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Time inhomogeneity in longest gap and longest run problems

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Abstract

Consider an inhomogeneous Poisson process and let D be the first of its epochs which is followed by a gap of size $\ell > 0$. We establish a criterion for $D < \infty$ a.s., as well as for D being long-tailed and short-tailed, and obtain logarithmic tail asymptotics in various cases. These results are translated into the discrete time framework of independent non-stationary Bernoulli trials where the analogue of D is the waiting time for the first run of ones of length ℓ. A main motivation comes from computer reliability, where $D + \ell$ represents the actual execution time of a program or transfer of a file of size ℓ in presence of failures (epochs of the process) which necessitate restart.

Keywords: Bernoulli trials, heads runs, tail asymptotics, Poisson point process, delayed differential equation, computer reliability

1 Introduction

This paper is concerned with the study of the time

$$D = \min\{T_n : T_{n+1} - T_n \geq \ell\}$$

of occurrence of the first gap of length ℓ in an inhomogeneous Poisson process $\mathcal{N}$ on $(0, \infty)$ with epochs $0 < T_1 < T_2 < \cdots$ (where we use the convention $T_0 = 0$). In particular, we study the logarithmic asymptotics of the tail $\mathbb{P}(D > t)$ as $t \to \infty$ subject to a variety of forms of the rate function $\mu(t)$ of $\mathcal{N}$.

The tail probability $\mathbb{P}(D > t)$ can alternatively be written as $\mathbb{P}\big(L(t + \ell) < \ell\big)$ where

$$L(t) = \sup\{T_{n+1} \wedge t - T_n : T_n < t\}$$

is the longest gap between epochs before t. In this formulation, the time-homogeneous problem where $\mu(t) \equiv \mu$ has a classical discrete time parallel as the study of the

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longest success run L_n of n i.i.d. Bernoulli trials $\xi_1, \ldots, \xi_n$, with $\mathbb{P}(\xi_k = 0)$ taking the role of μ. This is an old and a well-studied problem with applications to insurance, finance, traffic and reliability, see [7, 10, 9, 16], nevertheless there is hardly any literature on the inhomogeneous case. In the body of the paper, we consider the continuous time Poisson framework but outline the translation to the inhomogeneous Bernoulli case (where zeroes take the role of epochs) in Section 7.

The asymptotics of the tail $\mathbb{P}(D > t)$ is fairly easy to obtain in the homogeneous case where a renewal argument easily gives $\mathbb{P}(D > t) \sim c\mathrm{e}^{-\gamma t}$ as $t \to \infty$, with γ being a root of a certain equation, see Proposition 5.1 below. In contrast, the behaviour is more diverse in the inhomogeneous case, and it may even happen that $\mathbb{P}(D < \infty) < 1$. In Section 3, we show that the critical rate of increase of $\mu(t)$ for this phenomenon is $\ell \log t$. Thus the rate of increase to ∞ of $\mu(t)$ can only be allowed to be very modest for D to be finite a.s. In addition, in Section 4 we show that if $\mu(t) \to \infty$ then D has a long-tailed distribution, i.e. $\mathbb{P}(D > t + s)/\mathbb{P}(D > t) \to 1$ as $t \to \infty$, whereas $\mathbb{P}(D > t + s)/\mathbb{P}(D > t) \to 0$ when $\mu(t) \to 0$.

Our asymptotic study is presented in Section 5, where we separately discuss the following three cases: (i) $\mu(t) = \mu$, (ii) $\mu(t) \to 0$ and (iii) $\mu(t) \to \infty$. Note that (ii) includes the case where $\int_0^\infty \mu(t)\,\mathrm{d}t < \infty$ so that there is a last epoch $T^* < \infty$ of $\mathcal{N}$. It could then happen that $D = T^*$, but our results (based on a bounding argument) show that typically the tail of D is lighter than that of T^*. Particular examples studied are $\mu(t) = a \log^{-b} t$, $\mu(t) = at^{-b}$ and $\mu(t) = a\mathrm{e}^{-bt}$. The long-tailed case (iii) is analyzed using a delayed differential equation derived in Section 2; a particular example is $\mu(t) = b \log t$ for $b \in (0, 1/\ell)$. Section 6 deals with what provided our initial motivation, the study of the tail of the total execution time X of a task like program or file transmission in a fault-tolerant computing environment working under the RESTART protocol, where the task needs to be completely restarted after failure. Here ℓ takes the role of the ideal task time, failures occur at the epochs of $\mathcal{N}$ and so $X = \ell + D$. Earlier studies of similar problems are in Asmussen *et al.* [4, 5, 6] and Jelenković *et al.* [14, 13]. The novelty here is the time-inhomogeneity. We also discuss a related RESTART problem with homogeneous failures, but time-varying service rate $r(t)$. A main idea is to use a simple time change to transform $\mathcal{N}$ to a homogeneous Poisson(μ) process.

Finally, Section 7 gives the corresponding results for the discrete time inhomogeneous Bernoulli case. Intuitively, this is connected to the Poisson framework by $p_i = \mathbb{P}(\xi_k = 1) = \mathrm{e}^{-\mu(i)}$, which is roughly the probability of no failures in $(i-1, i]$, and with one exception, the analysis is indeed a straightforward translation. Classical references such as [7, 9, 16] only treat time-homogeneity. Time-inhomogeneity only seems to have been studied in the framework of Markovian regime switching which is somewhat different from the models of this paper by being asymptotically stationary rather than exhibiting a trend. Some references are [1, 12, 3, 6] ([3] also contains some early and in part unprecise version of a few of the results of this paper).

Preliminaries

We represent the Poisson point process $\mathcal{N}$ on $(0, \infty)$ as a random subset of $(0, \infty)$. The intensity measure is denoted $M(\mathrm{d}t)$ and taken absolutely continuous with respect to Lebesgue measure on $(0, \infty)$, i.e. $M(\mathrm{d}t) = \mu(t)\mathrm{d}t$ for some rate function $\mu(t)$; we also write $M(a,b) = \int_a^b \mu(t)\mathrm{d}t$. If we write $\mathcal{N}(a,b)$ for the number of poiints in (a,b), we thus have

$$\mathbb{P}\big(\mathcal{N}(a,b)=0\big) = \mathrm{e}^{-M(a,b)}, \quad \mathbb{P}\big(\mathcal{N}(a,b)\ge 1\big) = 1-\mathrm{e}^{-M(a,b)} \le M(a,b)\,. \tag{1.1}$$

Moreover, it is assumed that $M(0,t) < \infty$ for any t, so that $\mathcal{N}(a,b) < \infty$ a.s. when $b < \infty$, and that $\mu(t) > 0$ for (Lebesgue) almost all $t \ge 0$. The latter assumption guarantees that $\mathbb{P}(D > t) > 0$ for any $t > 0$, and can be replaced by a weaker one.

Remark 1.1. Given the intensity measure $M(\mathrm{d}t)$ and ℓ, we may scale down time by ℓ and consider the new point process $\mathcal{N}' = \mathcal{N}/\ell$. That is, we may take $\ell' = 1$ and $M'(0,t) = M(0,t\ell)$, yielding $\mu'(t) = \ell\mu(t\ell)$ and (in obvious notation)

$$\mathbb{P}\big(D > t \,\big|\, \ell, \mu(\cdot)\big) = \mathbb{P}(D > t) = \mathbb{P}(D' > t/\ell) = \mathbb{P}\big(D > t/\ell \,\big|\, 1, \ell\mu(\cdot\ell)\big) \tag{1.2}$$

Nevertheless, we formulate all our results for a general ℓ, but sometimes switch to $\ell = 1$ in the proofs. □

Finally, the relation $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ and $f(x) \approx_{\log} g(x)$ logaritmic asymptotics as in large deviations theory, i.e. $\log f(x)/\log g(x) \to 1$.

2 First calculations

We start by recalling the famous Slivnyak's formula of Palm theory, see e.g. [15], which is the basic tool in most of our calculations. For any non-negative (measurable) function h it states that

$$\mathbb{E}\sum_{t\in\mathcal{N}} h(t, \mathcal{N}\backslash\{t\}) = \int_0^\infty \mathbb{E}h(t,\mathcal{N})\mu(s)\,\mathrm{d}s.$$

In this setting the indicator that there are no gaps in $(0,t)$ will often be useful. It corresponds to

$$g(t,\mathcal{N}) = \mathbf{1}\{L(t) < \ell\} = \mathbf{1}\{T_{n+1}\wedge t - T_n < \ell, \forall T_n \in [0,t)\} \tag{2.1}$$

and so $\mathbb{E}g(t,\mathcal{N}) = \mathbb{P}(D > t-\ell)$. Note that $g(t,\mathcal{N})$ depends only on the points of $\mathcal{N}$ in $[0,t)$ and hence it is independent of $\mathcal{N}\cap[t,\infty)$.

We first present a delayed differential equation for the tail probabilities. It will be used for a crucial estimate in Section 5.3 and is also potentially useful for computations of exact values of the $\mathbb{P}(D > t)$.

Proposition 2.1. *It holds for $t \ge 0$ that*

$$\mathbb{P}(D \in (t,\infty)) = \int_t^\infty \mathrm{e}^{-M(s,s+\ell)}\mathbb{P}(D > s-\ell)\mu(s)\,\mathrm{d}s.$$

Proof. Consider the event $D \in (t, \infty)$, which means that there is a point $s > t$ followed by a gap, and additionally there are no gaps in $(0, s)$. There can be only one such location s and hence only one such point of $\mathcal{N}$ a.s. Hence by Slivnyak's formula we readily obtain

$$\begin{aligned}\mathbb{P}(D \in (t, \infty)) &= \mathbb{E}\sum_{s\in\mathcal{N}} \mathbf{1}\{s > t\}\, \mathbf{1}\{\mathcal{N} \cap (s, s+\ell) = \emptyset\} g(s, \mathcal{N}) \\ &= \int_t^\infty \mathbb{P}(\mathcal{N} \cap (s, s+\ell) = \emptyset) \mathbb{E} g(s, \mathcal{N}) \mu(s)\, \mathrm{d}s \\ &= \int_t^\infty \mathrm{e}^{-M(s,s+\ell)} \mathbb{P}(D > s - \ell) \mu(s)\, \mathrm{d}s,\end{aligned}$$

because the above indicators are independent and stay unchanged when we remove s from $\mathcal{N}$, since $(0, s)$ and $(s, s+\ell)$ are disjoint and do not contain s. □

Differentiating the result of Proposition 2.1 at t, we obtain the delayed differential equation

$$\mathbb{P}(D > t)' = -\mathrm{e}^{-M(t,t+\ell)} \mathbb{P}(D > t - \ell) \mu(t), \quad t \geq 0. \tag{2.2}$$

This may be solved in the intervals $\big[k\ell, (k+1)\ell\big)$ by using recursion in k and the initial condition

$$\mathbb{P}(D > t) = 1 - \int_0^t \mathrm{e}^{-M(s,s+\ell)} \mu(s) \mathrm{d}s - \mathrm{e}^{-M(0,\ell)}, \qquad t \in [0, \ell), \tag{2.3}$$

which follows from $\mathbb{P}(D = 0) = \mathrm{e}^{-M(0,\ell)}$ and the consequence

$$\mathbb{P}(D \in (0, t]) = \int_0^t \mathrm{e}^{-M(s,s+\ell)} M(\mathrm{d}s)\,, \quad t \in [0, \ell]$$

of Proposition 2.1. Letting $f(t) = -\log \mathbb{P}(D > t)$ we also have

$$f'(t) = \mathrm{e}^{-M(t,t+\ell)} \mu(t) \mathrm{e}^{f(t) - f(t-\ell)}, \tag{2.4}$$

which may be more suitable for numerical computation.

3 Finiteness of D

First, we present an integral test.

Theorem 3.1. *If $M(0, \infty) < \infty$ then $D < \infty$ a.s. Otherwise, let*

$$I = \int_0^\infty \mathrm{e}^{-M(t,t+\ell)} \mu(t) \mathrm{d}t\,.$$

Then $D < \infty$ a.s. if $I = \infty$ and $\mathbb{P}(D = \infty) > 0$ if $I < \infty$.

Proof. Assume $M(0,\infty)=\infty$ since otherwise $T^*<\infty$ so that the result is obvious as noted in the Introduction. According to Remark 1.1

$$I' = \int_0^\infty \mathrm{e}^{-M'(t,t+1)}\mu'(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-M(t\ell,(t+1)\ell)}\ell\mu(t\ell)\mathrm{d}t = I$$

and so it is enough to show the claim for $\ell = 1$. Let E_t be the event that $\mathcal{N}$ has a point in $[t,t+1)$ followed by a gap; there can be only one such point. By Slivnyak's formula,

$$\begin{aligned}\mathbb{P}(E_t) &= \mathbb{E}\sum_{s\in\mathcal{N}} \mathbf{1}\{s\in[t,t+1)\}\,\mathbf{1}\{\mathcal{N}\cap(s,s+1)=\emptyset\}\\ &= \int_t^{t+1}\mathbb{P}(\mathcal{N}\cap(s,s+1)=\emptyset)\mu(s)\,\mathrm{d}s = \int_t^{t+1}\mathrm{e}^{-M(s,s+1)}\mu(s)\,\mathrm{d}s.\end{aligned}$$

Hence

$$I=\sum_{n=0}^\infty \mathbb{P}(E_n) = \sum_{n=0}^\infty \mathbb{P}(E_{2n}) + \sum_{n=0}^\infty \mathbb{P}(E_{2n+1}).$$

If $I=\infty$ then at least one sum on the right is infinite; suppose the first one. But the events E_{2n} are independent, and so by Borel-Cantelli lemma infinitely many of them occur with probability 1. In particular, $D<\infty$ a.s.

Assume conversely $I<\infty$. Observe that the probability of a gap in (T,∞) is bounded by the expected number $\mathbb{E}N(T)$ of gaps in (T,∞), where

$$\mathbb{E}N(T) = \exp(-M(T,T+1)) + \int_T^\infty \exp(-M(t,t+1))M(\mathrm{d}t)\,. \tag{3.1}$$

We choose T so large that the last term is smaller than $1/4$, which is possible according to $I<\infty$. Now if the first term is smaller than $1/4$ then the probability of no gaps in (T,∞) is at least $1/2$. Hence we define $A=\{t\,:\,\exp(-M(t,t+1))\le 1/4\}$ and note that the Lebesgue measure of A is infinite since otherwise we have a contradiction with the assumptions $I<\infty$ and $M(0,\infty)=\infty$. To complete the proof, consider the stopping time $\tau=\inf\{t>T\,:\,t\in\mathcal{N}\cap A\}$ which is finite a.s. and hence $\mathbb{P}(\text{no gaps in }(0,\tau))=\delta>0$. Applying the strong Markov property at τ (or using Slivnyak's formula), we finally obtain $\mathbb{P}(D=\infty)\ge\delta/2>0$. □

Proposition 3.2. *For a different inhomogeneous Poisson process $\mathcal{N}'$ with rate function $\mu'(t)>0$, it holds that:*

(i) *If $\mu\le\mu'$ where $I'=\infty$ or $M'(0,\infty)<\infty$, then $\mathbb{P}(D<\infty)=1$.*

(ii) *If $\mu\ge\mu'$ where $I'<\infty$ and $M'(0,\infty)=\infty$, then $\mathbb{P}(D=\infty)>0$.*

Proof. This follows by standard coupling arguments. In (i), write $\mathcal{N}'=\mathcal{N}+\mathcal{N}''$ where $\mathcal{N},\mathcal{N}''$ are independent and $\mathcal{N}''$ Poisson with rate function $\mu'-\mu$. Then $\mathcal{N}\subseteq\mathcal{N}'$ which immediately implies $D\le D'<\infty$. The proof of (ii) is similar, noting that $M'(0,\infty)<\infty$ has to be excluded since then $\mathbb{P}(D'<\infty)=1$. □

Remark 3.3. Note that always $I < \infty$ when $M(0,\infty) < \infty$.

The difficulty in applying Theorem 3.1 directly is that the integrand in I is not necessarily monotonic in μ since $\mathrm{e}^{-M(t,t+\ell)}$ is decereasing in μ. Nevertheless, the converse parts of Theorem 3.1 give that also $I = \infty$ or $M(0,\infty) < \infty$ under the conditions of (i) and $I < \infty$ and $M(0,\infty) = \infty$ under the conditions of (i). □

Corollary 3.4. *It holds that:*

(i) *If* $\limsup_{t\to\infty} \mu(t)/\log t < 1/\ell$ *then* $D < \infty$ *a.s.*

(ii) *If* $\liminf_{t\to\infty} \mu(t)/\log t > 1/\ell$ *then* $\mathbb{P}(D = \infty) > 0$.

Proof. Consider for some $h > 0$ the particular rate function

$$\mu'(t) = \mathbf{1}\{t > T\} h \log t/\ell + \mathbf{1}\{t \le T\}\mu(t)$$

with h, T chosen such that $\mu \le \mu'$, $h < 1$ in (i) and $\mu \ge \mu'$, $h > 1$ in (ii). Observe that $M'(t,t+\ell) = h\log t + \mathrm{o}(1)$. Hence $\mathrm{e}^{-M'(t,t+\ell)} = t^{-h}(1+\mathrm{o}(1))$, and so I'' is finite for $h > 1$ and infinite for $h < 1$. Reference to Proposition 3.2 completes the proof. □

4 When does D have a long tail?

In the heavy-tailed area, it is customary to call a r.v. X for *long-tailed* if $\mathbb{P}(X > t+u)/\mathbb{P}(X > t) \to 1$ for any $u > 0$ as $t \to \infty$. This contrasts typical light-tailed r.v.'s such as the gamma or inverse Gaussian where the limit is in $(0,1)$, or the Gaussian or light-tailed Weibull, i.e. $\mathbb{P}(X > t) = \mathrm{e}^{-t^\beta}$ with $\beta > 1$, where it is 0. In the present context, we have:

Proposition 4.1. *Let* $u > 0$. *Then for* $t \to \infty$ *it holds that:*

(i) *If* $\mu(t) \to \infty$ *then* $\lim \mathbb{P}(D > t+u)/\mathbb{P}(D > t) = 1$.
Moreover, if $\liminf M(t, t+\ell-h) > 0$ *for some* $h > 0$ *then* $\liminf \mathbb{P}(D > t+u)/\mathbb{P}(D > t) > 0$.

(ii) *Conversely, let* $u \ge \ell$. *Then* $\mathbb{P}(D > t+u)/\mathbb{P}(D > t) \to 0$ *if* $\mu(t) \to 0$. *Moreover, if* $\limsup M(t,t+\ell) < \infty$, *then* $\limsup \mathbb{P}(D > t+u)/\mathbb{P}(D > t) < 1$.

Proof. We may assume that $\ell = 1$. In (ii), $u \ge 1$ and $D > t+u$ imply that at least $D > t$ and there is a point in $(t+1, t+2)$. Thus

$$\mathbb{P}(D > t+u) \le \mathbb{P}(D > t)(1 - \mathrm{e}^{-M(t+1,t+2)}),$$

where independence follows from the fact that $D > t$ is determined by the points of $\mathcal{N}$ in $(0, t+1]$. From this both assertions of (ii) follow, noting that if $\mu(t) \to 0$ then also $M(t+1, t+2) \to 0$.

For (i), we first note that for $u_1, u_2 > 0$ we have $u_2 \in \big[(k-1)u_1, ku_1\big]$ for some $k = 1, 2, \ldots$ and so

$$\frac{\mathbb{P}(D > t+u_2)}{\mathbb{P}(D > t)} \ge \frac{\mathbb{P}(D > t+ku_1)}{\mathbb{P}(D > t)} = \prod_{i=1}^{k} \frac{\mathbb{P}(D > t+iu_1)}{\mathbb{P}(D > t+(i-1)u_1)}.$$

Thus if any of the two assertions hold for u_1, it holds also for u_2 and so we can take $u = h \in (0, 1)$.

For a given t, let $\tau = \inf\bigl\{s \in \mathcal{N} : s \in (t, t+1], \text{no gap in } (0, s)\bigr\}$, $\tau = \infty$ if no such s exists. Assume $M(t, t+1-h) > \delta > 0$ for $t \geq T$. The event $D > t+h$ will hold if either $\tau \in (t, t+h]$ and there is a point in $(t+h, t+1]$ (the first candidate for D is then the first such point), or if $\tau \in (t+h, t+1]$ (τ is then the first candidate for D). Thus

$$\begin{aligned}\mathbb{P}(D > t+h) &\geq \mathbb{P}\bigl(\tau \in (t, t+h]\bigr)(1 - \mathrm{e}^{-\delta}) + \mathbb{P}\bigl(\tau \in (t+h, t+1]\bigr) \\ &\geq \mathbb{P}\bigl(\tau \in (t, t+1]\bigr)(1 - \mathrm{e}^{-\delta}) = \mathbb{P}(D > t)(1 - \mathrm{e}^{-\delta})\end{aligned}$$

for $t \geq T$. This immediately gives the second assertion in (i), and for the second we just need to note that $\mathbb{P}(D > t+1) \leq \mathbb{P}(D > t)$ and that δ can be taken arbitrarily large if $\mu(t) \to \infty$. □

5 Asymptotics

5.1 The homogeneous case

The following result occurs in [4] and its discrete time analogue can be found in e.g. [11, Ch. XIII], but we present it here for completeness.

Proposition 5.1. *Assume $\mu(t) \equiv \mu$ and let $\gamma > 0$ denote the unique root of*

$$1 = \int_0^\ell \mu \mathrm{e}^{(\gamma_0 - \mu)s}\,\mathrm{d}s\,. \tag{5.1}$$

Then $\mathbb{P}(D > t) \sim c\mathrm{e}^{-\gamma t}$ as $t \to \infty$ for some $0 < c < \infty$.

Proof. Conditioning on the first epoch we write

$$\mathbb{P}(D > t) = \int_0^{t \wedge \ell} \mathbb{P}(D > t-s)\,\mu\mathrm{e}^{-\mu s}\,\mathrm{d}s + \mathbf{1}\{t < \ell\}\mathbb{P}(T_1 \in (t, \ell)).$$

Putting $Z(t) = \mathbb{P}(D > t)\mathrm{e}^{\gamma t}$ and $z(t) = \mathbf{1}\{t < \ell\}\mathbb{P}(T_1 \in (t, \ell))\mathrm{e}^{\gamma t}$ we obtain the renewal equation

$$Z(t) = z(t) + \int_0^t Z(t-s)G(\mathrm{d}s), \tag{5.2}$$

where G is the measure with density $\mu\mathrm{e}^{(\gamma-\mu)s}$ for $s \leq \ell$. Observe that G is nonlattice and proper according to (5.1), and so the key renewal theorem, see [2, Thm. V.4.3], shows that $Z(t) \to \int_0^\infty z(s)\mathrm{d}s / \int_0^\infty sG(\mathrm{d}s) = c \in (0, \infty)$. This completes the proof. □

5.2 Short tail

The material of this section is based on the following two assumptions. Firstly, we assume that $M(t, t+\ell)$ is bounded, which immediately excludes the case of long tails, see Proposition 4.1. Secondly, supposing that the correct logarithmic asymptotic is given by a function $f(t)$, that is $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-f(t)}$, we assume that $t/f(t) \to 0$ as $t \to 0$. This latter property of $f(t)$ turns out to be crucial for tightening the gap between the bounds for $\log \mathbb{P}(D > t)$ proposed below . Moreover, under this assumption it is reasonable to expect short tails, and thus we mainly think about the case $\mu(t) \to 0$. Finally, Remark 1.1 allows to take $\ell = 1$ for notational simplicity.

Proposition 5.2. *Assume that $\ell = 1$, $M(t, t+1)$ is bounded for large t and that there exist positive functions f, g such that $t/f(t) \to 0$ as $t \to \infty$ and $g(\epsilon) \to 1$ as $\epsilon \downarrow 0$. If for any small enough $\epsilon > 0$ and some k*

$$\limsup_{t\to\infty} \frac{1}{f(t)} \sum_{i=k}^{\lfloor t \rfloor} \log M(i, i+1) \le -1, \tag{5.3}$$

$$\liminf_{t\to\infty} \frac{1}{f(t)} \sum_{i=k}^{\lceil t/(1-\epsilon) \rceil} \log M\big(i(1-\epsilon), i(1-\epsilon)+\epsilon\big) \ge -g(\epsilon) \tag{5.4}$$

then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-f(t)}$ as $t \to \infty$.

Proof. Notice that $\{D > t\}$ is contained in the event that $\mathcal{N}$ has a point in each of the intervals $[i, i+1), i = k, \dots, \lfloor t \rfloor$, and so

$$\mathbb{P}(D > t) \le \prod_{i=k}^{\lfloor t \rfloor} \big(1 - \exp\{-M(i, i+1)\}\big) \le \prod_{i=k}^{\lfloor t \rfloor} M(i, i+1).$$

by the independence property of $\mathcal{N}$ and (1.1). Taking logarithms we get

$$\log \mathbb{P}(D > t) \le \sum_{i=k}^{\lfloor t \rfloor} \log M(i, i+1).$$

Fix $\epsilon \in (0, 1/2)$ and let $h = 1 - \epsilon$. Consider the intervals $[hi, hi + \epsilon)$ and assume that there is a point of $\mathcal{N}$ in each of these intervals for $i = 0, \dots, n$ with $n = \lceil t/h \rceil$. Then necessarily $D > t$, see Figure 1. Hence we have a bound

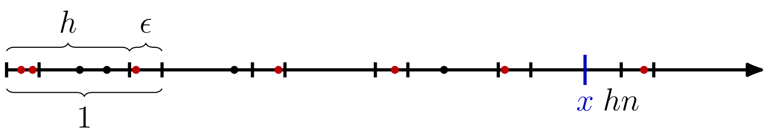

Figure 1: Points of $\mathcal{N}$ and the intervals $[hi, hi + \epsilon)$.

$$\mathbb{P}(D > t) \ge \prod_{i=0}^{\lceil t/h \rceil} \big(1 - \exp\{-M(hi, hi+\epsilon)\}\big) \ge c_1 \prod_{i=k}^{\lceil t/h \rceil} c_2 M(hi, hi+\epsilon)$$

for some $c_1, c_2 > 0$, because $M(t, t+1)$ is eventually bounded. Hence we obtain

$$\log \mathbb{P}(D > t) \ge \mathrm{O}(t) + \sum_{i=k}^{\lceil t/h \rceil} \log M(hi, hi+\epsilon).$$

Finally, from (5.3) and (5.4) we get

$$-g(\epsilon) \le \liminf \log \mathbb{P}(D > t)/f(t) \le \limsup \log \mathbb{P}(D > t)/f(t) \le -1,$$

because $t/f(t) = \mathrm{o}(1)$ according to the assumptions. Choosing ϵ arbitrarily small we get $\log \mathbb{P}(D > t)/f(t) \to -1$, which completes the proof. □

Remark 5.3. The choice of the lower bound for $\mathbb{P}(D > t)$ is not initially obvious: it requires a point near each natural number up to $\lceil t \rceil$. A more natural way is to require that each of the intervals $[i/2, i/2 + 1/2), i = 0, \dots, 2\lceil t \rceil$ has a point. In the short tail case it turns out that it is essential to keep the number of terms in the product close to t, which is achieved by the former bound. □

Let us give a simplified version of Proposition 5.2 for an important special case.

Corollary 5.4. *Assume that $\mu(t)$ is eventually non-increasing and let f be a function which is regularly varying at ∞ with index $\alpha \ge 1$ and satisfies $t/f(t) \to 0$. If*

$$\sum_{i=1}^{n} \log(\mu(ic)) \sim -c^{\rho} f(n) \qquad \text{as } n \to \infty \tag{5.5}$$

for all $c > 0$ and some $\rho \in \mathbb{R}$, then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-\ell^{\rho-\alpha} f(t)}$ as $t \to \infty$.

Note that the assumption $t/f(t) \to 0$ is automatic if $\alpha > 1$.

Proof. First we fix $\ell = 1$ and show that $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-f(t)}$. Regular variation implies that $f(t+h)/f(t) \sim 1$ uniformly in $h \in [-1, 1]$ (use e.g. the representation theorem [8, Thm. 1.3.1]). Next, we note that $M(i, i+1) \le \mu(i)$ and $M\big(i(1-\epsilon), i(1-\epsilon)+\epsilon\big) \ge \epsilon\mu\big((i+1)(1-\epsilon)\big)$ for $\epsilon < 1/2$ and $i \ge i_0$. So we have

$$\sum_{i=i_0}^{\lfloor t \rfloor} \log M(i, i+1) \le \frac{1}{-f(\lfloor t \rfloor)} \sum_{i=i_0}^{\lfloor t \rfloor} \log(\mu(i)) \frac{-f(\lfloor t \rfloor)}{f(t)} \sim -1.$$

Similarly,

$$\frac{1}{f(t)} \sum_{i=i_0}^{\lceil t/(1-\epsilon) \rceil} \log M\big(i(1-\epsilon), i(1-\epsilon)+\epsilon\big) \ge \frac{1}{f(t)} \sum_{i=i_0+1}^{\lceil t/(1-\epsilon) \rceil + 1} \big(\log \mu\big(i(1-\epsilon)\big) + \log \epsilon\big).$$

It only remains to note that

$$f\big(\lceil t/(1-\epsilon) \rceil + 1\big)/f(t) \sim f\big(t/(1-\epsilon)\big)/f(t) \sim (1-\epsilon)^{-\alpha},$$

and so we take $g(\epsilon) = (1-\epsilon)^{\rho-\alpha}$, which concludes the proof for $\ell = 1$.

For arbitrary $\ell > 0$, we take $\mu'(t) = \ell\mu(t\ell)$ according to Remark 1.1 and so

$$\sum_{i=1}^{n} \log \mu'(ic) = n \log \ell + \sum_{i=1}^{n} \log \mu(ic\ell) \sim -c^{\rho} \ell^{\rho} f(n)$$

implying $\mathbb{P}(D > t) = \mathbb{P}(D' > t/\ell) \approx_{\log} \mathrm{e}^{-\ell^{\rho} f(t/\ell)} \approx_{\log} \mathrm{e}^{-\ell^{\rho-\alpha} f(t)}$. □

Specializing to some important forms of $\mu(t)$, we obtain:

Corollary 5.5. *Let $a, b > 0$. Then it holds as $t \to \infty$ that:*

(i) *If $\mu(t) = a \log^{-b} t$ for large t then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt \log\log t/\ell}$.*

(ii) *If $\mu(t) = at^{-b}$ for large t then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt \log t/\ell}$.*

(iii) *If $\mu(t) = a\mathrm{e}^{-bt}$ for large t then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt^2/(2\ell)}$.*

Proof. (i) Note that $\sum_{i=1}^n \log\log(ic) \sim n \log\log n$, where one can use e.g. the discrete L'Hospital's rule. Hence

$$\sum_{i=1}^n \log \mu(ic) = n \log a - b \sum_{i=1}^n \log\log(ic) \sim -bn \log\log n = -f(n).$$

Hence $\rho = 0, \alpha = 1$ and $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt \log\log t/\ell}$.

(ii) Observe that

$$\sum_{i=1}^n \log \mu(ic) = n(\log a - b \log c) - b \sum_{i=1}^n \log i \sim -bn \log n = -f(n).$$

Hence $\rho = 0, \alpha = 1$ and $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt \log t/\ell}$.

(i) Follows by an argument which is similar and hence omitted.

(iii) Here

$$\sum_{i=1}^n \log \mu(ic) = n \log a - bc \sum_{i=0}^n i \sim -bcn^2/2 = -cf(n).$$

Hence $\rho = 1$, $\alpha = 2$ and so $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-bt^2/(2\ell)}$. □

5.3 Long tail

In this section we focus on the case of long tails, and hence we mainly think about examples where $\mu(t) \to \infty$ but so slowly that $D < \infty$ a.s.. Recall from Corollary 3.4 that this essentially only allows $\mu(t)$ to grow at rate $\log t$. This makes it reasonable to assume that $M(t, t+\ell)$ is approximately $\ell\mu(t)$. It turns out that in the long-tailed case the delayed differential equation (2.2) readily provides the asymptotics.

Proposition 5.6. *Assume that $\mu(t)$ is continuous almost everywhere, tending to ∞, and satisfies $M(t, t+\ell) = \ell\mu(t) + \mathrm{o}(1)$ as $t \to \infty$ together with the conditions of Theorem 3.1 for $\mathbb{P}(D < \infty) = 1$. Let c be arbitrary and define $f(t) = \int_c^t \mathrm{e}^{-\ell\mu(s)}\mu(s)\,\mathrm{d}s$. Then $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-f(t)}$.*

Proof. Let $\xi(t) = -\log \mathbb{P}(D > t)$. According to (2.4) we may write

$$\xi'(t) = \mathrm{e}^{-M(t,t+\ell)} \frac{\mathbb{P}(D > t - \ell)}{\mathbb{P}(D > t)} \mu(t) \sim \mathrm{e}^{-\ell\mu(t)}\mu(t),$$

where we also employ Proposition 4.1 giving that $\mathbb{P}(D > t - \ell)/\mathbb{P}(D > t) \to 1$. Hence for any $\epsilon > 0$ there exists T such that

$$(1-\epsilon)\mathrm{e}^{-\ell\mu(t)}\mu(t) \le \xi'(t) \le (1+\epsilon)\mathrm{e}^{-\ell\mu(t)}\mu(t), \quad t > T.$$

Thus by the fundamental theorem of calculus we have for $t > T$:

$$(1-\epsilon)\big(f(t) - f(T)\big) \le \xi(t) - \xi(T) \le (1+\epsilon)\big(f(t) - f(T)\big).$$

But $\xi(t) \to \infty$ since $\mathbb{P}(D < \infty) = 1$, and so it must be that $f(t) \to \infty$. This gives $\xi(t) \sim f(t)$. □

Corollary 5.7. *The following logarithmic asymptotics hold:*

(i) *If $\mu(t) = b \log t$ for large t with $b \in (0, 1/\ell)$ then*

$$\mathbb{P}(D > t) \approx_{\log} \exp\Big\{-\frac{b}{1-b\ell} t^{1-b\ell} \log t\Big\}.$$

(ii) *If $\ell = 1$ and $\mu(t) = \log t - b \log\log t - \log a$ for large t with $a > 0, b > -2$ then*

$$\mathbb{P}(D > t) \approx_{\log} \exp\Big\{-\frac{a}{2+b} \log^{2+b} t\Big\}.$$

Proof. In (i), the conditions of Proposition 5.6 concerning $\mu(t)$ are easily verified. Finally, $\mathrm{e}^{-\ell\mu(t)}\mu(t) = bt^{-b\ell} \log t$ with primitive

$$f(t) = \frac{b}{1-b\ell} t^{1-b\ell}(\log t - 1/(1-b\ell)) \sim \frac{b}{1-b\ell} t^{1-b\ell} \log t\,.$$

For (ii), we get

$$\int_c^t \mathrm{e}^{-\mu(s)}\mu(s)\,\mathrm{d}s \sim \int_c^t \frac{a \log^b s}{s} \log s\,\mathrm{d}s = \Big[\frac{a \log^{2+b} s}{2+b}\Big]_c^t$$

The assumption $b > -2$ ensures that this expression has limit ∞ as $t \to \infty$, and therefore both that $\mathbb{P}(D < \infty) = 1$ and that $\mathbb{P}(D > t)$ has the asserted logarithmic asymptotics. □

Remark 5.8. Note that in (ii), the result for $b = -1$ corresponds to a power tail $1/t^a$, the one for $b = 0$ to a lognormal tail etc.

Comparing to the homogeneous case $\mu \equiv a$, (i) in Corollary 5.7 gives the asymptotic tail of D when $\mu(\cdot)$ is of slightly bigger order. The complementary result for slightly smaller order is (i) in Corollary 5.5. □

Finally, we note that the problems corresponding to long and short tail asymptotics are essentially different: the delayed differential equation (2.2) seems to be of no help in the short tail case, whereas bounding ideas do not work in the long tail case. E.g. for $\mu(t) = b \log t$, $\ell = 1$ they only give

$$C_1 t^{1-b} \le -\log \mathbb{P}(D > t) \le C_2 t^{1-b/2},,$$

twhile he correct answer is $Ct^{1-b} \log t$ according to (i) in Corollary 5.7.

6 Translation to RESTART problems

Tasks such as the execution of a computer program or the transfer of a file on a communications link may fail. There is a considerable literature on protocols for handling such failures. We mention in particular RESUME where the task is resumed after repair, REPLACE where the task is abandoned and a new one taken from the pile of waiting tasks, RESTART where the task needs to be restarted from scratch, and CHECKPOINTING where the task contains checkpoints such that performed work is saved at checkpoint times and that upon a failure, the task only needs to be restarted from the last checkpoint.

The model of Asmussen *et al.* [4] assumes that failures occur at a time after each restart with the same distribution G for each restart (a particular important case is of course the exponential distribution). However, it is easy to imagine situations where the model behaviour is rather determined by the time of the day (the clock on the wall) rather than the time elapsed since the last restart. Think, e.g., of a time-varying load in the system which may influence the failure rate and/or the speed at which the task is performed. For example, the load could be identified with the number of busy tellers in a call center or the number of users in a LAN (local area network) currently using the central server, both exhibitng rush-hours. We provide here some first insight in the behaviour of such models.

The emphasis in [4] is on the more difficult case of a random rather than a constant ideal task time. However, as a first attempt it seems reasonable to assume a constant ideal task time of length ℓ. Then total task time is simply $X = \ell + D$, and the results of Sections 3–5 immediately apply to show that the critical rate of increase of $\mu(t)$ for X to be finite is $\log t/\ell$, that $\mathbb{P}(X > x) \approx_{\log} \mathrm{e}^{-bx\log x/\ell}$ as $x \to \infty$ when $\mu(t) = a/t^b$ etc.

A model of equal interest is the one with a time-varying processing rate $r(t) \geq 0$. For convenience, we will assume that $r(t)$ is a continuous, strictly positive function satisfying $\int_0^\infty r(t)\mathrm{d}t = \infty$, and that failures occur according to a Poisson process with constant rate μ^*. The quantity of interest is again the delay D^* (sum of times of unsuccessful attempts).

Note that $R(t) = \int_0^t r(s)\,\mathrm{d}s$ is the amount of work that has been spent on the task up to time t provided the task has not been completed and the task time X^* in absence of failures is given by $R(X^*) = \ell$, i.e. $X^* = R^{-1}(\ell)$. More generally, if the task is not completed at the time T^*_{n-1} of the $(n-1)$th failure, then the task is still uncompleted at T^*_n if and only if $R(T^*_n) - R(T^*_{n-1}) < \ell$. Hence the results for this model follow in a straightforward way from the preceding analysis by using the time change $T_n = R(T^*_n)$ to transform $\mathcal{N}^*$ into an inhomogeneous Poisson process $\mathcal{N}$. Then $D^* = R^{-1}(D)$ and $X^* = R^{-1}(D+\ell)$, and $M(0, R(t)) = M^*(0,t) = \mu^* t$ implying $\mu(R(t)) = \mu^*/r(t)$. We do not spell out this translation for all cases, but for example:

Corollary 6.1. (i) *If* $\limsup_{t\to\infty} \mu^*/(r(t)\log R(t)) < 1/\ell$, *then* $X^* < \infty$ *a.s. Conversely,* $\mathbb{P}(X^* = \infty) > 0$ *provided* $\liminf_{t\to\infty} \mu^*/(r(t)\log R(t)) > 1/\ell$.

(ii) *Assume* $r(t) = at^b$ *with* $b > 0$. *Then*

$$\mathbb{P}(X^* > x) \;\approx_{\log}\; \mathbb{P}(D^* > x) \;\approx_{\log}\; \exp\Big\{-\frac{ab}{b+1}t^{b+1}\log t/\ell\Big\}.$$

Proof. According to Corollary 3.4 we need to look at the limiting behaviour of $\mu(t)/\log t$, which is the same as that of $\mu(R(t))/\log R(t) = \mu^*/(r(t)\log R(t))$. But $X^* < \infty$ iff $D < \infty$ which concludes the proof of (i).

For (ii), note that $R(t) = a/(b+1)t^{b+1}$ and $R^{-1}(t) = ((b+1)/at)^{1/(b+1)}$ yielding

$$\mu(t) = \frac{\mu^*}{r(R^{-1}(t))} = \frac{\mu^*}{a^{1/(b+1)}((b+1)t)^{b/(b+1)}} = ct^{-b/(b+1)}.$$

According to Corollary 5.5 we have $\mathbb{P}(D > t) \approx_{\log} \mathrm{e}^{-b/(b+1)t\log t/\ell}$. Hence

$$\mathbb{P}(D^* > t) = \mathbb{P}(D > R(t)) \approx_{\log} \exp\Big\{-\frac{b}{b+1}at^{b+1}\log t/\ell\Big\}.$$

Finally, $\mathbb{P}(X^* > t) = \mathbb{P}(D + \ell > R(t))$ showing that $\mathbb{P}(X^* > t)$ has the same logarithmic asymptotic. □

7 Discrete time version

The following can be considered as an analogue of our gap problem in discrete time. Consider a (non-stationary) sequence ξ_i, $i = 1, 2, \ldots$, of independent Bernoulli variables with $\mathbb{P}(\xi_i = 1) = p_i$, and let for some fixed integer $\ell \geq 1$

$$D = \min\{n : \xi_n = \cdots = \xi_{n+\ell-1} = 1\}$$

be the time of the first run of ℓ ones. Write $q_i = 1 - p_i$ and assume $0 < p_i < 1$ for all i. Most results and proofs for this model is a straightforward translation from the inhomogeneous Poisson case so we give only some selected analogues.

Corollary 7.1. (i) $\mathbb{P}(D = n+1) = q_n \mathbb{P}(D > n - \ell) \prod_{j=n+1}^{n+\ell} p_j$.

(ii) *If* $\sum_{i=1}^{\infty} q_i < \infty$ *or* $I = \sum_{i=1}^{\infty} q_i \prod_{j=i+1}^{i+\ell} p_j < \infty$ *then* $D < \infty$ *a.s. Otherwise* $\mathbb{P}(D = \infty) > 0$.

(iii) *If* $\limsup_{i\to\infty} \frac{-\log p_i}{\log i} < \frac{1}{\ell}$ *then* $I = \infty$ *and if* $\liminf_{i\to\infty} \frac{-\log p_i}{\log i} > \frac{1}{\ell}$ *then* $I < \infty$.

(iv) $p_i \to 1$ *implies short tail, i.e.* $\mathbb{P}(D > i + n)/\mathbb{P}(D > i) \to 0$, *and* $p_i \to 0$ *implies long tail, i.e.* $\mathbb{P}(D > i + n)/\mathbb{P}(D > i) \to 1$ *as* $i \to \infty$, *where* $n \geq \ell$.

(v) *In the homogeneous case* $p_i \equiv p \in (0, 1)$, $\mathbb{P}(D > n) \sim c_1 z^{-n}$, *where* z *solves* $(1 - p)z \sum_{i=0}^{\ell-1} (pz)^i = 1$.

(vi) *Short tail: if* $p_i = \exp(-ai^{-b})$ *then* $\mathbb{P}(D > n) \approx_{\log} \exp(-bn \log n/\ell)$.

(vii) *Long tail: If* $p_i = i^{-b}, b \in (0, 1/\ell)$ *then* $\mathbb{P}(D > n) \approx_{\log} \exp\{-n^{1-b\ell}/(1 - b\ell)\}$.

Proof. The arguments are basically an easy adaptation of the ones for the inhomogeneous Poisson case to discrete time setting. In fact, some steps are even simpler and in particular, Slivnyak's formula is replaced by elementary conditioning arguments. Thus we only provide some crucial steps.

(i): Define $B_n = \{\xi_{n-1} = 0, \xi_n = \cdots = \xi_{n+\ell-1} = 1\}$ as the event that a run starts at n. Observe that $D = n + 1$ if and only if B_{n+1} occurs and there is no sequence of

ℓ ones in $1,\dots,n-1$. The latter event is independent of $\xi_n,\xi_{n+1},\dots$ and hence of B_{n+1}. Moreover, it coincides with $D>n-\ell$, which concludes the proof of (i).

(ii): Let $E_n=\cup_{\ell n+1}^{\ell(n+1)}B_i$ be the event that there is a zero in $\ell n,\dots,\ell(n+1)-1$ followed by ℓ ones. Since such a zero is unique, we have $\mathbb{P}(E_n)=\sum_{\ell n+1}^{\ell(n+1)}\mathbb{P}(B_i)$. Noting that $I=\sum_1^\infty\mathbb{P}(B_i)$ and that E_n,E_m are independent if $|m-n|>1$, arguments similar as for Theorem 3.1 give the first part. For the second, note that I is also the total expected number of runs starting at $i>1$, and use arguments similar to the ones based on (3.1).

(iii): Considering the special $p_i=i^{-h/\ell}$ leads to $I=\infty$ for $h<1$ and $I<\infty$ for $h>1$. The result then follows by a similar coupling argument as in the proof of Proposition 3.2.

(iv): The short tail part is just as for Proposition 4.1. The long tail one is even simpler since $\mathbb{P}(D>i+n)\ge\mathbb{P}(D>i)q_{i+\ell}\cdots q_{i+n}$.

(v): Shown in [11, Ch. XIII.7] by noting that the probability generating function of D is rational and performing fractional expansions. Alternatively, the renewal equation approach of Section 5.1 applies; not surprisingly, the equation determining z in [11, Ch. XIII.7] is simply the discrete version of (5.1).

(vi): Use a bounding argument similar to that in Proposition 5.2, and note that the lower bound on $\mathbb{P}(D>n)$ is now obtained by placing zeros at each $i\ell$ where $i=1,\dots\lceil n/\ell\rceil$.

(vii): Here (iv) implies $\mathbb{P}(D>n-\ell)\sim\mathbb{P}(D>n)$ and so by (i)

$$\frac{\mathbb{P}(D=n+1)}{\mathbb{P}(D>n)}\sim\prod_{j=n+1}^{n+\ell}\frac{1}{j^b}\sim\frac{1}{n^{b\ell}}\,.$$

Therefore for $n>N$

$$\begin{aligned}\frac{\mathbb{P}(D>n)}{\mathbb{P}(D>N)}&=\prod_{k=N}^{n-1}\frac{\mathbb{P}(D>k+1)}{\mathbb{P}(D>k)}=\prod_{k=N}^{n-1}\Bigl(1-\frac{\mathbb{P}(D=k+1)}{\mathbb{P}(D>k)}\Bigr)\\&=\exp\Bigl\{-(1+r'_{n,N})\sum_{k=N}^{n-1}\frac{1}{k^{b\ell}}\Bigr\}=\exp\Bigl\{-(1+r'_{n,N})\frac{n^{1-b\ell}-N^{1-b\ell}}{1-b\ell}\Bigr\},\end{aligned}$$

where $r'_{n,N}\to0$ as $N\to\infty$ unformly in $n>N$. From this the result follows. □

Remark 7.2. To link the continuous and discrete setups one may think of the correspondence $p_i\leftrightarrow\mathrm{e}^{-\mu(i)}$, where the latter is roughly the probability of no failures in $(i-1,i]$. That is, we partition the real line and group failures together. Interestingly, the only substantial difference in the results appears in the long tail asymptotics (compare (i) in Corollary 5.7 to (vii) in Corollary 7.1). In this regard note that a run of ℓ ones in the discretized framework implies a gap of size ℓ, but the opposite is not always true. This discrepancy becomes more pronounced when the rate of failures increases. This intuitively explains the fact that $\mathbb{P}(D>n)$ decays faster in the continuous setup.

If $\mu_i\to0$, the correspondance $p_i\leftrightarrow\mathrm{e}^{-\mu(i)}$ is equivalent to $q_i\leftrightarrow\mu(i)$ where $q_i,\mu(i)$ can be interpreted as the rate of separators of runs (gaps). □

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