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Abstract

The Markov renewal equation

$$Z_i(x) = z_i(x) + \sum_{j \in \mathcal{E}} \int_0^x Z_j(x-y) F_{ij}(dy), \quad i \in \mathcal{E},$$

is considered in the subcritical case where the matrix of total masses of the F_{ij} has spectral radius strictly less than one, and the asymptotics of the $Z_i(x)$ is found in the heavy-tailed case involving a local subexponential assumption on the F_{ij} . Three cases occur according to the balance between the $z_i(x)$ and the tails of the F_{ij} . A crucial step in the analysis is obtaining multivariate and local versions of a lemma due to Kesten on domination of subexponential tails. These also lead to various results on tail asymptotics of sums of a random number of heavy-tailed random variables in models which are more general than in the literature.

1 Introduction

The occurrence of heavy tails has been argued repeatedly in a variety of application areas covering insurance and finance ([1]), telecommunications and internet traffic ([2]), optics ([3]), cell proliferation ([4]) and many more; a broad overview is in [5]. Correspondingly, performance analysis of models for such situations has triggered a vast literature on probabilistic features of heavy tails.

This paper deals with two particular problems in this area. The first is asymptotics of renewal-type equations of the form

$$Z_i(x) = z_i(x) + \sum_{j \in \mathcal{E}} \int_0^x Z_j(x-y) F_{ij}(dy), \quad i \in \mathcal{E}, \quad (1.1)$$

where \mathcal{E} is a finite index set, $(Z_i)_{i \in \mathcal{E}}$ a set of unknown functions defined on $[0, \infty)$, $(z_i)_{i \in \mathcal{E}}$ a set of non-negative known functions, and $(F_{ij})_{i,j \in \mathcal{E}}$ a set of non-negative heavy-tailed measures on $[0, \infty)$. The second is the tail behaviour of a random sum

$$S = \sum_{i=1}^N X_i \quad (1.2)$$

where $N \in \mathbb{N}$ is a light-tailed r.v. and the X_i are non-negative heavy-tailed r.v.'s, such that certain types of dependence to be specified later may occur.

The simple renewal equation

$$Z(x) = z(x) + \int_0^x Z(x-y)F(dy) \quad (1.3)$$

is a classical structure in applied probability and occurs for example in branching processes ([6]), ruin problems ([7]) and ergodicity problems for possibly non-Markovian processes ([8, VI.1]). The emphasis is usually on asymptotic properties of $Z(x)$ as $x \rightarrow \infty$ where the simplest situation is existence of a limit¹ when F is a probability measure, i.e. when $\|F\| = 1$ where $\|F\|$ is the total mass of F . However, in the branching process example $\|F\|$ is the expected number of children of an individual so obviously the case $\|F\| \neq 1$ is of interest. One then typically has an exponential order $e^{\gamma x}$ of $Z(x)$. Here $\gamma > 0$ when $\|F\| > 1$ and $\gamma < 0$ when $\|F\| < 1$ and F is light-tailed (this last situation also occurs in Cramér-Lundberg asymptotics for ruin probabilities and queues, cf. [8, V.7]). For $\|F\| < 1$ and F heavy-tailed, the order depends on a delicate balance between the tails of F and z and in fact the results are more recent, [9] (see also [10] for a closely related results).

The system (1.1) goes under the name of the *Markov renewal equation*, cf. [8]; this terminology stems from the case of

$$\mathbf{P} = (\|F_{ij}\|)_{i,j \in \mathcal{E}}$$

being stochastic, i.e. the transition matrix of a Markov chain ξ_0, ξ_1, \dots . In branching processes, it occurs when individuals have several types, and in insurance and finance, it relates to regime-switching; another relevant example comes from computer reliability problems, [11]. The known asymptotic results on the $Z_i(x)$ depend crucially on the spectral radius ρ of \mathbf{P} : if $\rho = 1$, in particular if \mathbf{P} is stochastic, limits exists whereas otherwise the order is exponential, $e^{\gamma x}$, where $\gamma > 0$ when $\rho > 1$ and $\gamma < 0$ when $\rho < 1$ and the F_{ij} are light-tailed [again, regularity conditions are required]. The gap is the case $\rho < 1$ and heavy tails. The main contribution of the paper in the setting of (1.1) is to fill this gap. The result is stated as Theorem 2.2 below. Three cases occur depending on whether the tail of the z_i or the F_{ij} dominate, or if they are of same order. The conditions involve the non-standard concept of local subexponentiality.

One main example of random sums like (1.2) is total claims distributions in insurance: N is the number of claims in a given period and X_1, X_2, \dots the claim sizes which classically are taken i.i.d. and independent of N which is assumed light-tailed (for example, the negative binomial distribution is popular because of its interpretation as a gamma mixture of Poissons). In finance, S could be the payoff of a portfolio consisting of assets with values X_1, X_2, \dots . With light-tailed X_i , $\mathbb{P}(S > x)$ decays roughly exponential, with heavy tails the asymptotic form is $\mathbb{E}N \cdot \mathbb{P}(X_1 > x)$. Our main result in that direction, Theorem 2.3 below, is an extension to sums of the form $\sum_{k=1}^d \sum_1^{N_k} X_{i,k}$ where N_1, \dots, N_d are dependent and the distribution of

¹This and the following statements require regularity conditions which we omit; see the citations given

$X_{i,k}$ depends on i . For example in the insurance setting, (N_1, \dots, N_d) could be conditionally independent given (τ_1, \dots, τ_d) with Poisson rate λ_k for N_k , where τ_k is the time spent by some environmental process in state k in the period $[0, T]$ (dependence occurs because $\tau_1 + \dots + \tau_d = T$). The main step in the proof of Theorem 2.3 is a version of a classical lemma due to Kesten on tail domination of subexponential sums; for the proof of Theorem 2.2 we also need a local version of this.

The paper is organised as follows. Section 2 starts with some necessary background, in particular on local subexponentiality, and proceeds to state the two main results of the paper, Theorems 2.2 and 2.3 referred to above. The rest of the paper is then proofs supplemented with miscellaneous results of some independent interest. Sections 3–5 give the random sum part in various settings. This together with a number of additional steps then allows to conclude the proof of Theorem 2.2 in Section 6. This proceeds by first assuming \mathbf{P} to be substochastic, thereby allowing Markov chain interpretations, and finally reduce the general case to this by a Perron-Frobenius type transformation.

2 Preliminaries and statement of main results

For a distribution F on \mathbb{R}^+ , let $F(x) = F(0, x]$ be the distribution function (c.d.f.) and $\bar{F}(x) = 1 - F(x) = F(x, \infty)$ the tail. If F and G are distributions and $X \sim F$ and $Y \sim G$ independent random variables, then $F * G$ denotes the convolution of F and G ,

$$F * G(x) = \int_0^x G(x - y)F(dy) = \int_0^x F(x - y)G(dy),$$

which is the distribution of the sum $X + Y$.

We next briefly mention the most standard definitions and relations in the heavy-tailed area. For a more detailed and thorough treatment, see [1], [12]. A distribution F is said to be heavy-tailed if F does not possess any positive exponential moments, that is $\int_{-\infty}^{\infty} \exp(\lambda x)F(dx) = \infty$ for all $\lambda > 0$. It is long-tailed, written $F \in \mathcal{L}$, if it has unbounded support and $\bar{F}(x + y)/\bar{F}(x) \rightarrow 1$ as $x \rightarrow \infty$ for any fixed $y \in \mathbb{R}$. Any $F \in \mathcal{L}$ is especially heavy-tailed. F is said to be subexponential if $\bar{F}^{*n}(x)/\bar{F}(x) \rightarrow n$ for all n (actually, it is sufficient that this holds for $n = 2$). The intuition is that the only significant way the sum of independent F -distributed random variables can exceed some large threshold x is if the maximum of the random variables exceeds that x . This is also known as the principle of a single big jump. Similarly, a density f is long-tailed if $f(x) > 0$ for all sufficiently large x and $f(x + t)/f(x) \rightarrow 1$ for any fixed t . If F and G have densities f, g , respectively, the convolution $F * G$ then has density $f * g$ given by $f * g(x) = \int_0^x f(x - y)g(y)dy$. The density f of F is said to be subexponential, written $F \in \mathcal{S}_{ac}$, if f is long-tailed and $f^{*2}(x) = \int_0^x f(x - y)f(y)dy \sim 2f(x)$ as $x \rightarrow \infty$. Note that \mathcal{S}_{ac} is a subclass of \mathcal{S} . Also, since subexponentiality is a tail property it is actually sufficient for a distribution F to only have a density $f(x)$ for sufficiently large x .

We proceed to the less standard concepts of local long-tailedness and subexponentiality, as introduced in [9]. The local property can be viewed as intermediate

between being long-tailed (subexponential) and having a long-tailed (subexponential) density. First, we need to introduce some notation. For a fixed $T > 0$ define $\Delta = \Delta_T = (0, T]$ and let $x + \Delta = \{x + y \mid y \in \Delta\} = (x, x + T]$.

Definition 2.1. A distribution F is said to be Δ -long-tailed, written $F \in \mathcal{L}_\Delta$, if $F(x + \Delta) > 0$ for all sufficiently large x and $F(x + y + \Delta)/F(x + \Delta) \rightarrow 1$ as $x \rightarrow \infty$ for any fixed y . It is Δ -subexponential, written $F \in \mathcal{S}_\Delta$, if $F \in \mathcal{L}_\Delta$ and $F^{*2}(x + \Delta)/F(x + \Delta) \rightarrow 2$.

Notice that if we allow $T = \infty$ then the class \mathcal{L}_Δ corresponds to the ordinary longtailed distributions \mathcal{L} ; if $T < \infty$, then $\mathcal{L}_\Delta \subset \mathcal{L}$. Similarly, for any finite T the class $\mathcal{S}_\Delta \subset \mathcal{S}$, and if we allow $T = \infty$ the two classes coincides. It appears that Δ -subexponential distributions possess many similar properties as the ordinary subexponential distributions. A main and crucial difference is that the tail function $\bar{F}(x)$ is monotone whereas $F(x + \Delta)$ may not be. Also, it is worth noticing that a distribution F with a subexponential density f is Δ -subexponential for any $T > 0$.

We are now ready to state our main results. Let \mathbf{P} be the matrix with ij th element $p_{ij} = \|F_{ij}\|$.

Theorem 2.2. Consider the Markov renewal equation (1.1). Assume that \mathbf{P} is irreducible with $\text{spr}(\mathbf{P}) < 1$ and that $F_{ij}(x + \Delta) \sim p_{ij}c_{ij}G(x + \Delta)$ where $c_{ij} > 0$ and G is a Δ -subexponential distribution function for all $T > 0$. Let $g(x) = G(x, x + 1)$, $I_j = \int_0^\infty z_j(y) dy$, and let d_{ij} be the ij th element of the matrix $(\mathbf{I} - \mathbf{P})^{-1}\mathbf{M}(\mathbf{I} - \mathbf{P})^{-1}$ with $\mathbf{M} = (p_{k\ell}c_{k\ell})_{k,\ell \in \mathcal{E}}$, k_{ij} the ij th element of $\sum_0^\infty \mathbf{P}^n = (\mathbf{I} - \mathbf{P})^{-1}$. Then three cases occur:

i) Assume that z_j is directly Riemann integrable and $z_j(x)/g(x) \rightarrow 0$ for all $j \in \mathcal{E}$. Then

$$Z_i(x) \sim \sum_{j \in \mathcal{E}} I_j d_{ij} g(x).$$

ii) Assume that z_j is directly Riemann integrable and $z_j(x)/g(x) \rightarrow a_j$ where $a_j > 0$ for at least one $j \in \mathcal{E}$. Then

$$Z_i(x) \sim \sum_{j \in \mathcal{E}} (I_j d_{ij} + k_{ij} a_j) g(x).$$

iii) Assume that $z_j(y)/I_j$ has a subexponential density and that $z_j(x)/g(x) \rightarrow \infty$ for all j . Then

$$Z_i(x) \sim \sum_{j \in \mathcal{E}} k_{ij} z_j(x).$$

Theorem 2.3. Let N_1, \dots, N_d be non-negative integer-valued r.v.'s and

$$S = \sum_{i=1}^d \sum_{j=1}^{N_i} X_{ij}$$

where the X_{ij} are independent given N_1, \dots, N_d with distribution F_i of X_{ij} . Assume $\overline{F}_i(x) \sim c_i \overline{F}(x)$ for some $F \in \mathcal{S}$ and some $c_1, \dots, c_d \geq 0$ and that there are $\varepsilon_i > 0$ such that $\mathbb{E}[(1 + \varepsilon_i)^{N_i}] < \infty$ for all $i = 1, \dots, d$. Then

$$\mathbb{P}(S > x) \sim (c_1 \mathbb{E}[N_1] + \dots + c_d \mathbb{E}[N_d]) \overline{F}(x).$$

Note that N_1, \dots, N_d are not assumed independent. A local version of Theorem 2.3 is given below as Theorem 4.3.

3 Random sums of subexponentials

A classical bound due to Kesten ([13] pp. 148–9, [1] p. 41) states that for a subexponential distribution F_0 and $\varepsilon > 0$ there exists a $D_0 = D_0(\varepsilon) > 0$

$$\overline{F_0^{*n}}(x) \leq D_0(1 + \varepsilon)^n \overline{F_0}(x) \quad \text{for any } x \geq 0 \text{ and } n \in \mathbb{N}. \quad (3.1)$$

The following result is a similar result involving convolution of the tails of multiple subexponential distribution functions.

Proposition 3.1. *Let $F \in \mathcal{S}$, let $c_1, \dots, c_d \geq 0$ and let F_1, \dots, F_d be distributions with $\overline{F}_i(x) \sim c_i \overline{F}(x)$. Then for every $\varepsilon > 0$ there exists a $D = D(\varepsilon)$ such that the following inequality holds*

$$\overline{F_1^{*n_1} * \dots * F_d^{*n_d}}(x) \leq D(1 + \varepsilon)^n \overline{F}(x) \quad \text{for all } x \geq 0 \text{ and } n_1, \dots, n_d \in \mathbb{N} \quad (3.2)$$

where $n = n_1 + \dots + n_d$. Furthermore

$$\overline{F_1^{*n_1} * \dots * F_d^{*n_d}}(x) \sim (c_1 n_1 + \dots + c_d n_d) \overline{F}(x). \quad (3.3)$$

Proof. Define the distribution function F_0 by having tail

$$\overline{F_0}(x) = \sup_{i=1, \dots, d} \{\overline{F}_i(x)\} \sim \sup_{i=1, \dots, d} \{c_i \overline{F}(x)\} = \sup_{i=1, \dots, d} \{c_i\} \overline{F}(x).$$

Notice that taking the supremum will preserve properties as being cadlag, decreasing, and having values between $[0, 1]$. Since the tail of F_0 is the supremum of the tails of some c.d.f.'s, F_0 is indeed a c.d.f. itself. Let $k = \sup_{i=1, \dots, d} c_i$. The tail of F_0 is then asymptotically of the form $k \overline{F}(x)$, and since \mathcal{S} is closed under tail equivalence we must have $F_0 \in \mathcal{S}$. From the classical Kesten bound it follows that for every $\varepsilon > 0$ there is a $D_0 = D_0(\varepsilon)$ such that (3.1) holds. Now for $i = 0, 1, \dots, d$ let

$$X_{i,1}, \dots, X_{i,n_i} \sim F_i \quad \text{where } n_0 = n$$

Then by construction we have the stochastic ordering

$$X_{i,j} \leq_{st} X_{0,k} \quad \text{for all } i = 1, \dots, d, j = 1, \dots, n_i, k = 1, \dots, n,$$

which implies that the sums also have the corresponding stochastic ordering

$$X_{1,1} + \dots + X_{1,n_1} + \dots + X_{d,1} + \dots + X_{d,n_d} \leq_{st} X_{0,1} + \dots + X_{0,n}.$$

Written in terms of convolutions of the tails of the distribution functions this means that

$$\overline{F_1^{*n_1} * \dots * F_d^{*n_d}}(x) \leq \overline{F_0^{*n}}(x).$$

By (3.1) it is therefore sufficient to show that there exists a $D = D(\varepsilon)$ such that

$$D_0 \overline{F_0}(x) \leq D \overline{F}(x).$$

From the definition of F_0 and $\overline{F}_i(x) \sim c_i \overline{F}(x)$ it follows that whenever $K > \max_{1, \dots, d} \{c_i\}$, there exists a $x_0 \geq 0$ such that $\overline{F_0}(x)/\overline{F}(x) \leq K$ for all $x > x_0$. Replacing K by a larger K if necessary, we may assume this holds for all x . Thus

$$\begin{aligned} D_0 \overline{F_0}(x) &= D_0 \frac{\overline{F_0}(x)}{\overline{F}(x)} \overline{F}(x) \leq D_0 \sup_x \left\{ \frac{\overline{F_0}(x)}{\overline{F}(x)} \right\} \overline{F}(x) \\ &\leq D_0 \left(\sup_{x \leq x_0} \left\{ \frac{\overline{F_0}(x)}{\overline{F}(x)} \right\} + \sup_{x > x_0} \left\{ \frac{\overline{F_0}(x)}{\overline{F}(x)} \right\} \right) \overline{F}(x) \\ &\leq D_0 (K + 2 \max_{i=1, \dots, d} \{c_i\}) \overline{F}(x) \equiv D \overline{F}(x) \end{aligned}$$

with $D = D_0(K + 2 \max_{i=1, \dots, d} \{c_i\})$. The asymptotics in (3.3) now easily follows. To this end, just notice that $\overline{F_i^{*n_i}}(x) \sim n_i c_i \overline{F}(x)$ for all i by standard subexponential theory and proceed by induction, using that $\overline{G_1 * G_2}(x) \sim (\gamma_1 + \gamma_2) \overline{F}(x)$ if $\overline{G_i}(x) \sim \gamma_i \overline{F}(x)$ for $i = 1, 2$ and $F \in \mathcal{S}$. \square

Kesten's bounds is commonly used as majorant in dominated convergence to find the asymptotics of a randomly stopped sum of i.i.d. subexponential random variables. Proposition 3.1 can be used correspondingly for the multidimensional case which we use to give the proof of Theorem 2

Proof. The law of total probability yields

$$\begin{aligned} \frac{\mathbb{P}(S > x)}{\overline{F}(x)} &= \sum_{n_1, n_2, \dots, n_d=1}^{\infty} \mathbb{P}(N_1 = n_1, \dots, N_d = n_d) \frac{\overline{F_1^{*n_1} * \dots * F_d^{*n_d}}(x)}{\overline{F}(x)} \\ &\rightarrow \sum_{n_1, n_2, \dots, n_d=1}^{\infty} \mathbb{P}(N_1 = n_1, \dots, N_d = n_d) (c_1 n_1 + \dots + c_d n_d) \\ &= c_1 \mathbb{E}[N_1] + \dots + c_d \mathbb{E}[N_d] \end{aligned}$$

using Proposition 3.1 and dominated convergence; this is justified since Hölder's inequality implies that $\mathbb{E}[(1 + \varepsilon)^{N_1 + \dots + N_d}] < \infty$ if we take (say) $1 + \varepsilon = \min\{1 + \varepsilon_1, \dots, 1 + \varepsilon_d\}^{1/d}$. \square

4 Random sums of local subexponentials

The objective of the following Lemma is to obtain an upper bound for convolution of local subexponential distribution functions, which is needed to expand the local version of Kesten's bounds stated as Proposition 4 in [9] to include the convolution of several local subexponential distribution functions.

Lemma 4.1. *Let $H \in \mathcal{S}_\Delta$. Assume that $G_i, i = 1, 2$ are distributions satisfying*

$$G_i(x + \Delta) \leq b_i H(x + \Delta) \quad \text{for all } x \geq x_0$$

for some $b_i \geq 1$ and x_0 . Then for some constant a depending only on H , it holds that

$$G_1 * G_2(x + \Delta) \leq ab_1 b_2 H(x + \Delta) \quad \text{for all } x \geq x_0. \quad (4.1)$$

Proof. Define

$$a_1 = \sup_{u, v \geq x_0 : |u-v| \leq 1} \frac{H(u + \Delta)}{H(v + \Delta)}$$

and notice that a_1 is finite for sufficiently large x_0 since H is Δ -subexponential, and therefore especially Δ -longtailed. We have the representation

$$\begin{aligned} G_1 * G_2(x + \Delta) &= \int_0^x G_2(x - y + \Delta) G_1(dy) + \int_x^{x+T} G_2(0, x + T - y] G_1(dy) \\ &\equiv P_1(x) + P_2(x). \end{aligned}$$

Consider the first term. Let k be the smallest integer such that $x/k \leq 1$ and $1/k \leq T$. Split interval $(0, x]$ into k disjoint equally sized parts. For $x \geq x_0$ the first term can then be written as a sum and assessed as follows

$$\begin{aligned} P_1(x) &= \sum_{i=0}^{k-1} \int_{xi/k}^{x(i+1)/k} G_2(x - y + \Delta) G_1(dy) \\ &\leq b_2 \sum_{i=0}^{k-1} \int_{xi/k}^{x(i+1)/k} H(x - y + \Delta) G_1(dy) \\ &\leq a_1 b_2 \sum_{i=0}^{k-1} \int_{xi/k}^{x(i+1)/k} H(x - xi/k + \Delta) G_1(dy) \\ &\leq a_1 b_2 \sum_{i=0}^{k-1} H(x - xi/k + \Delta) G_1(xi/k, x(i+1)/k) \\ &\leq a_1 b_1 b_2 \sum_{i=0}^{k-1} H(x - xi/k + \Delta) H(xi/k + \Delta) \end{aligned}$$

where the summands can be evaluated backwards

$$\begin{aligned} \int_{xi/k}^{x(i+1)/k} H(x - y + \Delta) H(dy) &\geq \frac{1}{a_1} \int_{xi/k}^{x(i+1)/k} H(x - ix/k + \Delta) H(dy) \\ &= \frac{1}{a_1} H(x - ix/k + \Delta) H(xi/k + \Delta). \end{aligned}$$

Inserting this upper bound for the summands yields the inequality

$$\begin{aligned} P_1(x) &\leq a_1^2 b_1 b_2 \sum_{i=0}^{k-1} \int_{xi/k}^{x(i+1)/k} H(x - y + \Delta) H(dy) \\ &= a_1^2 b_1 b_2 \int_0^x H(x - y + \Delta) H(dy) \\ &\leq a_1^2 b_1 b_2 H^{*2}(x + \Delta). \end{aligned}$$

Since H is Δ -subexponential there must be a finite δ such that

$$H^{*2}(x + \Delta) \leq (2 + \delta)H(x + \Delta) \quad \text{for all } x \geq x_0$$

if x_0 is sufficiently large. This provides the final inequality for the first term

$$P_1(x) \leq (2 + \delta)a_1^2 b_1 b_2 H(x + \Delta).$$

Now, consider the second term. It follows directly that

$$P_2(x) \leq \int_x^{x+T} G_1(dy) \leq b_1 H(x + \Delta) \leq b_1 b_2 H(x + \Delta).$$

Altogether we now have

$$G_1 * G_2(x + \Delta) \leq ((2 + \delta)a_1^2 + 1)b_1 b_2 H(x + \Delta)$$

so we can take $a = (2 + \delta)a_1^2 + 1$. □

We can use this to obtain a local version of Proposition 3.1.

Proposition 4.2. *Let $F \in \mathcal{S}_\Delta$ for some Δ , and $F_i(x + \Delta) \sim c_i F(x + \Delta)$ for $c_1, \dots, c_d \geq 0$. Then for every $\varepsilon > 0$ there exists a $D = D(\varepsilon)$ and a $x_0 = x_0(\varepsilon)$ such that the following inequality holds*

$$F_1^{*n_1} * \dots * F_d^{*n_d}(x + \Delta) \leq D(1 + \varepsilon)^n F(x + \Delta) \quad (4.2)$$

for all $x \geq x_0$ and $n_1, \dots, n_d \in \mathbb{N}$ where $n = n_1 + \dots + n_d$. Furthermore

$$F_1^{*n_1} * \dots * F_d^{*n_d}(x + \Delta) \sim (c_1 n_1 + \dots + c_d n_d) F(x + \Delta).$$

Proof. Let $\varepsilon > 0$ be given. Notice that from Proposition 1 it follows that for $i = 1, \dots, d$ there is a $V_i = V_i(\varepsilon)$ such that

$$F_i^{*n_i}(x + \Delta) \leq V_i(1 + \varepsilon)^{n_i} F(x + \Delta).$$

(4.2) can then be proven by induction using Lemma 4.1. For $d = 2$ it follows directly from the Lemma that there is an a such that

$$F_1^{*n_1}(x + \Delta) * F_2^{*n_2}(x + \Delta) \leq aV_1V_2(1 + \varepsilon)^{n_1+n_2} F(x + \Delta).$$

Letting $D = aV_1V_2$ we have the desired inequality. Assume that (4.2) holds for d distribution functions with constant D . Now consider the case with $d+1$ distribution functions, then we have

$$\begin{aligned} F_1^{*n_1} * \dots * F_{d+1}^{*n_{d+1}}(x + \Delta) &= (F_1^{*n_1} * \dots * F_d^{*n_d}) * F_{d+1}^{*n_{d+1}}(x + \Delta) \\ &\leq aDV_{d+1}(1 + \varepsilon)^{n_1+\dots+n_d}(1 + \varepsilon)^{n_{d+1}} F(x + \Delta) \\ &= aDV_{d+1}(1 + \varepsilon)^{n_1+\dots+n_{d+1}} F(x + \Delta). \end{aligned}$$

From Corollary 2 in [9] it follows that $F_i^{*n_i}(x + \Delta) \sim n_i c_i F(x + \Delta)$ for $i = 1, \dots, d$, and using Proposition 3 in [9] one can further deduce that

$$F_1^{*n_1} * \dots * F_d^{*n_d}(x + \Delta) \sim (n_1 c_1 + \dots + n_d c_d) F(x + \Delta). \quad \square$$

Theorem 4.3. *Assume in addition to the assumptions of Theorem 2.3 that $F \in \mathcal{S}_\Delta$ for some Δ and $F_i(x + \Delta) \sim c_i F(x + \Delta)$ for $c_1, \dots, c_d \geq 0$. Then*

$$F^{*N_1} * \dots * F^{*N_d}(x + \Delta) \sim (c_1 \mathbb{E}[N_1] + \dots + c_d \mathbb{E}[N_d]) F(x + \Delta).$$

Proof. Use dominated convergence justified by Proposition 4.2. □

5 Random sums with subexponential densities

In [9] it is shown that when $F \in \mathcal{S}_{ac}$ then, for any $\varepsilon > 0$, there exist $x_0 = x_0(\varepsilon)$ and $V = V(\varepsilon)$ such that

$$f^{*n}(x) \leq V(1 + \varepsilon)^n f(x) \quad \text{for any } x \geq x_0 \text{ and } n \in \mathbb{N}.$$

We now seek to obtain a version of Theorem 2.3 involving densities instead. To do this we need an upper bound for convolutions, as for the local subexponential case.

Lemma 5.1. *Let $F \in \mathcal{S}_{ac}$. Let f_1 and f_2 be densities with the properties*

$$f_i(x) \leq b_i f(x) \quad \text{with } b_i > 1 \text{ for } i = 1, 2 \text{ and all } x \geq x_0.$$

Then there is a constant A independent of f_1 and f_2 such that

$$f_1 * f_2(x) \leq A b_1 b_2 f(x) \quad \text{for all } x \geq x_0.$$

Proof. Since f is longtailed then $f(x) > 0$ for all $x \geq x_0$ for x_0 sufficiently large, hence

$$a \equiv \sup_{y \in (0, x-x_0)} \frac{f(x-y)}{f(x)}$$

is finite. Consider the partition

$$\begin{aligned} f_1 * f_2(x) &= \int_0^x f_1(x-y)f_2(y)dy \\ &= \int_0^{x-x_0} f_1(x-y)f_2(y)dy + \int_{x-x_0}^{x_0} f_1(x-y)f_2(y)dy + \int_{x_0}^x f_1(x-y)f_2(dy) \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Now each term is assessed individually, starting with the first for $x \geq x_0$,

$$I_1 \leq b_1 \int_0^{x-x_0} f(x-y)F_2(dy) \leq ab_1 \int_0^{x-x_0} f(x)F_2(dy) \leq ab_1 b_2 f(x).$$

Analogously for the third term,

$$I_3 = \int_0^{x-x_0} f_2(x-y)f_1(y)dy \leq ab_1 b_2 f(x).$$

Only the second term is left to be evaluated. Now let $x \geq 2x_0$, then it follows

$$I_2 = b_1 b_2 \int_{x-x_0}^{x_0} f(x-y)f(y)dy \leq b_1 b_2 \int_0^x f(x-y)f(y)dy = b_1 b_2 f^{*2}(x).$$

Since f is a subexponential density, if x_0 is sufficiently large there must be a $\delta > 0$ such that

$$f^{*2}(x) \geq (2 + \delta)f(x) \quad \text{for all } x \geq x_0.$$

Hence,

$$I_2 \leq b_1 b_2 (2 + \delta)f(x).$$

To conclude the proof, let $A = 2a + 2 + \delta$. □

Theorem 5.2. *Let $F \in \mathcal{S}_{ac}$ and f_1, \dots, f_d be densities with $f_i(x) \sim c_i f(x)$ for some $c_i \geq 0$, $i = 1, \dots, d$. Then for any $\varepsilon > 0$ there is a $D = D(\varepsilon)$ and $x_0 = x_0(\varepsilon)$ such that*

$$f_1^{*n_1} * \dots * f_d^{*n_d}(x) \leq D(1 + \varepsilon)f(x)$$

for any $x \geq x_0$ and $n_1, \dots, n_d \in \mathbb{N}$. Furthermore

$$f_1^{*n_1} * \dots * f_d^{*n_d}(x) \sim (n_1 c_1 + \dots + n_d c_d)f(x).$$

Proof. Analogous to Proposition 4.2. □

6 Proof of Theorem 2.2

Recall that $\mathbf{P} = (p_{ij})_{i,j \in \mathcal{E}}$ where the entries are defined by $p_{ij} = \|F_{ij}\|$. Let $\lambda = \text{spr}(\mathbf{P})$ be the Perron-Frobenius root of \mathbf{P} and \mathbf{v} the corresponding right eigenvector. Hence, $\mathbf{P}^n \mathbf{v} = \lambda^n \mathbf{v}$, which by strict positivity of \mathbf{v} implies that the n 'th element $p_{ij}^{(n)}$ of \mathbf{P}^n will decay at rate λ^n . We will first treat the case where in addition to $\lambda < 1$ the matrix \mathbf{P} is substochastic. This means that the row sums satisfy $\sum_j p_{ij} \leq 1$, with strict inequality for at least one i .

Introduce an irreducible absorbing Markov chain $(\xi_n)_{n \in \mathbb{N}}$ with state space $\mathcal{E} \cup \{\dagger\}$ where \dagger is the coffin state. Let $\mathbf{Q} = (q_{ij})_{i,j \in \mathcal{E} \cup \{\dagger\}}$ be the transition matrix with $q_{ij} = p_{ij}$ for $i, j \in \mathcal{E}$, $q_{\dagger i} = 0$, $q_{\dagger \dagger} = 1$, and $q_{i\dagger} = 1 - \sum_{j \in \mathcal{E}} p_{ij}$. Recall that there must be at least one index i such that $q_{i\dagger} > 0$. Then $N = \inf\{n \geq 0 : \xi_n = \dagger\}$ will be the time until absorption.

Let T_n denote the waiting time between jump n and $n + 1$. Then T_0, T_1, \dots are conditionally independent given $\mathcal{F} = \sigma(\xi_0, \xi_1, \dots)$ with conditional distribution function satisfying

$$G_{ij}(t) = \mathbb{P}(T_n \leq t \mid \mathcal{F}) = \mathbb{P}(T_n \leq t \mid \xi_n, \xi_{n+1})$$

on $\{\xi_n = i, \xi_{n+1} = j\}$, and we have

$$F_{ij}(t) = \mathbb{P}(\xi_{n+1} = j, T_n \leq t \mid \xi_n = i) = q_{ij} G_{ij}(t).$$

Henceforth we will only consider $i, j \neq \dagger$. Thus, $F_{ij}(t) = p_{ij} G_{ij}(t)$.

The solution of the Markov renewal equation (1.1) is given by Proposition 4.4 in [8] as

$$Z_i(x) = \sum_{j \in \mathcal{E}} Z_{ij}(x) \quad \text{where } Z_{ij}(x) = \int_0^x z_j(x-y) U_{ij}(dy) \quad (6.1)$$

with Markov renewal kernel U_{ij} being the expected number of returns to state $j \in \mathcal{E}$ before time t given that the Markov chain starts in state $i \in \mathcal{E}$. That is,

$$U_{ij}(t) = \sum_{n=0}^{\infty} (\mathbf{F}^{*n})_{ij}(t) = \sum_{n=0}^{\infty} \mathbb{P}_i(\xi_n = j, S_{n-1} \leq t),$$

where $S_{n-1} = T_0 + \dots + T_{n-1}$. First, it is necessary to consider the local asymptotics of U_{ij} .

Lemma 6.1. *Under the assumptions of Theorem 2.2,*

$$U_{ij}(t + \Delta) \sim d_{ij}G(t + \Delta). \quad (6.2)$$

Proof. Notice that the n 'th convolution of the semi-Markov kernel also can be defined locally

$$(\mathbf{F}^{*n})_{ij}(t + \Delta) = \mathbb{P}_i(\xi_n = j, S_{n-1} \in t + \Delta).$$

Rewriting the right-hand side using conditional expectations yields

$$\begin{aligned} \mathbb{P}_i(\xi_n = j, S_{n-1} \in t + \Delta) &= \mathbb{P}_i(\xi_n = j, T_0 + \dots + T_{n-1} \in t + \Delta) \\ &= p_{ij}^{(n)} \mathbb{P}(T_0 + \dots + T_{n-1} \in t + \Delta \mid \xi_0 = i, \xi_n = j) \\ &= p_{ij}^{(n)} \mathbb{E} \left[\mathbb{E}[\mathbb{P}(T_0 + \dots + T_{n-1} \in t + \Delta \mid \xi_0 = i, \xi_n = j) \mid \xi_1, \dots, \xi_{n-1}] \right] \\ &= p_{ij}^{(n)} \mathbb{E}[G_{ij}^{(n)}(t + \Delta)], \end{aligned}$$

where $G_{ij}^{(n)}$ is the distribution of the sum $T_0 + \dots + T_{n-1}$ conditioned on the Markov states $\xi_0 = i, \xi_1, \dots, \xi_{n-1}, \xi_n = j$.

Let $N_{k\ell}^{(n)}$ be the random variable counting the number of jumps from state $k \in \mathcal{E}$ to state $\ell \in \mathcal{E}$ before the n 'th jump, that is

$$N_{k\ell}^{(n)} = \sum_{m=0}^{n-1} \mathbf{1}_{\{\xi_m = \ell, \xi_{m+1} = k\}}.$$

Correspondingly, let $N_{k\ell|ij}^{(n)}$ be the random variable representing the number of jumps from $k \in \mathcal{E}$ to $\ell \in \mathcal{E}$ before jump n given that $\xi_0 = i$ and $\xi_n = j$, i.e. it is distributed as $N_{k\ell}^{(n)}$ conditioned on $\xi_0 = i, \xi_n = j$. This has expected value

$$\begin{aligned} \mathbb{E}[N_{k\ell|ij}^{(n)}] &= \mathbb{E}[N_{k\ell}^{(n)} \mid \xi_0 = i, \xi_n = j] = \frac{\mathbb{E} \left[\sum_{m=0}^{n-1} \mathbf{1}_{\{\xi_0 = i, \xi_m = k, \xi_{m+1} = \ell, \xi_n = j\}} \right]}{\mathbb{P}(\xi_0 = i, \xi_n = j)} \\ &= \frac{\sum_{m=0}^{n-1} \mathbb{P}(\xi_0 = i, \xi_m = k, \xi_{m+1} = \ell, \xi_n = j)}{\mathbb{P}(\xi_0 = i, \xi_n = j)} = \frac{\sum_{m=0}^{n-1} p_{ik}^{(m)} p_{k\ell} p_{\ell j}^{(n-m-1)}}{p_{ij}^{(n)}} \\ &= \frac{\sum_{m=0}^{n-1} (\mathbf{P}^m)_{ik} p_{k\ell} (\mathbf{P}^{n-m-1})_{\ell j}}{(\mathbf{P}^n)_{ij}}. \end{aligned}$$

Recall that $G_{ij}^{(n)}$ is the distribution of the sum of the random variables T_0, T_1, \dots, T_{n-1} conditioned on the Markov chain until the n th jump. The distribution of T_m merely depends on ξ_m and ξ_{m+1} . Therefore, $G_{ij}^{(n)}$ can be interpreted as the random convolution

$$G_{ij}^{(n)}(t + \Delta) = \bigstar_{k, \ell \in \mathcal{E}} G_{k\ell}^{*N_{k\ell|ij}^{(n)}}(t + \Delta).$$

Applying Theorem 4.3 the local asymptotics of $G_{ij}^{(n)}$ can then be specified as

$$\mathbb{E} \left[\bigstar_{k, \ell \in \mathcal{E}} G_{k\ell}^{*N_{k\ell|ij}^{(n)}}(t + \Delta) \right] \sim \left(\sum_{k, \ell \in \mathcal{E}} \mathbb{E}[N_{k\ell|ij}^{(n)}] c_{k\ell} \right) G(t + \Delta).$$

This asymptotic specification transfers to the semi-Markov kernel as follows

$$(\mathbf{F}^{*n})_{ij}(t + \Delta) \sim \sum_{k, \ell \in \mathcal{E}} \sum_{m=0}^{n-1} (\mathbf{P}^m)_{ik} p_{k\ell} (\mathbf{P}^{n-m-1})_{\ell j} c_{k\ell} G(t + \Delta)$$

and on to the Markov renewal kernel

$$\begin{aligned} U_{ij}(t + \Delta) &\sim \sum_{n=0}^{\infty} \sum_{k, \ell \in \mathcal{E}} \sum_{m=0}^{n-1} (\mathbf{P}^m)_{ik} p_{k\ell} (\mathbf{P}^{n-m-1})_{\ell j} c_{k\ell} G(t + \Delta) \\ &= \sum_{k, \ell \in \mathcal{E}} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} (\mathbf{P}^m)_{ik} p_{k\ell} (\mathbf{P}^{n-m-1})_{\ell j} c_{k\ell} G(t + \Delta). \end{aligned}$$

Since \mathbf{P} has spectral radius strictly less than 1 the infinite series above converges with limits

$$\sum_{n=m+1}^{\infty} (\mathbf{P}^{n-m-1})_{\ell j} = \sum_{k=0}^{\infty} (\mathbf{P}^k)_{\ell j} = (\mathbf{I} - \mathbf{P})_{\ell j}^{-1}, \quad \sum_{m=0}^{\infty} (\mathbf{P}^m)_{ik} = (\mathbf{I} - \mathbf{P})_{ik}^{-1}.$$

Thus

$$U_{ij}(t + \Delta) \sim \left(\sum_{k, \ell \in \mathcal{E}} (\mathbf{I} - \mathbf{P})_{ik}^{-1} p_{k\ell} (\mathbf{I} - \mathbf{P})_{\ell j}^{-1} c_{k\ell} \right) G(t + \Delta),$$

concluding the proof. \square

For the proof of Theorem 2.2, it suffices in view of (6.1) to find the asymptotics of the summands $Z_{ij}(x) = \int_0^x z_j(x-y) U_{ij}(dy)$ in the solution of the Markov renewal equation. As an introductory remark, notice that $G \in \mathcal{S}_\Delta$ for all $T > 0$ has the implications

$$G(x, x + 1/n] \sim \frac{g(x)}{n} \quad \text{for all } n \quad \text{and} \quad \frac{g(x+y)}{g(x)} \rightarrow 1 \quad \text{for all } |y| < y_0 < \infty$$

for some appropriate y_0 . For a suitable $A < x/2$ decompose $Z_{ij}(x)$ into three parts, namely

$$\begin{aligned} Z_{ij}(x) &= \left(\int_0^A + \int_A^{x-A} + \int_{x-A}^x \right) z_j(x-y) U_{ij}(dy) \\ &\equiv J_1(x, A) + J_2(x, A) + J_3(x, A). \end{aligned}$$

We now consider and evaluate the three parts separately. In case i) of the Theorem we have

$$\begin{aligned} J_1(x, A) &= \int_0^A z_j(x-y) U_{ij}(dy) \\ &= g(x) \int_0^A \frac{z_j(x-y)}{g(x-y)} \frac{g(x-y)}{g(x)} U_{ij}(dy) = o(g(x)) \end{aligned}$$

when $x \rightarrow \infty$. Also

$$\begin{aligned} J_2(x, A) &= \int_A^{x-A} z_j(x-y)U_{ij}(\mathrm{d}y) = \int_A^{x-A} \frac{z_j(x-y)}{g(x-y)}g(x-y)U_{ij}(\mathrm{d}y) \\ &= o(1) \int_A^{x-A} g(x-y)U_{ij}(\mathrm{d}y) = o(g(x)) \int_A^{x-A} \frac{g(x-y)}{g(x)}U_{ij}(\mathrm{d}y). \end{aligned}$$

From this we can conclude

$$\lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{J_2(x, A)}{g(x)} = \lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{o(g(x))}{g(x)}(U(x-A) - U(A)) = 0.$$

Finally, consider a finite partition of the interval $(x-A, x]$ into n equally sized intervals. A is now assumed to be an integer. Furthermore, let

$$\bar{z}_n(x) = \sup_{|y-x| \leq \frac{1}{n}} z(y).$$

Then we can bound J_3 from above by the upper Riemann sum

$$\begin{aligned} J_3(x, A) &= \int_{x-A}^x z_j(x-y)U_{ij}(\mathrm{d}y) \leq \sum_{k=0}^{An-1} \bar{z}_n\left(\frac{k}{n}\right) U_{ij}\left(x - \frac{k+1}{n}, x - \frac{k}{n}\right) \\ &\sim d_{ij} \sum_{k=0}^{An-1} \bar{z}_n\left(\frac{k}{n}\right) G\left(x - \frac{k+1}{n}, x - \frac{k}{n}\right) \\ &\sim g(x) \frac{d_{ij}}{n} \sum_{k=0}^{An-1} \bar{z}_n\left(\frac{k}{n}\right) \sim g(x) \frac{d_{ij}}{n} \sum_{k=0}^{\infty} \bar{z}_n\left(\frac{k}{n}\right) \end{aligned}$$

as $x \rightarrow \infty$ and $A \rightarrow \infty$. Since z is assumed to be directly Riemann integrable, we have

$$\frac{1}{n} \sum_{k=0}^{\infty} \bar{z}_n\left(\frac{k}{n}\right) \rightarrow I_j$$

as n tends to infinity. So we have now obtained

$$\limsup_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{J_3(x, A)}{g(x)} \leq d_{ij} I_j.$$

Using the same approach with \liminf a similar bound from below is obtained using lower Riemann sums. This finishes case i).

Now consider case ii). The first of the decomposed parts has the asymptotics

$$\begin{aligned} J_1(x, A) &= \int_0^A z_j(x-y)U_{ij}(\mathrm{d}y) \sim a_j g(x) \int_0^A \frac{g(x-y)}{g(x)}U_{ij}(\mathrm{d}y) \\ &\sim a_j g(x)U_{ij}(A) \end{aligned}$$

which leads to

$$\lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{J_1(x, A)}{g(x)} = \lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} a_j U_{ij}(A) = k_{ij} a_j.$$

The second part in case ii) can also be evaluated as follows

$$\begin{aligned} J_2(x, A) &= \int_A^{x-A} z_j(x-y)U_{ij}(\mathrm{d}y) = \int_A^{x-A} \frac{z_j(x-y)}{g(x-y)}g(x-y)U_{ij}(\mathrm{d}y) \\ &= O(1) \int_A^{x-A} g(x-y)U_{ij}(\mathrm{d}y). \end{aligned}$$

Notice the similarity to J_2 in case i). Corresponding calculations show that $J_2(x, A) = o(g(x))$. J_3 is also similar to case i), which concludes case ii).

In case iii) we are no longer able to consider the decomposition we have so far. Instead, let K_j denote the probability measure with density $z_j(x)/I_j$. Recall that $K_j \in \mathcal{S}_\Delta$. Now let $\Delta = (0, 1]$ and consider a new decomposition

$$\begin{aligned} \int_0^x z_j(x-y)U_{ij}(\mathrm{d}y) &= \int_0^{x-A} z_j(x-y)U_{ij}(\mathrm{d}y) + \int_{x-A}^x z_j(x-y)U_{ij}(\mathrm{d}y) \\ &\equiv I_1(x, A) + I_2(x, A), \end{aligned}$$

where the second term satisfies

$$I_2(x, A) \leq A \sup_{y \leq A} |z_j(y)| U_{ij}(x-A, x] = o(z_j(x)).$$

Letting A tend to infinity in a slower rate than x (recall that we have chosen A to be less than $x/2$) will preserve this inequality. Now consider a corresponding decomposition of the convolution $K_j * U_{ij}$.

$$\begin{aligned} (K_j * U_{ij})(x + \Delta) &= \int_0^x K_j(x-y+\Delta)U_{ij}(\mathrm{d}y) \\ &= \int_0^{x-A} K_j(x-y+\Delta)U_{ij}(\mathrm{d}y) + \int_{x-A}^x K_j(x-y+\Delta)U_{ij}(\mathrm{d}y) \\ &\equiv I_1'(x, A) + I_2'(x, A). \end{aligned}$$

As for I_2 we can evaluate $I_2'(x, A) = o(z_j(x))$. Since $z_j \in \mathcal{L}$ we have $z_j(x) \sim I_j \cdot K_j(x + \Delta)$ for $\Delta = (0, 1]$ and therefore $I_1(x, A) \sim I_j \cdot I_1'(x, A)$. This yields

$$\int_0^x z_j(x-y)U_{ij}(\mathrm{d}y) \sim I_j \cdot (K_j * U_{ij})(X + \Delta).$$

Due to the assumption $z_j(x)/g(x) \rightarrow \infty$ the use of Proposition 3 in [9] with $c_1 = 1$ and $c_2 = 0$ gives

$$I_j \cdot (K_j * U_{ij})(x + \Delta) \sim k_{ij}I_jK(x + \Delta) \sim k_{ij}z_j(x)$$

which concludes the proof of Theorem 2.2 in the substochastic case.

If instead of \mathbf{P} being substochastic it merely satisfies $\text{spr}(\mathbf{P}) < 1$, we define the measure $\tilde{F}_{ij}(dx) = F_{ij}(dx)v_i/v_j$ for $i, j \in \mathcal{E}$ and let $\tilde{\mathbf{P}}$ be the matrix with elements $\|\tilde{F}_{ij}\| = \|F_{ij}\|v_i/v_j$. $\tilde{\mathbf{P}}$ will then be a substochastic matrix with row sums λ and

therefore it must also have spectral radius less than one. Letting $\tilde{Z}_i(x) = v_i Z_i(x)$ and $\tilde{z}_i(x) = v_i z_i(x)$ another renewal equation occurs

$$\tilde{Z}_i(x) = \tilde{z}_i(x) + \sum_{j \in \mathcal{E}} \int_0^x \tilde{Z}_j(x-y) \tilde{F}_{ij}(dy),$$

which has the same properties as analyzed previously in the substochastic case with Markov kernel $\tilde{U}_{ij} = U_{ij}v_i/v_j$. What we have shown already can therefore be applied with coefficients $\tilde{a}_j, \tilde{d}_{ij}, \tilde{k}_{ij}$ and \tilde{I}_j . The assumption of the asymptotic properties of G_{ij} stated in Theorem 2.2 implies that $\tilde{G}_{ij}(t + \Delta) = v_i G_{ij}(t + \Delta)/v_j \sim \tilde{c}_{ij} G(t + \Delta)$, where $\tilde{c}_{ij} = c_{ij}v_i/v_j$ and same relation transfers to $\tilde{d}_{ij} = d_{ij}v_i/v_j$. Correspondingly for $\tilde{k}_{ij} = \tilde{U}_{ij}(0, \infty) = U_{ij}(0, \infty)v_i/v_j = k_{ij}v_i/v_j$. Last to mention is $\tilde{I}_j = \int_0^\infty \tilde{z}_j(y)dy = v_j \int_0^\infty z_j(y)dy = v_j I_j$ and $\tilde{a}_j = v_j a_j$. Inserting these into the asymptotics of \tilde{Z}_i gives the same result as Theorem 2.2. For example in case ii), the substochastic results gives $\tilde{Z}_i(x) \sim \sum_{j \in \mathcal{E}} (\tilde{I}_j \tilde{d}_{ij} + \tilde{a}_j \tilde{k}_{ij}) g(x)$ which translates to

$$\begin{aligned} Z_i(x) &= \frac{1}{v_i} \tilde{Z}_i(x) \sim \frac{1}{v_i} \sum_{j \in \mathcal{E}} \left(v_j I_j \frac{v_i}{v_j} d_{ij} + v_j a_j \frac{v_i}{v_j} k_{ij} \right) g(x) \\ &= \sum_{j \in \mathcal{E}} \left(I_j d_{ij} + a_j k_{ij} \right) g(x). \end{aligned}$$

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