, and so the process is given entirely by the ambit set and the Lévy basis. Specifically, suppose that L is a homogeneous Lévy basis on $\mathbb{R}^d \times \mathbb{R}$ and that $A \subseteq \mathbb{R}^d \times \mathbb{R}$ is a Borel subset with finite Lebesgue measure, then we obtain a trawl process Y by letting $A(t) = A + (0, t)$ and

$$Y(t) = \int_{A(t)} L(d\xi\, ds) = \int 1_A(\xi, t-s)\, L(d\xi\, ds) = L(A(t)) \tag{3.3}$$

The process is by construction stationary. Despite their apparent simplicity, trawl processes possess enough flexibility to be of use. If L' denotes the seed of L, then the cumulant function of Y is given by

$$C\{\zeta \ddagger Y(t)\} = |A| C\{\zeta \ddagger L'\}. \tag{3.4}$$

For the mean, variance, autocovariance and autocorrelation it follows that

$$\begin{aligned} \mathbb{E}[Y(t)] &= |A| \mathbb{E}[L'], \\ \operatorname{var}(Y(t)) &= |A| \operatorname{var}(L'), \\ r(t) &:= \operatorname{cov}(Y(t), Y(0)) = |A \cap A(t)| \operatorname{var}(L'), \\ \rho(t) &:= \frac{\operatorname{cov}(Y(t), Y(0))}{\operatorname{var}(Y(0))} = \frac{|A \cap A(t)|}{|A|}. \end{aligned}$$

From this we conclude the following. The one-dimensional marginal distribution is determined entirely in terms of the size (not shape) of the ambit set and the distribution of the Lévy seed; given any infinitely divisible distribution there exists trawl processes having this distribution as the one-dimensional marginal; and the autocorrelation is determined entirely by the size of the overlap of the ambit sets, that is, the shape of the ambit set A. Thus we can specify the autocorrelation and marginal distribution independently of each other. It is, for example, easy to construct a trawl process with the same autocorrelation as the OU process, see [3] and [10] for more results and details. By using integer-valued Lévy bases, integer-valued trawl processes are obtained. These processes are studied in detail in [6] and applied to high frequency stock market data.

We remark, that $Y(x, t) = L(A + (x, t))$ is an immediate generalisation of trawl processes to trawl fields. It has the same simple properties as the trawl process.

Trawl processes can be used to directly model an object of interest, for example, the exponential of the trawl process has been used to model the energy dissipation, see next subsection, or they can be used as a component in a composite model, for example to model the volatility/intermittency in a Brownian semistationary process.

3.2 The energy dissipation

In [34] it has been shown that exponentials of trawl processes are able to reproduce the main stylized features of the (surrogate) energy dissipation observed for a wide range of datasets. Those stylized features are the one-dimensional marginal distributions, the structure of the correlators and the scaling and self-scaling of the correlators. There the energy dissipation ε is modelled as

$$\varepsilon(t) = \exp(L(A(t))), \tag{3.5}$$

where L is a homogeneous Lévy basis on $\mathbb{R} \times \mathbb{R}$ and $A(t) = A + (0, t)$ for a bounded set $A \subset \mathbb{R} \times \mathbb{R}$. The ambit set A is given as

$$A = \{(x, t) \mid 0 \leq t \leq T, -f(t) \leq x \leq f(t)\}, \tag{3.6}$$

where $T > 0$. For $T > 0$, $k > 1$, and $\theta > 0$, the function f is defined as

$$f(t) = \left(\frac{1 - (t/T)^\theta}{1 + (kt/T)^\theta} \right)^{1/\theta}, \qquad 0 \leq t \leq T. \tag{3.7}$$

Finally, the correlator of order (p, q) is defined by

$$c_{p,q}(s) = \frac{\mathbb{E}[\varepsilon(t)^p \varepsilon(t+s)^q]}{\mathbb{E}[\varepsilon(t)^p]\mathbb{E}[\varepsilon(t+s)^q]}. \tag{3.8}$$

In [34] it is shown that the one-dimensional marginal of the logarithm of the energy dissipation is well described by a normal inverse Gaussian distribution, i.e. $\log \varepsilon(t) \sim \mathrm{NIG}(\alpha, \beta, \mu, \delta)$, where the shape of these distributions is the same for all datasets (independent of the so-called Reynolds number). This means that L should be a normal inverse Gaussian Lévy basis whose parameters are given by the observed distribution of $\log \varepsilon(t)$. The shape of the ambit set is chosen to reproduce the correlator $c_{1,1}$. This completely specifies the parameters of (3.5). The model is able to reproduce the behaviour of the correlators of other orders, including scaling,

$$c_{p,q}(s) \propto s^{\tau(p,q)}, \tag{3.9}$$

where $\tau(p, q)$ is the scaling exponent, and self-scaling,

$$c_{p,q}(s) \propto c_{p',q'}(s)^{\tau(p,q;p',q')}, \tag{3.10}$$

where $\tau(p, q; p', q')$ is the self-scaling exponent. Furthermore, self-scaling exponents are predicted from the shape of the one-point distribution of the energy dissipation alone.

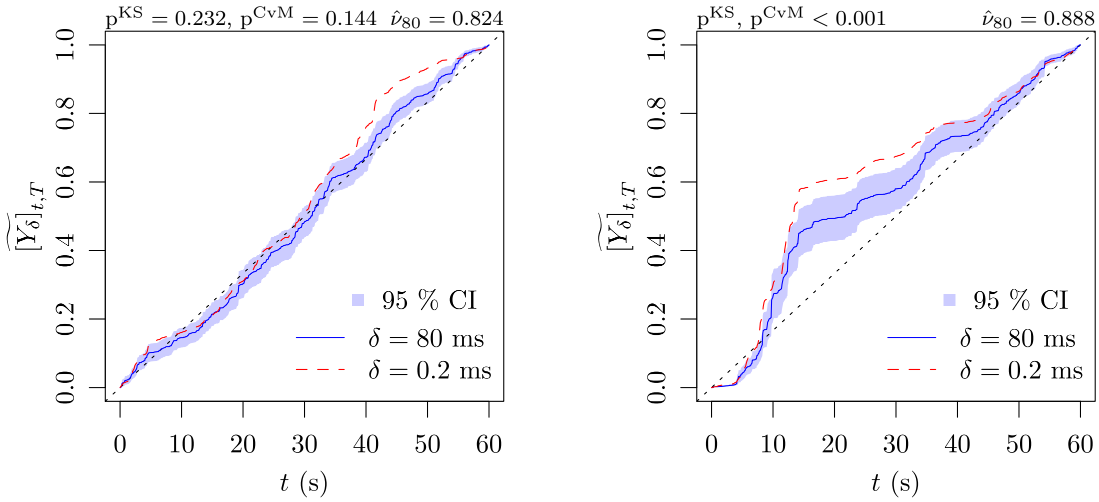

Figure 2: Brookhaven turbulence data periods 18 and 25 – RRQV and 95 % confidence intervals

3.3 Realised relative volatility/intermittency/energy dissipation

By its very nature, volatility/intermittency is a relative concept, delineating variation that is relative to a conceived, simpler model. But also in a model for volatility/intermittency in itself it is relevant to have the relative character in mind, as will be further discussed below. We refer to this latter aspect as relative volatility/intermittency and will consider assessment of that by *realised relative volatility/intermittency* which is defined interms of quadratic variation. The ultimate purpose of the concept of relative volatility/intermittency is to assess the volatility/intermittency or energy dissipation in arbitrary subregions of a region C of space-time relative to the total volatility/intermittency/energy dissipation in C. In the purely temporal setting the *realised relative volatility/interrmittency* is defined by

$$[Y_\delta]_t/[Y_\delta]_T$$

where $[Y_\delta]_t$ denotes the realised quadratic variation of the process Y observed with lag δ over a time interval $[0, t]$. We refer to this quantity as RRQV (for realised relative quadratic volatility). An important aspect of the definition of the realised relative volatility/interrmittency is that its calculation does not require estimation of ν under the gamma specification (2.5) of g.

Convergence in probability and a central limit theorem for the RRQV is established in [12]. Figure 2 illustrates its use, for two sections of the "Brookhaven" dataset, one where the volatility effect was deemed by eye to be very small and one where it appeared strong. (The "Brookhaven" dataset consist of 20 million one-point measurements of the longitudinal component of the wind velocity in the atmospheric boundary layer, 35 meters above ground. The measurements were performed using a hot-wire anemometer and sampled at 5 kHz. The time series can be assumed to be stationary. We refer to [27] for further details on the dataset; the dataset is called no. 3 therein).

3.4 Role of selfdecomposability

If τ is given by (3.1) it is automatically infinitely divisible, and selfdecomposable provided L has that property; whereas if τ is defined by (3.2) it will only in exceptional cases be infinitely divisible. For a general discussion of selfdecomposable fields we refer to [14]. See also [35] which provides a survey of when a selfdecomposable random variable can be represented as a stochastic integral, like in (3.1). Representations of that kind allow, in particular, the construction of field-valued processes of OU or supOU type that may be viewed as propagating, in time, an initial volatility/intermittency field defined on the spatial component of space-time for a fixed time, say $t = 0$. Similarly, suppose that a model has been formulated for the time-wise development of a stochastic field at a single point in space. One may then seek to define a field on space-time such that at every other point of space the time-wise development of the field is stochastically the same as at the original space point and such that the field is stationary and selfdecomposable.

Example 3.1 (One dimensional turbulence)**.** Let Y denote an ambit field in the tempo-spatial case where the spatial dimension is 1, and assume that for a preliminary purely temporal model X of the same turbulent phenomenon a model has been formulated for the squared amplitude volatility component, say ω. It may then be desirable to devise Y such that the volatility/intermittency field $\tau = \sigma^2$ is stationary and infinitely divisible.and such that for every spatial position x the law of $\tau(x, \cdot)$ is identical to that formulated for the temporal setting, i.e. ω. If the temporal process is selfdecomposable then, subject to a further weak condition, such a field can be constructed.

To sketch how this may proceed, recall first that the classical definition of selfdecomposability of a process X says that all the finite-dimensional marginal distributions of X should be selfdecomposable. Accordingly, due to a result by [42], for any finite set of time points $\hat{u} = (u_1, \ldots, u_n)$ the selfdecomposable vector variable $X(\hat{u}) = (X(u_1), \ldots, X(u_n))$ has a representation

$$X(\hat{u}) = \int_0^\infty e^{-\xi} \, L(d\xi, \hat{u})$$

for some n-dimensional Lévy process $L(\,\cdot\,, \hat{u})$, provided only that the Lévy measure of $X(\hat{u})$ has finite log-moment. We now assume this to be the case and that X is stationary

Next, for fixed $\hat{u}$, let $\{\tilde{L}(x, \hat{u}) \mid x \in \mathbb{R}\}$, be the n-dimensional Lévy process having the property that the law of $\tilde{L}(1, \hat{u})$ is equal to the law of $X(\hat{u})$. Then the integral

$$X(x, \hat{u}) = \int_{-\infty}^x e^{-\xi} \tilde{L}(d\xi, \hat{u})$$

exists and the process $\{X(x, \hat{u}) \mid x \in \mathbb{R}\}$ will be stationary – of Ornstein-Uhlenbeck type – while for each x the law of $X(x, \hat{u})$ will be the same as that of $X(\hat{u})$.

However, off hand the Lévy processes $\tilde{L}(\,\cdot\,, \hat{u})$ corresponding to different sets $\hat{u}$ of time points may have no dynamic relationship to each other, while the aim is to obtain a stationary selfdecomposable field $X(x, t)$ such that $X(x, \,\cdot\,)$ has the same law as X for all $x \in \mathbb{R}$. But, arguing along the lines of theorem 3.4 in [11], it is possible to choose the representative processes $\tilde{L}(\,\cdot\,, \hat{u})$ so that they are all defined on a single probability space and are consistent among themselves (in analogy to Kolmogorov's consistency result); and that establishes the existence of the desired field $X(x,t)$. Moreover, $X(\,\cdot\,,\,\cdot\,)$ is selfdecomposable, as is simple to verify.

The same result can be shown more directly by arguing on the general level of master Lévy measures and the associated Lévy-Ito representations, cf. [14].

Example 3.2. Assume that X has the form

$$X(u) = \int_{-\infty}^{u} g(u-\xi)\, L(d\xi) \tag{3.11}$$

where L is a Lévy process.

It has been shown in [14] that, in this case, provided g is integrable with respect to the Lebesgue measure, as well as to L, and if the Fourier transform of g is non-vanishing then X, as a process, is selfdecomposable if and only if L is selfdecomposable. When that holds we may, as above, construct a selfdecomposable field $X(x,t)$ with $X(x,\,\cdot\,) \sim X(\,\cdot\,)$ for every $x \in \mathbb{R}$ and $X(\,\cdot\,,t)$ of OU type for every $t \in \mathbb{R}$.

As an illustration, suppose that g is the gamma kernel (2.5) with $\nu \in (1/2, 1)$. Then the Fourier transform of g is

$$\hat{g}(\zeta) = (1 - i\zeta/\lambda)^{-\nu}.$$

and hence, provided that L is such that the integral (3.11) exists, the field $X(x,u)$ is stationary and selfdecomposable, and has the OU type character described above.

4 Time change and universality in turbulence and finance

4.1 Distributional Collapse

In [5], O.E. Barndorff-Nielsen, P. Blæsild and J. Schmiegel demonstrate two properties of the distributions of increments $\Delta_\ell X(t) = X(t) - X(t-\ell)$ of turbulent velocities. Firstly, the increment distributions are *parsimonious*, i.e., they are described well by a distribution with few parameters, even across distinct experiments. Specifically it is shown that the four-parameter family of normal inverse Gaussian distributions ($\mathrm{NIG}(\alpha,\beta,\mu,\delta)$) provides excellent fits across a wide range of lags ℓ,

$$\Delta_\ell X \sim \mathrm{NIG}(\alpha(\ell), \beta(\ell), \mu(\ell), \delta(\ell)). \tag{4.1}$$

Secondly, the increment distributions are *universal*, i.e., the distributions are the same for distinct experiments, if just the scale parameters agree,

$$\Delta_{\ell_1} X_1 \sim \Delta_{\ell_2} X_2 \quad \text{if and only if} \quad \delta_1(\ell_1) = \delta_2(\ell_2), \tag{4.2}$$

provided the original velocities (not increments) have been non-dimensionalized by standardizing to zero mean and unit variance. Motivated by this, the notion of stochastic equivalence class is introduced.

In [17], the analysis is extended to many more data sets, and the notion of stochastic equivalence class is extended to

$$\frac{\Delta_{\ell_1} X_1}{g_1(\ell_1)} \sim \frac{\Delta_{\ell_2} X_2}{g_2(\ell_2)} \quad \text{if and only if} \quad F_1(\ell_1) = F_2(\ell_2), \tag{4.3}$$

where g_i and F_i are deterministic with F_i monotone. The role of g_i is to normalize by the size of the large scales of the process, and the role of F_i is to change time by the small scales of the process. In the context of turbulence is was found that $g_i(\ell) = \sqrt{\mathrm{var}(X_i)}$ and $F_i(\ell) = \delta_i(\ell)/\sqrt{\mathrm{var}(X_i)}$.

Finally, in [22], O.E. Barndorff-Nielsen, J. Schmiegel and N. Shephard extend the analysis from fluid velocities in turbulence to currency and metal returns in finance and demonstrate that

$$\Delta_{\ell_1} X_1 \sim \Delta_{\ell_2} X_2 \quad \text{if and only if} \quad \mathrm{var}(\Delta_{\ell_1} X_1) \sim \mathrm{var}(\Delta_{\ell_2} X_2), \tag{4.4}$$

where X_i denotes the log-price, so increments are log-returns. Now $g_i(\ell) = 1$ and $F_i(\ell) = \mathrm{var}(\Delta_\ell X_i)$.

A conclusion from the cited works is that within the context of turbulence or finance there exists a one-parameter family of distributions such that for many distinct experiments and a wide range of lags, the corresponding increments are distributed according to a member of this family. Moreover, the index into this family is uniquely determined by the size of the large scale and the size of the small scales.

4.2 A first look at financial data from SP500

Motivated by all this, we extend the analysis even further to 29 assets from SP500 (AA, AIG, AXP, BA, BAC, C, CAT, CVX, DD, DIS, GE, GM, HD, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PG, SPY, T, UTX, VZ, WMT, XOM). For each asset, between 7 and 12 years of data is available, but only data from before the financial crisis is retained in order to simplify this initial analysis. This excludes approximately 40 % of each data set. The data has been provided by A. Lunde (Aarhus University), see also [31].

Figure 3 shows that the distributions of log-returns across a wide range of lags ranging from 1 second to approximately 4.5 hours are quite accurately described by normal inverse Gaussian distributions. This is not surprising given that numerous publications have demonstrated the applicability of the generalised hyperbolic distribution, in particular the subfamily consisting of the normal inverse Gaussian distributions, to describe financial datasets. See for example [43, 15, 16, 2, 29]. We note the transition from a highly peaked distribution towards the Gaussian as the lag increases.

Next, we see on figure 4 that the distributions at the same lag of the log-returns for the 29 dataset are quite different, that is, they do not collapse onto the same

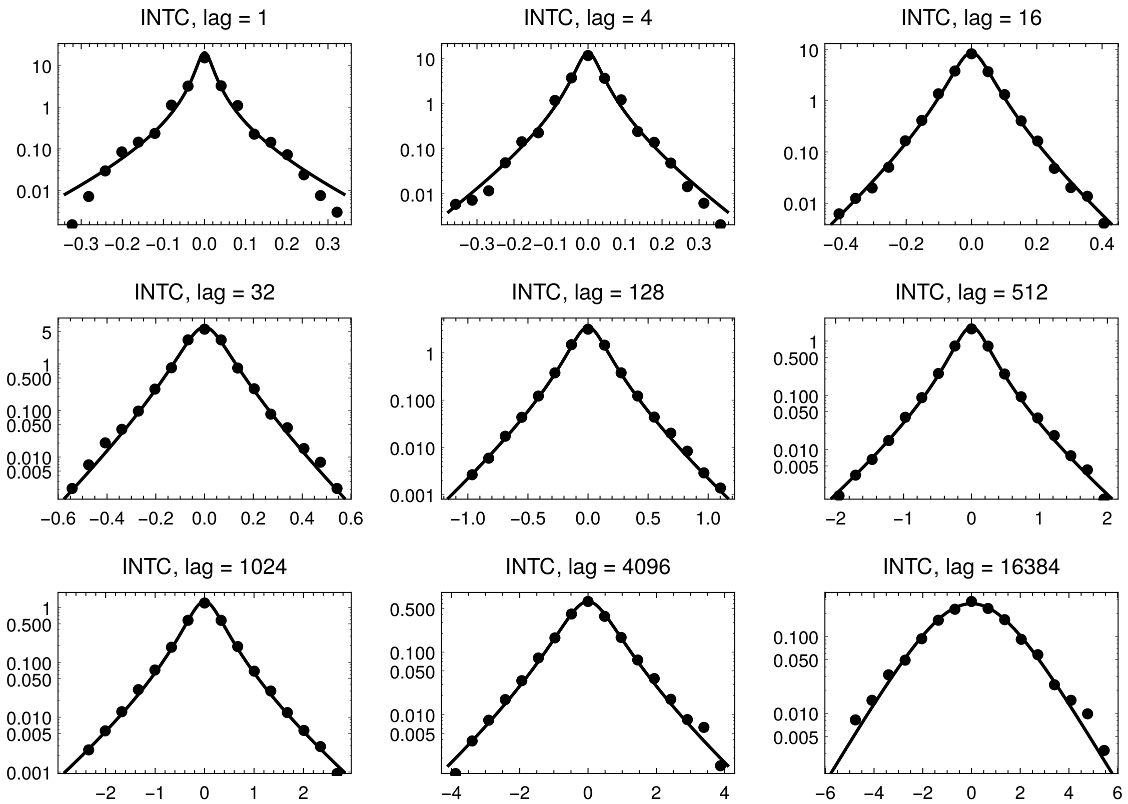

Figure 3: Probability densities on a log-scale for the log-returns of INTC at various lags ranging from 1 second to 16384 seconds. The dots denote the data and the solid line denotes the fitted NIG distribution. The log-returns have been multiplied with 100 in order to un-clutter the labeling of the x-axes.

curve. However, the transition from a highly peaked distribution at small lags towards a Gaussian at large lags hints that a suitable change of time may cause such a collapse. Motivated by the observations in [22] we therefore consider the variance of the log-returns as a function of the lag. Figure 5 shows how the variance depends on the lag. Except at the smallest lags, a clear power law is observed. The behaviour at the smallest lags is due to market microstructure noise [31]. Nine variances have been selected to represent most of the variances observed in the 29 datasets. For each selected variance and each dataset the corresponding lag is computed. We note that for the smallest lags/variances this is not without difficulty since for some of the datasets the slope approaches zero.

Finally, figure 6 displays the distributions of log-returns where the lag for each dataset has been chosen such that the variance is the one specified in each subplot. The difference between figure 4 and figure 6 is pronounced. While the distributions in figure 6 do not collapse perfectly onto the same curve for all the chosen variances we are invariable led to the the preliminary conclusion that also in the case of the analysed assets from SP500, a one-parameter family of distributions exists such that all distributions of log-returns are members of this family and such that the variance is the parameter which indexes into the family. The lack of collapse at the smaller variances may in part be explained by the difficulty in reading off the corresponding lags.

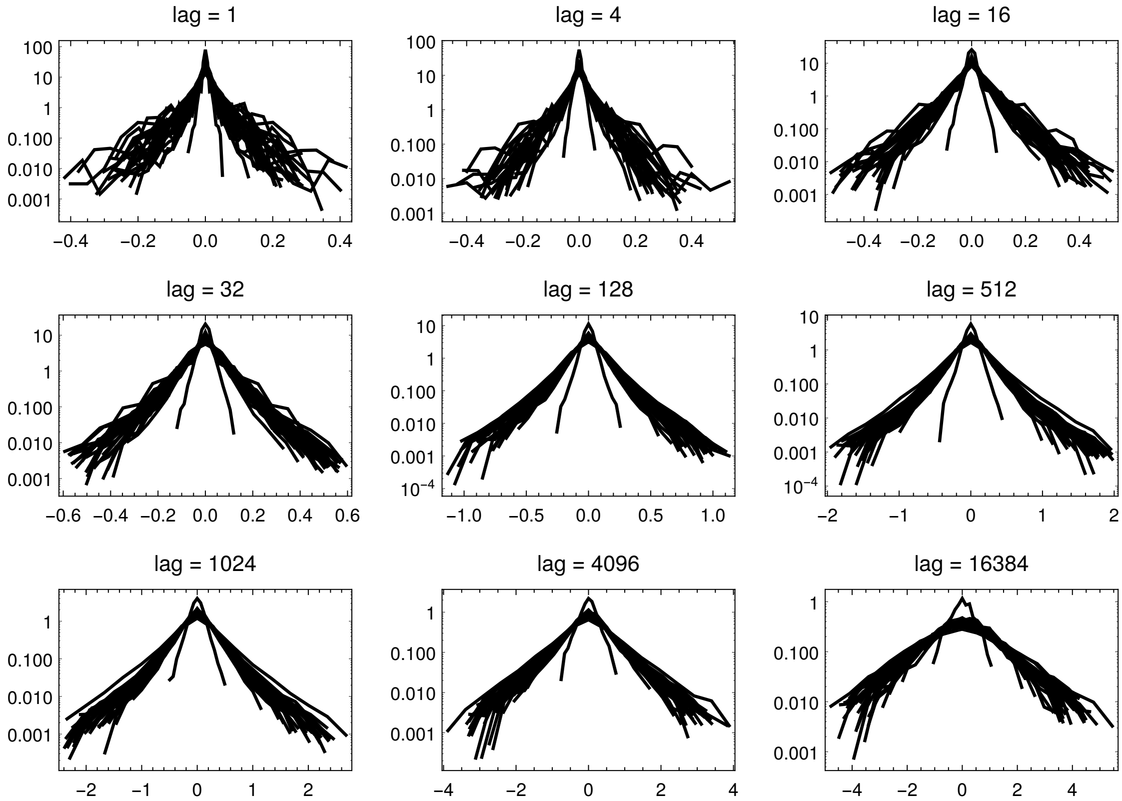

Figure 4: Probability densities on a log-scale for the log-returns of all 29 datasets at various lags ranging from 1 second to 16384 seconds. The log-returns have been multiplied with 100 in order to un-clutter the labeling of the x-axes.

The observed parsimony and in particular universality has implications for modelling since any proper model should possess both features. Within the context of turbulence, BSS-processes have been shown to be able to reproduce many features, see [37] and the following subsection for a recent example. The extent to which BSS-processes in general possess universality is still ongoing research [21] but preliminary results indicate that BSS-processes and in general LSS-process are good candidates for models where parsimony and universality are desired features.

4.3 Modelling turbulent velocity time series

A specific time-wise version of (2.1), called Brownian semistationary processes has been proposed in [19, 20] as a model for turbulent velocity time series. It was shown that BSS processes in combination with continuous cascade models (exponentials of certain trawl processes) are able to qualitatively capture some main stylized features of turbulent time series.

Recently this analysis has been extended to a quantitative comparison with turbulent data [37, 34]. More specifically, based on the results for the energy dissipation oulined in section 3.2, BSS processes have been analyzed and compared in detail to turbulent velocity time series in [37] by directly estimating the model parameters from data. Here we briefly summarize this analysis.

Time series of the main component v_t of the turbulent velocity field are modelled

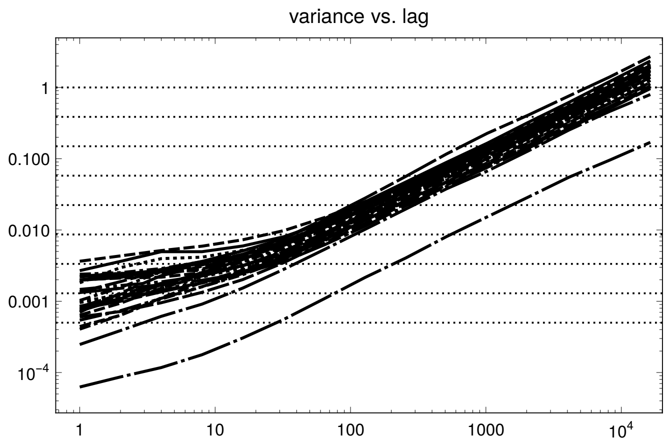

Figure 5: The variance of the log-returns for the 29 datasets as a function of the lag displayed in a double logarithmic representation.

as a BSS process of the specific form

$$\begin{aligned} v(t) = v(t; g, \sigma, \beta) &= \int_{-\infty}^{t} g(t-s)\sigma(s)\, B(ds) + \beta \int_{-\infty}^{t} g(t-s)\sigma(s)^2\, ds \\ &=: R(t) + \beta S(t) \end{aligned} \tag{4.5}$$

where g is a non-negative $L^2(\mathbb{R}_+)$ function, σ is a stationary process independent of B, β is a constant and B denotes standard Brownian motion. An argument based on quadratic variation shows that σ^2 can be identified with the surrogate energy dissipation, $\sigma^2 = \varepsilon$, where ε is the process given by (3.5). The kernel g is specified as a convolution of gamma kernels [32],

$$g(t) = at^{\nu_1+\nu_2-1} \exp(-\lambda_2 t) {}_1F_1(\nu_1, \nu_1+\nu_2, (\lambda_2-\lambda_1)t) 1_{(0,\infty)}(t)$$

with $a > 0$, $\nu_i > 0$ and $\lambda_i > 0$. Here ${}_1F_1$ denotes the Kummer confluent hypergeometric function.

The data set analysed consists of one-point time records of the longitudinal (along the mean flow) velocity component in a gaseous helium jet flow with a Taylor Reynolds number $R_\lambda = 985$. The same data set is also analyzed in [34] and the estimated parameters there are used to specify $\sigma^2 = \varepsilon$ in (4.5). The remaining parameters for the kernel g and the constant β can then be estimated from the second and third order structure function, that is, the second and third order moments of velocity increments. In [37] it is shown that the second order structure function is excellently reproduced and that the details of the third order structure function are well captured. It is important to note that the model is completely specified from the energy dissipation statistics and the second and third order structure functions.

The estimated model for the velocity is then succesfully compared with other derived quantites, including higher order structure functions, the distributions of velocity increments and their evolution as a function of lag, the so-called Kolmogorov variable, and the energy dissipation as prediced by the model.

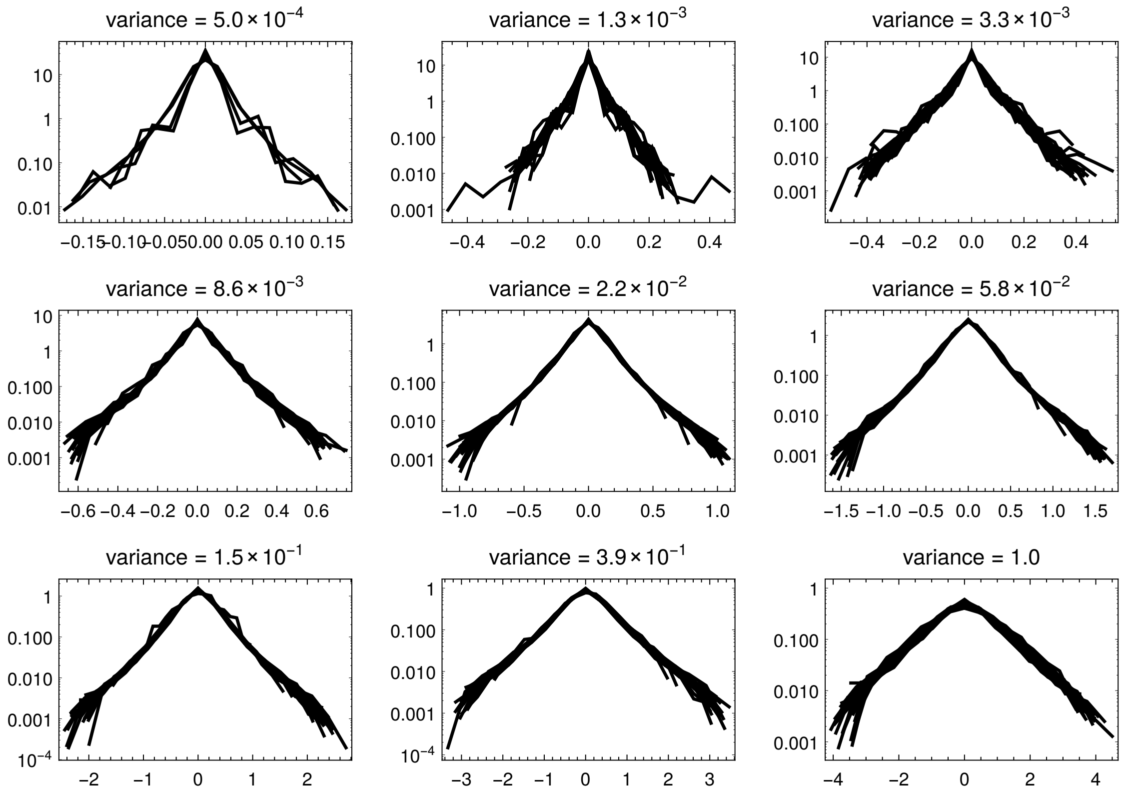

Figure 6: Probability densities on a log-scale of log-returns where the lag for each dataset has been chosen such that the variances of the datasets in each subplot is the same. The chosen variances are also displayed in figure 5 as horizonal dotted lines. For the smallest and largest variances, not all dataset are present since for some datasets those variances are not attained.

5 Stochastic integration for time changed volatility modulated Lévy-driven Volterra processes

In this section we discuss briefly the concept of integration with respect to purely temporal ambit processes. More precisely, we present the theory of stochastic integration with respect to doubly volatility modulated Lévy driven Volterra processes (DVVP). The theory presented below is an extension of the work of Barndorff-Nielsen et al. [9], where the authors introduced an integral with respect to volatility modulated Lévy driven Volterra processes (VMLV). The original integral of [9] has been further studied in the setting of white noise analysis (see [7]) and in the setting of Hilbert-valued stochastic processes (see [24]).

In [9] the authors define an integral with respect to VMLV processes, that is, processes of the form

$$X(t) = \int_0^t g(t,s)\sigma(s)\,dL(s), \tag{5.1}$$

where g is a deterministic function, σ is a stochastic process embodying the modulation of the amplitude of volatility and L is either a standard Brownian motion or a pure-jump, mean-zero, square-integrable Lévy process. The original motivation for studying VMLV processes is two-fold. They can be used to model a wide range of phenomena in a very flexible way and they are analytically tractable. When introducing DVVP processes, we kept the same ideas in mind as we discuss below.

Before we proceed, let us discuss some of the features of the VMLV integral that are inherited by the DVVP integral defined in what follows. The integral proposed in [9] allows for the kernel g to have a singularity at $t = s$ (equivalently, at zero for the shift-kernel of the form $g(t, s) = g(t - s)$). The VMLV integral extends previous approaches to integration with respect to Volterra processes in that it allows for amplitude volatility modulation via the stochastic process σ in equation (5.1). However, when comparing the integral defined in [9] with previously defined integrals such as [1], one crucial difference (even with $\sigma \equiv 1$) is the correction term with Malliavin derivative appearing in the definition. This term causes the expectation of the integral to be non-zero, as is to be expected since the integrator X is rarely a semimartingale.

A DVVP is a process of the form

$$X(t) = \int_0^t g(t, s)\sigma(s)\, L(\theta(s)), \tag{5.2}$$

where the only new component is θ, a stochastic "time-change" process that can also be thought of as embodying the modulation of the intensity of the volatility. Thus a DVVP process is a VMLV process with a time-change applied to the underlying driving noise. This small difference between equations (5.1) and (5.2) has a great impact on the modelling flexibility and on the class of driving noises that can be considered for the Volterra processes. From the modelling perspective, it allows for a separate modulation of intensity and amplitude of volatility. On the other hand, the use of Malliavin calculus in the formal derivation of the definition of the integral with respect to X as in equation (5.1) requires the driving process L to be either a standard Brownian motion or a pure-jump, mean-zero, square-integrable Lévy process. The presence of θ in equation (5.2) enables us to study much more general driving noises such as α-stable processes which have no second moment and for certain α are not even integrable.

We consider integrals of the form

$$\int_0^t Y(s)\, dX(s) \tag{5.3}$$

where X is as in equation (5.2) and Y satisfies certain regularity conditions discussed later. The definition of the above integral relies on a heuristic derivation which follows closely the steps taken by the authors of [9] with the necessary modifications to allow for the time-change in the driver of the integrator. Henceforth we assume that σ is measurable with respect to the σ-algebra generated by the underlying driver L and θ is independent of the underlying driver L. This allows us to condition on θ and apply the methods of [9] to obtain a meaningful expression for the integral in equation (5.3).

Although the driving noise in equations (5.1) and (5.2) can be either a Brownian motion or a pure-jump, mean-zero, square-integrable Lévy process, in this short survey we are going to discuss only the case of the Brownian motion. We refer an interested reader to [9] for the treatment of integration with respect to Lévy driven VMLV processes and to [8] for the treatment of integration with respect to Lévy driven DVVP processes.

5.1 Time-change

Definition 5.1. *By* time-change *on* $[0, \infty)$ *we understand any stochastic process* θ *with non-decreasing cádlág paths, continuous in probability and such that* $\theta(0) = 0$ *and* $\lim_{t\to\infty} \theta(t) = \infty$ *a.s..*

There are several standard and interesting examples of stochastic time-changes. First, let us mention subordinators, that is non-decreasing Lévy processes. Canonical examples of subordinators are the Gamma process, the Inverse Gaussian process and $\alpha/2$-stable processes with $\alpha \in (0, 2)$. A Brownian motion time-changed with the above leads to the Variance Gamma, the Normal Inverse Gaussian (NIG) and the α-stable process, respectively.

Another interesting class of time-changes are the so-called absolutely continuous time-changes. These arise from infinite lifetime Markov processes in the following way. Suppose that $M(t)$ is an infinite lifetime Markov process and $V : \mathbb{R} \to [0, \infty)$ is a deterministic function. Define

$$\theta(t) = \int_0^t V(M(s))\, ds.$$

Observe that θ defined above trivially satisfies definition 5.1 and in contrast with subordinators that are pure-jump processes, an absolutely continuous time-change has almost surely continuous paths.

Somewhere in between the subordinators and absolutely continuous time-changes lays yet another interesting class of singular time-changes. As discussed in [16], θ is a singular time-change if it is a continuous time-change such that $\frac{d\theta(t)}{dt} = 0$ almost everywhere. A canonical example of a deterministic singular time-change is the Devil's staircase (also called the Cantor function). The singular time-change can have abrupt changes not allowed in the absolutely continuous time-changes but not as violent as in the case of subordinators.

Before we proceed, we define the generalized inverse of a right continuous non-decreasing function $\theta : [0, T] \to [0, \infty)$ as

$$\theta^{-}(s) = \begin{cases} \inf\{t : \theta(t) > s\}, & s \in [0, \theta(T)), \\ T, & \text{otherwise.} \end{cases}$$

Now, suppose that B is a Brownian motion and $\{\mathcal{F}_t^B\}_{t\in[0,\infty)}$ is its natural right-continuous filtration augmented by $\mathbb{P}$-null sets. From now on, we assume that θ and B are independent and denote by $\{\mathcal{F}_t^{B(\theta)}\}_{t\in[0,T]}$ the right-continuous filtration generated by $B(\theta(t))$ and augmented by $\mathbb{P}$-null sets. We define in a similar manner the filtration $\{\mathcal{F}_t^{\theta}\}_{t\in[0,T]}$. In the end, the definition of the integral relies on conditioning, so we introduce notation for the conditional probability $\mathbb{P}_\theta(\,\cdot\,) = \mathbb{P}(\,\cdot \mid \mathcal{F}_T^{\theta})$.

Using the above notation, we have the following formula for the change of variables in stochastic integrals.

Lemma 5.2. *Suppose that f is an $\{\mathcal{F}_t^{B(\theta)}\}$-adapted cádlág process such that*

$$\mathbb{E}\left(\int_0^T f(s-)^2 \, d\theta(s)\right) < \infty.$$

Then

$$\int_0^T f(t-) \, dB(\theta(t)) = \int_0^{\theta(T)} f(\theta^-(s)-) \, dB(s) \qquad a.s..$$

Above, the integral on the left side is an integral with respect to an $\{\mathcal{F}_t^{B(\theta)}\}$ martingale and the one on the right side is with respect to an $\{\mathcal{F}_t^B\}$ martingale.

Proof. See lemma 1.3 of [23]. □

5.2 Malliavin Calculus

We are not going into the details of derivation of the results presented in this subsection, nor are we going to be very rigorous in its presentation and we refer the reader to the excellent monographs by Nualart [38] or Di Nunno et al. [28] for in-depth treatments of Malliavin calculus. We only intend to present the notation and the integration by parts formula that is the main tool used in the derivation of the definition of the integral with respect to DVVP processes.

Suppose that $F \in L^2(\Omega)$. It is a well known fact that any such random variable has a Wiener–Itô chaos expansion, say,

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where I_n denotes the n times iterated Wiener integral and f_n is a deterministic function such that $f_n \in \hat{L}^2(\mathbb{R}^n)$ with $\hat{L}^2$ denoting the space of square-integrable symmetric functions.

We denote by D_tF the Malliavin derivative of F. In terms of the chaos decomposition we have

$$D_tF = \sum_{n=0}^{\infty} nI_{n-1}(f_n(\,\cdot\,, t)),$$

and F is Malliavin differentiable (belongs to space $\mathbb{D}_{1,2}$) if and only if

$$\|D_tF\|_2^2 := \mathbb{E}\int_0^{\infty} (D_tF)^2 \, dt = \sum_{n=0}^{\infty} n\|f_n(\,\cdot\,, t)\|^2_{L^2(\mathbb{R}^{n-1})} < \infty.$$

The operator adjoint to the Malliavin derivative D_t is the Skorohod integral denoted by δ. Suppose that $Z(t)$ is a square-integrable stochastic process with the chaos expansion

$$Z(t) = \sum_{n=0}^{\infty} I_n(f_n(\,\cdot\,, t)),$$

where $f_n(t_1, \ldots, t_n, t)$ is deterministic, square-integrable and symmetric in the first n variables. Then (with $\hat{f}$ denoting symmetrization of f)

$$\int_0^\infty Z(t)\,\delta B(t) = \sum_{n=0}^\infty I_{n+1}(\hat{f}_n),$$

provided that

$$\left\| \int_0^\infty Z(t)\,\delta B(t) \right\|^2 := \mathbb{E}\left(\int_0^\infty Z(t)^2\,dt \right) = \sum_{n=0}^\infty n! \|f_n\|^2_{L^2(\mathbb{R}^{n+1})}.$$

The family of Skorohod integrable processes is denoted by $\mathrm{Dom}(\delta)$.

The fundamental result of Malliavin calculus used in the derivation of the integral with respect to DVVP is the integration by parts formula that we recall below.

Theorem 5.3. *Suppose that $Z \in \mathrm{Dom}(\delta)$, $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}(F^2 \int_0^\infty Z(t)^2\,dt) < \infty$. Then*

$$F \int_0^\infty Z(t)\,\delta B(t) = \int_0^\infty F Z(t)\,\delta B(t) + \int_0^\infty (D_t F) Z(t)\,dt.$$

5.3 The definition of the integral

The approach that we take in defining the integral with respect to DVVP follows that of [9] with the additional application of lemma 5.2 and an argument based on conditional probability. Below we only outline the idea and refer the interested reader to [8]. The definition of the DVVP integral is derived through a series of formal steps and starts with the observation that application of lemma 5.2 allows us to rewrite equation (5.2) as

$$X(t) = \int_0^t g(t,s)\sigma(s)\,dB(\theta(s)) = \int_0^{\theta(t)} g(t, \theta^-(s))\sigma(\theta^-(s))\,dB(s). \tag{5.4}$$

Observe that on the left side of the above expression t in $g(t,s)$ is merely a parameter and takes no part in the integration and θ is not applied to it on the right side of equation (5.4).

Now, upon conditioning on θ, we can use the tools of Malliavin calculus, as if θ was deterministic. Henceforth we assume that θ is a deterministic time-change. The heuristic derivation of the integral is sketched very briefly below.

First, application of the integration by parts formula yields

$$\int_0^t Y(s)\,dX(s) = Y(t)X(t) - \int_0^t \frac{dY(u)}{du} X(u)\,du. \tag{5.5}$$

We will concentrate on the integral on the right side of the above equation as similar steps are applied to the $Y(t)X(t)$ term. Using equation (5.4) and theorem 5.3, we have

$$\int_0^t \frac{dY(u)}{du} X(u)\,du = \int_0^t \frac{dY(u)}{du} \int_0^{\theta(u)} g(u,\theta^-(s))\sigma(\theta^-(s))\,dB(s)\,du$$
$$= \int_0^t \int_0^{\theta(u)} \frac{dY(u)}{du} g(u,\theta^-(s))\sigma(\theta^-(s))\,\delta B(s)\,du$$
$$+ \int_0^t \int_0^{\theta(u)} D_s\Big\{\frac{dY(u)}{du}\Big\} g(u,\theta^-(s))\sigma(\theta^-(s))\,ds\,du.$$

By stochastic Fubini theorem (see [40], theorem IV.65) and properties of Malliavin derivative we have

$$\int_0^t \frac{dY(u)}{du} X(u)\,du = \int_0^{\theta(t)} \left(\int_{\theta^-(s)}^t \frac{dY(u)}{du} g(u,\theta^-(s))\,du\right)\sigma(\theta^-(s))\,\delta B(s)$$
$$+ \int_0^{\theta(t)} D_s\Big\{\int_{\theta^-(s)}^t \frac{dY(u)}{du} g(u,\theta^-(s))\,du\Big\}\sigma(\theta^-(s))\,ds.$$

Applying similar procedure to the $Y(t)X(t)$ term in equation (5.5), combining the results and simplifying the result, we obtain the following definition.

Definition 5.4. *Suppose that θ is a stochastic time-change indepdendent of B and*

1. *$(\theta^-(s),t) \ni u \mapsto g(u,\theta^-(s))$ is a right-continuous function of bounded variation for all $s \in (0,\theta(t))$ $\mathbb{P}$-a.s.;*
2. *$g(t,\theta^-(s))$ is well defined for all $s \in (0,\theta(t))$ $\mathbb{P}$-a.s.;*

We define

$$\mathcal{K}_g^\theta(Y)(t,s) = Y(\theta^-(s))g(t,\theta^-(s)) + \int_{\theta^-(s)}^t \big(Y(u) - Y(\theta^-(s))\big)\,\mu_g^\theta(du,s), \tag{5.6}$$

where $\mu_g^\theta(\,\cdot\,,s)$ is a Lebesgue-Stieltjes measure arising from an outer measure given by

$$\hat{\mu}_g^\theta([a,b),s) = g(b,\theta^-(s)) - g(a,\theta^-(s)).$$

Suppose also that

3. *$\mathcal{K}_g^\theta(Y)(t,s)$ is Malliavin differentiable for all $s \in (0,\theta(t))$ $\mathbb{P}^\theta$-a.s.;*
4. *$s \mapsto \mathcal{K}_g^\theta(Y)(t,s)\sigma(\theta^-(s))$ is Skorohod integrable with respect to B on $(0,\theta(t))$ $\mathbb{P}^\theta$-a.s.;*
5. *$s \mapsto D_s\{\mathcal{K}_g^\theta(Y)(t,s)\}\sigma(\theta^-(s))$ is Lebesgue integrable on $(0,\theta(t))$ $\mathbb{P}^\theta$-a.s..*

Then Y is X-integrable and we define the dX-integral of Y as

$$\int_0^t Y(s)\,dX(s) := \int_0^{\theta(t)} \mathcal{K}_g^\theta(Y)(t,s)\sigma(\theta^-(s))\,\delta B(s) + \int_0^{\theta(t)} D_s\{\mathcal{K}_g^\theta(Y)(t,s)\}\sigma(\theta^-(s))\,ds \tag{5.7}$$

and understood on the space $(\Omega, \mathcal{F}_T^{B(\theta)}, \mathbb{P}_\theta)$.

Remark 5.5. Note that in definition 5.4 conditions 1 and 2 ensure that the operator $\mathcal{K}_g^\theta(Y)(t,s)$ in equation (5.6) is well-defined and conditions 3–5 ensure that the integral as in equation (5.7) is well-defined.

5.4 The properties of the integral

As any widely-accepted integral operator, the integral given in definition 5.4 is linear and the integral of 1 and 0 are X and 0 respectively. Moreover, the integral is additive in the domain of integration, that is $\int_0^t Y(s)\,dX(s) = \int_0^u Y(s)\,dX(s) + \int_u^t Y(s)\,dX(s)$ for any $0 < u < t$ such that all of the integrals above exist.

There is also an integration by parts formula for the integral defined in definition 5.4. Suppose that Z is a bounded random variable and Y is an X-integrable stochastic process such that ZY is also X-integrable. Then

$$\int_0^t ZY(s)\,dX(s) = Z\int_0^t Y(s)\,dX(s).$$

Putting together all of the above properties yields that the integral of a step function

$$Y(t) = \sum_{i=0}^{n} \mathbf{1}_{[t_i,t_{i+1})}(t)\xi_i,$$

where $0 = t_0 < \cdots < t_n = T$ and ξ_i are $\mathcal{F}_\infty^B$-measurable bounded random variables, is given by

$$\int_0^T Y(t)\,dX(t) = \sum_{i=0}^{n} \xi_i(X(t_{i+1}) - X(t_i)).$$

Suppose that $h(t)$ is a deterministic dX-integrable function. Then a simple computation yields

$$\int_0^t h(s)\,dX(s) = \int_0^t \mathcal{K}_g^{\mathrm{id}}(h)(t,s)\sigma(s)\,dB(\theta(s)),$$

where id denotes the identity function. Hence, a DVVP-integral of a deterministic function is again a DVVP process.

Now, suppose that $h(t,s)$ is a deterministic function such that $s \mapsto h(t,s)$ is dX-integrable for almost all t. Then we have that

$$\int_0^t h(t,s)\,dX(s) = \int_0^t \mathcal{K}_g^{\mathrm{id}}(h(t,\,\cdot\,))(t,s)\sigma(s)\,dB(\theta(s)).$$

Thus a DVVP-driven Volterra process is itself a DVVP process.

Observe that in the above, the operator $\mathcal{K}_g^{\mathrm{id}}$ is the same as the operator $\mathcal{K}_g$ used in the definition of the VMLV integral in [9]. Let us close this section with the observation that our integral is exactly the same as the one introduced in of [9] when $\theta = \mathrm{id}$.

5.5 An example regarding wind energy

In [33], a simple Langevin equation is used to model the power p produced by a wind turbine in response to a stochastic driving wind speed w,

$$dp(t) = -\alpha(p(t) - f(w(t)))\,dt + \beta\,dB(t), \tag{5.8}$$

where $f(v)$ is the power the wind turbine would produce if the driving wind speed was constant and equal to v, B is the standard Brownian motion, and α and β are constants. The solution to (5.8) is given by

$$p(t) = p(0)e^{-\alpha t} + \int_0^t \alpha e^{-\alpha(t-s)} f(w(s))\, ds + \beta \int_0^t e^{-\alpha(t-s)}\, dB(s).$$

The constant β can be estimated using quadratic variation and it was found in [33] to be not a constant but a variable depending on the driving wind speed w. In the light of the above, it is natural to replace $\beta\, dB(s)$ with $\beta(w(s))\, dB(s)$. Thus we obtain an amplitude volatility modulation. Moreover, since the wind turbine operates in a turbulent environment, the activity – or intensity – of the noise varies. Therefore it is natural to propose $\beta(w(s))\, dB(\theta(s))$ instead. Hence we obtain a model for the power curve given by

$$p(t) = p(0)e^{-\alpha t} + \int_0^t \alpha e^{-\alpha(t-s)} f(w(s))\, ds + \int_0^t e^{-\alpha(t-s)} \beta(w(s))\, dB(\theta(s)).$$

On the one hand, a model as the one above could aid the wind energy industry with better performance assessments. On the other hand, one might use such a model to learn more about the nature of the intensity modulation in turbulence using wind turbines as giant measurement devices.

References

[1] Alòs, E., Mazet, O. and Nualart., D. (2001): Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29**(2), 766–801.

[2] Barndorff-Nielsen, O.E. (1998): Processes of normal inverse Gaussian type. *Finance & Stoch.* **2**(1), 41–68.

[3] Barndorff-Nielsen, O.E. (2011): Stationary infinitely divisible processes. *Braz. J. of Probab. and Stat.* **25**(3), 294–322.

[4] Barndorff-Nielsen, O.E. (2012): Notes on the gamma kernel. *Research Report 3, Thiele Centre for Applied Mathematics in Natural Science, Aarhus University.*

[5] Barndorff-Nielsen, O.E., Blæsild, P. and Schmiegel, J. (2004): A parsimonious and universal description of turbulent velocity increments. *Eur. Phys. J. B* **41**, 345–363.

[6] Barndorff-Nielsen, O.E., Lunde, A., Shephard, N. and Veraart, A.E.D. (2013): Integer-valued trawl processes: A class of infinitely divisible processes. *Scand. J. Stat.* (To appear.)

[7] Barndorff-Nielsen, O.E., Benth, F.E. and Szozda, B. (2014): Stochastic integration for volatility modulated Brownian-driven processes via white noise analysis. *Infin. Dimens. Anal. and Ouantum Probab.* **17**(2), 1450011.

[8] Barndorff-Nielsen, O.E., Benth, F.E. and Szozda, B. (2015): Integration with respect to time-changed volatility modulated Lévy-driven Volterra processes. (In preparation.)

[9] Barndorff-Nielsen, O.E., Benth, F.E., Pedersen, J. and Veraart, A.E.D. (2013): On stochastic integration theory for volatility modulated Lévy-driven Volterra processes. *Stoch. Process. Appl.* **124**, 812–847.

[10] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A.E.D. (2015): Recent advances in ambit stochastics with a view towards tempo-spatial stochastic volatility/intermittency. *Banach Cent. Publ.* (To appear.)

[11] Barndorff-Nielsen, O.E., Maejima, M. and Sato, K. (2006): Infinite divisibility for stochastic processes and time change. *J. Theor. Probab.* **19**, 411–446.

[12] Barndorff-Nielsen, O.E., Pakkanen, M. and Schmiegel, J. (2014): Assessing relative volatility/intermittency/energy dissipation. *Electron. J. Stat.* **8**, 1996–2021.

[13] Barndorff-Nielsen, O., Pérez-Abreu, V. and Thorbjørnsen, S. (2013): Lévy mixing. *Lat. Am. J. of Probab. and Math. Stat.* **10**, 921–970.

[14] Barndorff-Nielsen, O.E., Sauri, O. and Szozda, B. (2015): Selfdecomposable fields. (In preparation.)

[15] Barndorff-Nielsen, O.E. and Shephard, N. (2001): Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Royal Stat. Soc.* **63**, 167–241.

[16] Barndorff-Nielsen, O.E., Shephard, N., Sauri, O., Hedevang, E. (2015): Levy-driven Models with Volatility. (In preparation.)

[17] Barndorff-Nielsen, O.E. and Schmiegel, J. (2006): Time change and universality in turbulence. *Research Report 15, Thiele Centre for Applied Mathematics in Natural Science, Aarhus University.*

[18] Barndorff-Nielsen, O.E. and Schmiegel, J. (2007): Ambit processes; with applications to turbulence and tumour growth. In Benth, F.E., Nunno, G.D., Linstrøm, T., Øksendal, B. and Zhang, T. (Eds.): *Stochastic Analysis and Applications: The Abel Symposium 2005.* Pp. 93–124. Heidelberg: Springer.

[19] Barndorff-Nielsen, O.E. and Schmiegel, J. (2008): A stochastic differential equation framework for the timewise dynamics of turbulent velocities. *Theory Probab. Its Appl.* **52**, 372–388.

[20] Barndorff-Nielsen, O.E. and Schmiegel, J. (2009): Brownian semistationary processes and volatility/intermittency. In Albrecher, H., Rungaldier, W. and Schachermeyer, W. (Eds.): *Advanced Financial Modelling.* Radon Ser. Comp. Appl. Math. **8**. Pp. 1–26. Berlin: W. de Gruyter.

[21] Barndorff-Nielsen, O.E., Hedevang, E., and Schmiegel, J. (2015): Distributional universality and BSS modelling. (In preparation.)

[22] Barndorff-Nielsen, O.E., Schmiegel, J. and Shephard, N. (2006): Time change and universality in turbulence and finance. *Research Report 18, Thiele Centre for Applied Mathematics in Natural Science, Aarhus University.*

[23] Barndorff-Nielsen, O.E. and Shiryaev, A.N. (2010): *Change of Time and Change of Measure.* Singapore: World Scientific.

[24] Benth, F.E. and Süß, A. (2013): Integration Theory for infinite dimensional volatility modulated Volterra processes. *arXiv: 1303.7143.*

[25] Basse-O'Connor, A., Graversen, S.E. and Pedersen, J. (2013): Stochastic integration on the real line. *Theory Probab. Its Appl.* **58**, 355–380.

[26] Chong, C. and Klüppelberg, C. (2013): Integrability conditions for space-time stochastic integrals: theory and applications. *arXiv: 1303.2468v2.*

[27] Dhruva, B.R. (2000): An experimental study of high Reynolds number turbulence in the atmosphere. *PhD Thesis, Yale University.*

[28] Di Nunno, G., Øksendal, B. and Proske, F. (2009): *Malliavin calculus for Lévy processes with applications to finance.* Berlin: Springer-Verlag.

[29] Eberlein, E. (2001): Application of generalized hyperbolic Lévy motions to finance. In Barndorff-Nielsen, O.E. et al. (Eds.): *Lévy Processes. Theory and Applications.* Birkhäuser, 319–336.

[30] Frisch, U. (1995): *Turbulence. The legacy of A.N. Kolmogorov.* Cambridge University Press.

[31] Hansen, P.R. and Lunde, A. (2006): Realized Variance and Market Microstructure Noise. *J. of Bus. & Econ. Stat.* **24**(2), 127–161.

[32] Hedevang, E. (2012): Stochastic modelling of turbulence: With applications to wind energy. *PhD Thesis. Aarhus University.*

[33] Hedevang, E. (2014): Wind turbine power curves incorporating turbulence intensity. *Wind Energy* **17**, 173–195.

[34] Hedevang, E. and Schmiegel, J. (2014): A causal continuous-time stochastic model for the turbulent energy cascade in a helium jet flow. *J. Turbul.* **14**(11), 1–26.

[35] Jurek, Z.J. (2011): The Random Integral Representation Conjecture a quarter of a century later. *Lith. Math. J.* **51**, 362–369.

[36] von Karman, T. (1948): Progress in the statistical theory of turbulence. *J. Marine Res.* **7**, 252-–264.

[37] Márquez, J.U. and Schmiegel, J. (2015): Modelling turbulent time series by BSS-processes. (Submitted.)

[38] Nualart, D. (2006): *The Malliavin calculus and related topics.* New York: Springer-Verlag.

[39] Podolskij, M. (2014): Ambit Fields: survey and new challenges. (To appear.)

[40] Protter, P.E. (2005): *Stochastic integration and differential equations.* Springer-Verlag, Berlin.

[41] Rajput, B.S. and Rosiński, J. (1989): Spectral representations of infinitely divisible processes. *Probab. Theory Rel. Fields* **82**, 451–487.

[42] Sato, K. and Yamazato, M. (1984): Operator selfdecomposable distributions as limits distributions of processes of Ornstein-Uhlenbeck type. *Stoch. Process. Appl.* **17**(1), 73–100.

[43] Veraart, A.E.D and Veraart, L.A.M (2014): Modelling electricity day-ahead prices by multivariate Lévy semistationary processes. In Benth, F.E., Kholodnyi, V., and Laurence, P. (Eds.): *Quant. Energy Finance*, Springer. (To appear.)