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Abstract. Interest rate futures are basic securities and at the same time highly liquid traded objects. Despite this observation, most models of the term structure of interest rate assume forward rates as primary elements. The processes of futures prices are therefore endogenously determined in these models. In addition, in these models hedging strategies are based on forward and/or spot contracts and only to a limited extent on futures contracts.

Inspired by the market model approach of forward rates by Miltersen, Sandmann, and Sondermann (1997), the starting point of this paper is a model of futures prices. Using the prices of futures on interest related assets as the input to the model, new no-arbitrage restricions on the volatility structure are derived. Moreover, these restrictions turn out to prevent an application of a market model based on futures prices.

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I. Introduction

The aim of this paper is to establish and analyze the no-arbitrage conditions originating from a term structure of interest rate model where the exogenous inputs consist of futures prices on zero coupon bonds, and the dynamics of these futures prices. The earliest models of the term structure of interest rates appearing in the literature were based on the short-term interest rate as the exogenously given input. The no-arbitrage condition derived claims that the expected excess return of a bond divided by its volatility, should equal the market price of risk function which is found to be independent of the maturity of the bond considered. Assuming the latter function known, interest rate derivatives could then be priced. This modelling was subsequently modified by increasing the set of inputs to the model. Firstly, the parameters of the stochastic differential equation were chosen in such a manner that the model determined prices would be in accordance with the today observed prices. Secondly, the number of state variables were extended, but the no-arbitrage implications on the drift and volatility terms were uninfluenced by these modifications.

A major step forward was made by the so-called Heath, Jarrow, and Morton (1992) modelling approach. Here the dynamics of the family of forward rates and not only the short-term interest rate is the input to the model. This modelling approach highlights the dynamic relationship between different interest rate depending objects, which has to be satisfied in a continuous time dynamic setting without arbitrage, like bonds of different maturities, yields, forward rates, etc. Whereas in earlier models the drift and the volatility terms could be chosen independently of each other, the degree of freedom was now reduced so that only one of these terms could be specified exogenously: an important restriction on the drift parameter in relation to a chosen volatility structure was established.

The strength and elegance of the Heath, Jarrow, and Morton model comes from the exogenous modelling of the instantaneous forward rate processes. However, this is also the most critical aspect of the model: The instantaneous interest rates are theoretical objects defined by taking the limit as the compounding interval approaches zero. These rates do not correspond in any simple way to interest rates observed in real financial markets. Observable rates like forward rates are endogenously determined within this modelling approach. The same hold true for futures prices and rates. What happens if we now exogenously specify the much richer family of futures prices? Richer in the sense that a futures has many more payment days than the corresponding forward contract. This question will be addressed in the following. Furthermore, we will analyze whether a consistent market model based on futures prices can be established to overcome the non-adequate behavior of the forward based market model where we cannot in a rational sense price both swaps and caps.

The paper is organized as follows: In Section II we recall some known results and present some definitions. Section III contains the main model of futures prices related to the term structure of interest rates. In this section new restrictions concerning the volatility structure of the futures price process are presented. In Section V we discuss the interrelationship between the Heath, Jarrow, Morton modelling and the futures based approaches.

II. Spot, Forward and Futures Prices

The construction of an arbitrage free model for financial instruments starts with the definition of the underlying securities. This step is usually not difficult. Nevertheless this modelling step asks for particular care in the case of an interest rate market. There is nothing like the interest rate. Instead, different concepts of interest rates have to be reflected.

The terminology in this paper includes spot, forward and futures prices as well as the concept of nominal and instantaneous interest rates on spot, forward and futures markets. A spot price is the amount of money that we have to pay now (today) to get immediate ownership of a specific good or security. Since we restrict ourselves to the interest rate market, the basic securities are coupon or zero coupon bonds. Setting the face value to one, the owner of a zero coupon bond holds the right to a payment of one unit of account at the maturity of the contract. Denote by $B(t,\tau)$ the spot price at time $t \leq \tau$ of a zero coupon bond with face value one and maturity τ.

In contrast to the spot contract, a forward contract is a binding agreement to deliver a specific good or security at some fixed point in time in the future. Consequently the forward price of a good or security is the amount of money payable at the delivery date. Since, in the case of a forward contract, the closure of the contract and the delivery date are not identical, the forward price differs from the spot price. In particular its dimension is not money today. With respect to the interest rate market we restrict ourselves to forward contracts on zero coupon bonds. Set $t \leq u \leq \tau$ and define by $F(t,u,\tau)$ the forward price of a zero coupon bond with maturity τ and delivery at time u. At time t the forward price $F(t,u,\tau)$ is determined in such a way that the value of the forward contract equals zero. Closely related to spot prices are spot and forward interest rates. A forward interest rate is an interest rate fixed by two parties for a specific compounding period in the future. Denote by $r_n(t,u,\tau)$ the nominal forward interest rate at time t for the compounding interval $[u,\tau]$ with $t \leq u \leq \tau$. The relationship between nominal forward rates on the one side and zero coupon bond spot and forward prices on the other side are given by

$$B(t,\tau) =: \frac{B(t,u)}{(1+(\tau-u)\cdot r_n(t,u,\tau))}, \tag{1}$$

$$F(t,u,\tau) = \frac{B(t,\tau)}{B(t,u)} =: \frac{1}{1+(\tau-u)\cdot r_n(t,u,\tau)}. \tag{2}$$

The nominal spot rate is obtained for $u = t$. The concept of nominal spot and forward rates is the main modelling instrument of the market model approach. The interest rate concept underlying this class of models is chosen to be close to observable interest rates. In contrast, most other models of the term structure of interest rates are defined on the concept of instantaneous spot and forward rates. Instantaneous interest rates are defined as the limiting concept of nominal interest rates as the length of the compounding period converges to zero. Assume that the spot prices of zero coupon bonds are differentiable with respect to the maturity date. The function of the instantaneous forward rate $f(t,\cdot) : [t,T] \longrightarrow \mathbb{R}$ at time t is defined by:

$$f(t,u) := \lim_{\tau \to u} r_n(t,u,\tau) = -\frac{\partial \ln B(t,u)}{\partial u}. \tag{3}$$

As a special case the instantaneous spot rate $r_c(\cdot)$ is defined by:

$$r_c(t) := f(t,t) = \lim_{\tau \to t} r_n(t,t,\tau) = -\left.\frac{\partial \ln B(t,u)}{\partial u}\right|_{u=t}. \tag{4}$$

The forward contract as well as the futures contract is an agreement between two parties to exchange a good or security at a specific price in the future. In contrast to the forward contract, the margin system of a futures contract implies a continuous cash flow between the two counterparts. The futures price is fixed in such a way that the value of the futures contract is equal to zero. This implies that the futures price changes over time and the cash flow between the counterparts is determined by the increments of the futures price. As for the forward price, the dimension of the futures price is not money today. Following Cox, Ingersoll, and Ross (1981), the futures price is equal to the amount of money necessary to implement a self-financing portfolio strategy with a payoff equal to the value of the underlying security times the rollover bank account. Denote by $H(t,u,\tau)$ the futures price at time t if the underlying security is a zero coupon bond with maturity τ, and delivery is at time u. Suppose that the marketed-to-market of a futures is continuous. In this case $H(t,u,\tau)$ is equal to the present value of the payoff

$$\exp\left\{\int_t^u r_c(s)ds\right\} B(u,\tau)$$

at time $u \geq t$. This implies that for $u > t$ the forward and futures prices only coincide if the interest rate is deterministic or if $\exp\left\{\int_t^u r_c(s)ds\right\}$ is orthogonal to $B(u,\tau)$. Furthermore, the difference between forward and futures prices is determined by the model of the term structure of interest rates. For $u = t$ we have

$$H(t,t,\tau) = B(t,\tau) = F(t,t,\tau).$$

In the limit $u = t$ we therefore have that

$$f(t,\tau) = -\frac{\partial \ln H(t,t,\tau)}{\partial \tau}, \tag{5}$$

$$r_c(t) = -\left.\frac{\partial \ln H(t,t,\tau)}{\partial \tau}\right|_{\tau=t}. \tag{6}$$

III. Stochastic Model of Futures Prices

Any of the mentioned definitions for prices and rates can be used as the starting point of the construction of a model for the term structure of interest rates. Within the Heath, Jarrow and Morton (1992) framework the exogenous assumptions are based on the concept of instantaneous forward rates. Consequently the stochastic evolution of spot and forward prices as well as nominal interest rates are determined endogenously within this model structure. The market model approach by Miltersen, Sandmann and Sondermann (1997) on the other hand is formulated with respect to nominal forward rates. The idea of this section is to use futures prices as the primary and exogenous objects of the modelling structure.

From a theoretical point of view one can argue that these approaches are equivalent to each other. Neglecting technical aspects this argument is to some extent valid and will be discussed in this section. Nevertheless two aspects should be stressed at this point.

- First, the information given by futures prices is richer than the one given by forward prices. In addition to the drift restriction determined by the initial forward price curve, further no-arbitrage restrictions should be expected.
- Second, the futures market is a highly liquid market. Banks, companies and institutional investors are managing large futures positions. Why is a futures based model not used to analyze the risk of these positions? Due to the marketed-to-market system the default risk of futures is less crucial than for forward contracts. To reduce the effects of default risk an empirical study of the term structure of interest rates should refer to the futures market data rather than to the forward market data.

These two aspects serve as the main intuitive justification for the following modelling approach.

A. **Term Structure of Futures Prices.** The futures price is equal to the present value of a self-financing financial strategy with a payoff equal to the value of the underlying security at delivery multiplied by the rollover return. Therefore an arbitrage free model of the term structure of interest rates implies that the futures price is equal to the expected value of the underlying security under the martingale measure. Let $(\Omega, \mathbb{F}, P, \mathbb{F}_t)$ be a filtered probability space and let P^* be a probability measure equivalent to P such that discounted spot price processes are martingales under P^*. The martingale property of spot prices implies for futures prices:

$$
\begin{aligned}
(7) \qquad H(t,u,\tau) &= E_{P^*}[B(u,\tau)|\mathbb{F}_t| = E_{P^*}\left[E_{P^*}[B(u,\tau)|\mathbb{F}_s]|\mathbb{F}_t\right] \\
&= E_{P^*}[H(s,u,\tau)|\mathbb{F}_t] \quad \forall t \leq s \leq u \leq \tau,
\end{aligned}
$$

which yields that the futures price is a martingale under P^*. Note that the martingale property of futures prices is not restricted to zero coupon bonds as underlying securities.

For the following we apply the usual modelling framework, i.e. the stochastic processes are defined as stochastic integrals. Assume that the filtration of the probability space is generated by a k-dimensional Brownian motion $\{W^*(t)\}_t$ under the martingale measure P^*. The futures price is therefore a solution of the stochastic differential equation

$$
(8) \qquad dH(t,u,\tau) = H(t,u,\tau)\sigma_H(t,u,\tau) \cdot dW^*(t),
$$

where $\sigma_H(\cdot,u,\tau) : [t_0,u] \to \mathbb{R}^k$ is the k-dimensional stochastic volatility function of the futures price. We have to impose the usual restrictions on the volatility structure guaranteeing the existence of the solution to the above stochastic differential equation. Furthermore the restrictions should allow for the application of the stochastic Fubini Theorem at several places in the paper.[1] Furthermore, the solution of the futures price process given by equation (8) is determined by

(9)

$$
H(t,u,\tau) = H(t_0,u,\tau) \cdot \exp\left\{-\frac{1}{2}\int_{t_0}^{t} ||\sigma_H(s,u,\tau)||^2 ds + \int_{t_0}^{t} \sigma_H(s,u,\tau) \cdot dW^*(s)\right\}.
$$

[1] For details see Appendix A.

B. **Implied Term Structure of Forward and Spot Prices.** Since the volatility structure of the futures prices is exogenously given, we are interested in the implied structure of spot and forward prices and interest rates. For $u = t$ the futures price is equal to the spot price of the underlying. Therefore the spot price is a solution of the stochastic differential equation

$$
\begin{aligned}
dB(t,\tau) &= \left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} dt + dH(\cdot,t,\tau)_t \\
&= \left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} dt + H(t,t,\tau)\ \sigma_H(t,t,\tau)\cdot dW^*(t) \\
&= B(t,\tau)\ \left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} dt + B(t,\tau)\ \sigma_H(t,t,\tau)\cdot dW^*(t).
\end{aligned} \tag{10}
$$

Under the equivalent martingale measure the drift of the zero coupon bond equals the spot rate. In addition to the definition of the instantaneous spot rate in equation (6) the no-arbitrage implies that

$$
\left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} = r_c(t) \quad \forall t \le \tau. \tag{11}
$$

We can know address our main question: Does the initial curve of futures prices restrict the volatility structure of the term structure of interest rates? Our main answer to this question is positive and can be summarized by the following proposition:

Proposition 1. *Suppose that the volatility function of the futures price satisfies the usual regularity conditions then:*

i) *Under the martingale measure the expected value of the instantaneous spot rate is determined by the futures price at time t_0, i.e.*

$$
E_{P^*}\left[r_c(t)|\mathbb{F}_{t_0}\right] = -\left.\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\right|_{\tau=t}.
$$

ii) *No-arbitrage conditions imply that the volatility of the futures price must satisfy a.s.*

$$
0 = \left.\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} + \left.\frac{\partial \sigma_H(t,u,\tau)}{\partial u}\right|_{u=t}
$$

and furthermore

$$
0 = \frac{\partial}{\partial t}\left(\left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t}\right)\Bigg|_{s=t} + \frac{\partial}{\partial t}\left(\frac{\partial \sigma_H(s,t,\tau)}{\partial t}\right)\Bigg|_{s=t}
$$

iii) *A weaker condition, derived by taking expectations, is*

$$
\int_t^\tau \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s}\, ds = E_{P^*}\left[\int_{t_0}^t \sigma_H(s,t,\tau)\cdot\frac{\partial \sigma_H(s,t,\tau)}{\partial t}\, ds \,\middle|\, \mathbb{F}_{t_0}\right].
$$

The statements in Proposition 1 concentrate on two aspects.

- First, the initial futures prices completely determine the expected spot rate under the martingale measure. In other words, the derivative of the futures price is the best predictor of the spot rate under the martingale measure.

- Second, the volatility function cannot be specified in an arbitrary manner. The derivative of the futures volatility with respect to settlement of the future and the maturity of the underlying bond respectively shares the same property as the futures prices. The formal relationship between the initial futures prices and the volatility is given by Proposition 1 ii). Unfortunately, there is no easy interpretation possible. The weaker condition iii) implies that the expected volatility under the martingale measure is determined by the initial futures prices. Observe that the left hand side of this equation is in principle given by the initial futures prices. Thus from an empirical point of view the expected volatility can be related to observable data. This result is of particular interest if in addition the volatility is assumed to deterministic.

IV. Applications of the Futures Market Model

To further understand the no-arbitrage restrictions given in Proposition 1 we have to consider specific situations. In the first model we analyze a specification, in similarity with the LIBOR market model. In the second model $\sigma_H(t,u,\tau)$ itself is assumed to have a deterministic development.

A. **Futures Market Model and Log-normality?** Within this section we consider a specific volatility structure which is in accordance with the LIBOR-Market model, i.e.

$$\sigma_H(t,u,\tau) = (1 - H(t,u,\tau))\ \gamma_H(t,u,\tau) \qquad \forall t \leq u \tag{12}$$

where the functions $\gamma_H(\cdot,u,\tau) : [t_0,u] \to \mathbb{R}^k$ are $\forall u \in [t_0,T]$ and $\forall \tau \in [t_0,T]$ with $u \leq \tau$ deterministic and bounded. Refering to the Bund Futures contract the implied futures rate in nominal terms $r_H(t,u,\tau)$ is defined by:

$$H(t,u,\tau) =: \frac{1}{1 + (\tau - u)\ r_H(t,u,\tau)}. \tag{13}$$

With this choice the implied futures rate process can be rewritten as the solution of the following stochastic differential equation:

$$\begin{aligned} dr_H(t,u,\tau) = &-r_H(t,u,\tau)\ \gamma_H(t,u,\tau) \\ &\cdot \left(dW^*(t) - (1 - H(t,u,\tau))\gamma_H(t,u,\tau) \cdot dt\right). \end{aligned} \tag{14}$$

Obviously the futures market structure is similar to the modelling assumptions within the LIBOR market model. Nevertheless, defining the volatility of the futures market in accordance with equation (12), implies that no forward rate process is a log-normal martingale under the appropriate forward risk adjusted measure. The log-normality for the corresponding nominal forward rate process under the τ-forward risk adjusted measure is fulfilled if a deterministic function $\gamma_F(.,u,\tau)$ exists such that

$$\sigma_H(t,t,\tau) - \sigma_H(t,t,u) = (1 - F(t,u,\tau)) \cdot \gamma_F(t,u,\tau).$$

Instead, the specification (12) implies that this function should equal

$$\gamma_F(t,u,\tau) = \frac{(1 - B(t,\tau))\ \gamma_H(t,t,\tau) - (1 - B(t,u))\ \gamma_H(t,t,u)}{B(t,\tau) - B(t,u)} \cdot B(t,u).$$

Even if we assume $\gamma_H(.,.,.)$ to be constant, a stochastic specification of the function $\gamma_F(.,.,.)$ is implied. In other words, the Black formula for caplets and floorlets are not satisfied. The reason for this strong property is that, assuming that (12) is valid for all τ, is a much stronger assumption than the corresponding one in the LIBOR market model. In particular log-normality within the LIBOR market model can only be imposed on one specific compounding period $\tau - u$. As we will see the same property holds within the futures market model.

We will now analyze whether the above log-normal volatility structure satisfies the conditions in Proposition 1. Inserting the expression for $\sigma_H(t,u,\tau)$ into ii) of Proposition 1, leads to the condition

$$\begin{aligned}
&\left.\frac{\partial H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} \gamma_H(t,t,t) \\
&\quad = -\left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} \gamma_H(t,t,\tau) + (1 - H(t,t,\tau)) \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t}
\end{aligned} \tag{15}$$

which can be reformulated to

$$B(t,\tau) = \frac{r_c(t)\ \gamma_H(t,t,t) + \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t}}{r_c(t)\ \gamma_H(t,t,\tau) + \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t}} \tag{16}$$

Observe that $B(t,t) = 1$. In addition we want to restrict the volatility structure such that $B(t,\tau)$ is a decreasing function in τ. Before the presentation of possible structures satisfying this requirement the implications of the second no-arbitrage condition in Proposition 1 will be derived. Inserting the expression for $\sigma_H(t,u,\tau)$ leads to the condition

$$\begin{aligned}
0 = {} & r_c(t) \left(\left.\frac{\partial \gamma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} + \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t,\tau=t} \right) \\
& - \gamma_H(t,t,t) \left(\left.\frac{\partial^2 H(t,t,\tau)}{\partial \tau^2}\right|_{\tau=t} + \left.\frac{\partial^2 H(t,u,\tau)}{\partial u \partial \tau}\right|_{u=t,\tau=t} \right) \\
& - \gamma_H(t,t,\tau) \left.\frac{\partial^2 H(t,u,\tau)}{\partial u^2}\right|_{u=t} - 2 r_c(t) B(t,\tau) \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t} \\
& + (1 - B(t,\tau)) \left.\frac{\partial^2 \gamma_H(t,u,\tau)}{\partial u^2}\right|_{u=t}
\end{aligned} \tag{17}$$

A large class of $\gamma-$functions could fulfill (16) with $B(t,\cdot)$ a decreasing function. However in equation (17) in particular the term $\left.\frac{\partial \gamma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t}$ restricts the possible choice of the volatility structure. A valid choice of the volatility structure concerning (16) and at the same time preventing the term $\left.\frac{\partial \gamma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t}$ to equal infinity is

$$\gamma_H(t,u,\tau) = A(t,\tau)\left(A(t,\tau) + G(t,u)\right)$$

with

i) $A(t,\tau)$ being nonnegative, concave and increasing in τ

ii) $\left.\frac{\partial A(t,\tau)}{\partial \tau}\right|_{\tau=t} < \infty$

iii) $G(t,t) = 1, G(t,u) > 0$ and increasing in u.

Choosing e.g. $A(t,\tau) = \ln(\tau - t + 1)$ and $G(t,u) = \exp(\alpha(u-t))$ with $\alpha > 0$ equation (16) is satisfied with $B(t,\cdot)$ being a positive, convex and decreasing function in τ. Equation (17) turns into

$$(18) \quad r_c(t) = \ln(\tau - t + 1)\ (\ln(\tau - t + 1) + 1)\ \gamma_H(t,t,\tau)\ \left.\frac{\partial^2 H(t,u,\tau)}{\partial u^2}\right|_{u=t} \\ + 2\ r_c(t)\ B(t,\tau)\ \alpha\ \ln(\tau - t + 1) - (1 - B(t,\tau))\ \alpha^2\ \ln(\tau - t + 1)$$

Observe now that this expression should be valid for all τ meaning that the parameters have to be chosen in such a manner that the right hand side is only a function of t. It might at a first glance seem possible to have (18) satisfied. However, the choice $\tau = t$ turns the right hand side into 0 with the nonsensical consequence that $r_c(t) = 0$. Based on this and other analyzed cases, we do not believe that it is possible to establish a rational futures market model. In other words the log-normal volatility specification in equation (12) cannot be satisfied for all t. Similar to the LIBOR Market model, log-normality of implied futures rates can only be satisfied for a finite set of Futures.

B. **Deterministic volatility model.** In the deterministic volatility model it is assumed that the futures prices are log-normal, i.e. that

$$dH(t,u,\tau) = H(t,u,\tau)\ \sigma_H(t,u,\tau) \cdot dW^*(t)$$

with $\sigma_H(t,u,\tau)$ being deterministic. As the futures price turns into a bond price at the maturity date of the futures, and as we want to model the possibility that the futures price deviates from the corresponding forward price an obvious choice of $\sigma_H(t,u,\tau)$ is

$$\sigma_H(t,u,\tau) = (\nu(t,\tau) - \nu(t,u))\left(1 - \xi \exp\left(-\frac{\alpha}{u-t}\right)\right),$$

with $\alpha > 0$. It is easily shown that this specification satisfy the two established no-arbitrage conditions on the volatility structure. Furthermore, the volatility $\sigma_F(t,u,\tau)$ of the corresponding forward contract is equal to

$$\sigma_F(t,u,\tau) := \sigma_H(t,t,\tau) - \sigma_H(t,t,u) = \nu(t,\tau) - \nu(t,u).$$

For $\beta = 0$ the volatility coincides with the volatility of the corresponding forward contract, whereas the situations $\xi < 0$ and $\xi > 0$ lead to that $\sigma_F(t,u,\tau)\cdot\sigma_H(t,u,\tau) < 0$ and $\sigma_F(t,u,\tau)\cdot\sigma_H(t,u,\tau) > 0$ respectively. As an example we consider an application of this structure leading to the Vasicek (1977) model.

Example 1. *Suppose that the futures prices at time t_0 are up to some constant determined by the following function:*

$$-\frac{\partial \ln H(t_0,t,s)}{\partial s} = f(t_0,s) + \sum_{i=1}^{k} \eta_i e^{-\beta_i(s-t_0)} \cdot [\cosh(\beta_i(t-t_0)) - 1],$$

where $f(t_0,s)$ is equal to the instantaneous forward rate. $\eta = (\eta_1, ...\eta_k)$ and $\beta := (\beta_1, ..., \beta_k)$ are k-dimensional constants. In addition, suppose that the volatility is *deterministic. Applying Proposition 1 the volatility of the futures price has to satisfy*

$$\begin{aligned}
\int_{t_0}^{t} \sigma_H(s,t,\tau) \cdot \frac{\partial \sigma_H(s,t,\tau)}{\partial t}\, ds &= \int_t^{\tau} \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s}\, ds \\
&= \int_t^{\tau} -\sum_{i=1}^{k} \eta_i \cdot \beta_i e^{-\beta_i(s-t_0)} \cdot \sinh(\beta_i(t-t_0))\, ds \\
&= -\sum_{i=1}^{k} \eta_i \cdot \sinh(\beta_i(t-t_0)) \left(e^{-\beta_i(t-t_0)} - e^{-\beta_i(\tau-t_0)}\right) \\
&= -\sum_{i=1}^{k} \frac{\eta_i}{2} \left[e^{-\beta_i t} - e^{-\beta_i \tau}\right] \cdot e^{-\beta_i t} \cdot \left[e^{2\beta_i t} - e^{2\beta_i t_0}\right] \\
&= -\sum_{i=1}^{k} \eta_i \beta_i \left[e^{-\beta_i t} - e^{-\beta_i \tau}\right] \cdot e^{-\beta_i t} \cdot \int_{t_0}^{t} e^{2\beta_i s}\, ds \\
&= -\sum_{i=1}^{k} \eta_i \beta_i \int_{t_0}^{t} \left[e^{-\beta_i(t-s)} - e^{-\beta_i(\tau-s)}\right] \cdot e^{-\beta_i(t-s)}\, ds
\end{aligned}$$

This implies that the volatility function should be of the form:

$$\sigma_H(s,t,\tau) = \left(\sqrt{\eta_1}\left(e^{-\beta_1(t-s)} - e^{-\beta_1(\tau-s)}\right), \ldots, \sqrt{\eta_k}\left(e^{-\beta_k(t-s)} - e^{-\beta_k(\tau-s)}\right)\right)$$

i.e. the above specification of the initial futures prices and the assumption of a deterministic volatility structure imply that the term structure of interest rates is determined by a k-factor, non-Markovian Vasicek type term structure model with the speed factor $\beta := (\beta_1, \ldots, \beta_k)$ *and the volatility of the instantaneous spot rate equal to*

$$\sigma := \left(\beta_1 \cdot \sqrt{\eta_1}, \ldots, \beta_k \cdot \sqrt{\eta_k}\right).$$

In other words Proposition 1 enables us to estimate the volatility structure from the initial futures prices in the deterministic case. If the volatility is stochastic, the initial futures prices restrict the volatility structure. In the Vasicek case the solution for the spot rate is given by

$$r_c(t) = -\left.\frac{\partial \ln H(t_0,t,s)}{\partial s}\right|_{s=t} - \sum_{i=1}^{k} \int_{t_0}^{t} \beta_i \sqrt{\eta_i} e^{-\beta_i(t-s)}\, dW_i^*(s).$$

V. HJM and the Futures Market model

Applying the relationship (5) between forward rates and futures prices the forward rates can be expressed as

$$\begin{aligned}
f(t,\tau) &= -\frac{\partial \ln H(t,t,\tau)}{\partial \tau} \\
&= -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} + \frac{1}{2}\int_{t_0}^{t} \frac{\partial ||\sigma_H(s,t,\tau)||^2}{\partial \tau}\, ds \\
&\quad - \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau} \cdot dW^*(s).
\end{aligned} \tag{19}$$

In addition the specification of the futures model endogenously fixes the dynamics of the forward prices and the nominal forward and futures rates. Applying Itô's Lemma, the forward price is given by:

$$
\begin{aligned}
(20) \qquad dF(t,u,\tau) &= d\left(\frac{B(t,\tau)}{B(t,u)}\right) \\
&= F(t,u,\tau)\left(\sigma_H(t,t,\tau) - \sigma_H(t,t,u)\right) \cdot dW^u(t),
\end{aligned}
$$

where $dW^u(t) := dW^*(t) - \sigma_H(t,t,u)dt$ defines a Brownian motion under the u-forward risk adjusted measure Q^u

$$
\begin{aligned}
(21) \qquad \left.\frac{dQ^u}{dP^*}\right|_t &:= \frac{\exp\left\{-\int_t^u r_c(s)ds\right\} B(u,u)}{E_{P^*}\left[\exp\left\{-\int_t^u r_c(s)du\right\} B(u,u)\middle|\, \mathbb{F}_t\right]} \\
&= \exp\left\{-\frac{1}{2}\int_t^u ||\sigma_H(s,s,u)||^2 ds + \int_t^u \sigma_H(s,s,u)\cdot dW^*(s)\right\} \quad .
\end{aligned}
$$

Since the forward price process $\{F(t,u,\tau)\}_t$ is a martingale under the u-forward risk adjusted measure, the nominal forward rate, $\{r_n(t,u,\tau)\}_t$, is a martingale under the τ-forward risk adjusted measure, i.e.

$$
\begin{aligned}
dr_n(t,u,\tau) &= d\frac{1}{\tau-u}\left(F(t,u,\tau)^{-1} - 1\right) \\
(22) \qquad &= -\frac{1}{\tau-u}\frac{1}{F(t,u,\tau)^2}dF(t,u,\tau) + \frac{1}{\tau-u}\frac{1}{F(t,u,\tau)^3}d\langle\, F(\cdot,u,\tau)\,\rangle_t \\
&= -\frac{r_n(t,u,\tau)}{1-F(t,u,\tau)}\left(\sigma_H(t,t,\tau) - \sigma_H(t,t,u)\right)\cdot dW^\tau(t).
\end{aligned}
$$

Similarly, we know that the futures price process $(H(t,u,\tau))_t$ is a martingale under P^*. With reference to the Bund Futures contract the implied futures rate in nominal terms $r_H(t,u,\tau)$ is a solution to the following stochastic differential equation

$$
(23) \qquad dr_H(t,u,\tau) = -\frac{r_H(t,u,\tau)}{1-H(t,u,\tau)}\left(\sigma_H(t,u,\tau)\cdot dW^*(t) - ||\sigma_H(t,u,\tau)||^2 dt\right).
$$

The process of the implied futures rate $(r_H(t,\tau,\alpha))_t$ is not a martingale under P^*. Define for $t \le u \le \tau$ by

$$
\begin{aligned}
(24) \qquad \left.\frac{dQ_H^{u,\tau}}{dP^*}\right|_t &= \frac{H(u,u,\tau)}{E_{P^*}[H(u,u,\tau)|\mathbb{F}_t]} \\
&= \exp\left\{-\frac{1}{2}\int_t^u ||\sigma_H(s,u,\tau)||^2 du + \int_t^u \sigma_H(s,u,\tau)dW^*(u)\right\}
\end{aligned}
$$

a new probability measure. Under this (u,τ)-futures risk adjusted measure, $Q_H^{u,\tau}$, the process $(W_H^{u,\tau}(t))_t$ with

$$
(25) \qquad dW_H^{u,\tau}(t) := dW^* - \sigma_H(t,u,\tau)\cdot dt
$$

is a standard Brownian motion and the implied futures rate is a martingale, i.e.

$$
(26) \qquad dr_H(t,u,\tau) = -\frac{r_H(t,u,\tau)}{1-H(t,u,\tau)}\,\sigma_H(t,u,\tau)\cdot dW_H^{u,\tau}(t).
$$

The martingale property of the implied futures rate corresponds to a change of numeraire. In the case of the forward rate adjusted measure a zero coupon bond is chosen to be the numeraire. The futures risk adjusted measure is derived by choosing a futures price as the numeraire. Note that, for $u = t$, the (u, τ)-futures risk adjusted measure and the τ-forward risk adjusted measure coincide. Furthermore, if the volatility function of the corresponding futures satisfies the log-normal assumption given by equation (12), the implied futures rates is a log-normal martingale under the corresponding futures risk adjusted measure. The following example illustrates the implication of the martingale property for the implied futures rate under the futures risk adjusted measure:

Example 2. *Following the analysis in section A we can impose the log-normal assumption (12) on one specific compounding period $(\tau^* - u^*)$, i.e assume that the implied futures rate is a solution of*

$$\begin{aligned} dr_H(t, u^*, \tau^*) &= -r_H(t, u, \tau)\ \gamma_H(t, u, \tau) \cdot (dW^*(t) - (1 - H(t, u, \tau))\ \gamma_H(t, u, \tau) \cdot dt) \\ &= -r_H(t, u^*, \tau^*)\gamma_H(t, u^*, \tau^*) \cdot dW_H^{u^*,\tau^+}(t). \end{aligned}$$

where $\gamma_H(\cdot, u^*, \tau^*) : [t_0, u^*] \to \mathbb{R}^k$ is a deterministic function. The implied futures rate $\{r_H(, u^*, \tau^*)\}_t$ is a log-normal martingale under the appropriate futures risk adjusted measure. In this case the no-arbitrage value of futures style options on the implied futures rate or the futures price are easy to calculate. As a first example consider the futures price of a futures style option on the implied futures rate. The no arbitrage price of the option with face value V, strike L on the implied futures rate $r_H(t, u^*, \tau^*)$ and maturity $s \geq t$ is equal to:

$$\begin{aligned} &V(\tau^* - u^*)E_{P^*}\left[[r_H(s, u^*, \tau^*) - L]^+ \mid \mathbb{F}_t\right] \\ &= V(\tau^* - u^*)\ H(t, u^*, \tau^*)E_{Q_H^{u^*,\tau^*}}\left[H(s, u^*, \tau^*)^{-1}[r_H(s, u^*, \tau^*) - L]^+ \mid \mathbb{F}_t\right] \\ &= V(\tau^* - u^*)H(t, u^*, \tau^*) \\ &\qquad \cdot E_{Q_H^{u^*,\tau^*}}\left[(1 + (\tau^* - u^*)r_H(s, u^*, \tau^*))[r_H(s, u^*, \tau^*) - L]^+ \mid \mathbb{F}_t\right] \\ &= V(\tau^* - u^*)H(t, u^*, \tau^*)\Big[r_H(t, u^*, \tau^*)N(d_1) - LN(d_2) \\ &\qquad + (\tau^* - u^*)r_H(t, u^*, \tau^*)\Big(r_H(t, u^*, \tau^*)e^{g^2}N(d_1 + g) - LN(d_1)\Big)\Big] \end{aligned}$$

with

$$d_{1/2} := \frac{\ln\left(\frac{r_H(t,u^*,\tau^*)}{L}\right) \pm \frac{1}{2}g^2}{g} \quad \text{and} \quad g^2 := \int_t^s ||\gamma_H(\theta, u^*, \tau^*)||^2 d\theta.$$

Formally the pricing formula of a futures style option on the implied futures rate corresponds to a sum of two caplet formulas. The main reason for this is that the implied futures rate is a martingale under the futures risk adjusted measure and not under the martingale measure. Furthermore, the margin system implies that the futures style option is equivalent to a contract situation with payment in advance. The above pricing formula is based on the definition of the nominal futures rate. This corresponds not to the definition of the Eurodollar Futures rate. The implied Eurodollar Futures rate is by definition linearly related to the futures price. From the relation between the futures price and the implied Eurodollar Futures rate $\tilde{r}_H$ is given by

$$H(t, u^*, \tau^*) =: 1 - (\tau^* - u^*)\tilde{r}_H(t, u^*, \tau^*).$$

Therefore the Eurodollar option corresponds to an option on the futures price. In a similar way, we can now compute the arbitrage price of a futures style option on a futures contract with exercise time $s \geq t$, $s < \tau^*$ and strike K:

$$\begin{aligned}
&E_{P^*}\left[[H(s,u^*,\tau^*)-K]^+ \mid I\!F_t\right] \\
&= E_{P^*}\left[H(s,u^*,\tau^*)\cdot\left[1-K\cdot H(s,u^*,\tau^*)^{-1}\right]^+ \mid I\!F_t\right] \\
&= H(t,u^*,\tau^*)\cdot E_{Q_H^{u^*,\tau^*}}\left[[1-K(1+(\tau^*-u^*)\cdot r_H(s,u^*,\tau^*)]^+ \mid I\!F_t\right] \\
&= H(t,u^*,\tau^*)\cdot[(1-K)\cdot N(e_1)-(\tau^*-u^*)\cdot K\cdot r_H(t,u^*,\tau^*)\cdot N(e_2)] \\
&= H(t,u^*,\tau^*)(1-K)N(e_1)-K(1-H(t,u^*,\tau^*))N(e_2),
\end{aligned}$$

with

$$e_{1/2} = \frac{\ln\left(\dfrac{(1-K)H(t,u^*,\tau^*)}{K(1-H(t,u^*,\tau^*))}\right) \pm \frac{1}{2}g^2}{g} \quad \textit{and} \quad g^2 := \int_t^s ||\gamma_H(\theta,u^*,\tau^*)||^2 d\theta.$$

The structure of this formula coincides with the arbitrage price of an option on a zero coupon bond in the LIBOR Market Model.

In general under the futures risk adjusted measure we obtain the following solutions:

$$-\frac{\partial \ln H(t,u,\tau)}{\partial \tau} = -\frac{\partial \ln H(t_0,u,\tau)}{\partial \tau} - \int_{t_0}^t \frac{\partial \sigma_H(s,u,\tau)}{\partial \tau}\cdot dW_H^{u,\tau}(s),$$

$$f(t,\tau) = -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} - \int_{t_0}^t \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\cdot dW_H^{t,\tau}(s), \tag{27}$$

$$r_c(t) = \frac{\partial \ln H(t_0,t,\tau)}{\partial t} + \int_{t_0}^t \frac{\partial \sigma_H(s,t,\tau)}{\partial t}\cdot dW_H^{t,\tau}(s).$$

This implies that $-\frac{\partial \ln H(t,u,\tau)}{\partial \tau}$, the intensity of the futures prices, is a martingale under the appropriate futures risk adjusted measure, i.e.

$$-d\frac{\partial \ln H(t,u,\tau)}{\partial \tau} = \left(-\frac{\partial \sigma_H(t,u,\tau)}{\partial \tau}\right)\cdot dW_H^{u,\tau}(t). \tag{28}$$

The instantaneous forward rate can be expressed as a solution to the following stochastic differential equation:

$$\begin{aligned}
df(t,\tau) &= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right)\cdot dW_H^{t,\tau}(t) + \frac{\partial}{\partial u}\left(-\frac{\partial \ln H(t,u,\tau)}{\partial \tau}\right)\Bigg|_{u=t} dt \\
&= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right)\cdot dW^\tau(t) - \frac{\partial}{\partial \tau}\left(\frac{\partial \ln H(t,u,\tau)}{\partial u}\Bigg|_{u=t}\right) dt \\
&= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right)\cdot dW^\tau(t) - \frac{\partial}{\partial \tau}\left(r_c(t)\right) dt \\
&= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right)\cdot dW^\tau(t).
\end{aligned} \tag{29}$$

The instantaneous forward rate is a martingale under the appropriate forward risk adjusted measure. Similarly, the spot rate is a solution to

$$
\begin{aligned}
(30) \qquad dr_c(t) &= df(t,t) + \left.\frac{\partial f(t,\tau)}{\partial \tau}\right|_{\tau=t} dt \\
&= \left(-\left.\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} \right) \cdot dW^*(t) - \left(\left.\frac{\partial^2 \ln H(t,t,\tau)}{\partial \tau^2}\right|_{\tau=t} \right) dt.
\end{aligned}
$$

Example 3. *Consider again Example 1 of the Vasicek term structure model and for simplicity assume the 1-factor version. From equation (29) the forward rate can be expressed by*

$$df(t,\tau) = -\beta\sqrt{\eta}e^{-\beta(\tau-t)}dW^{\tau}(t).$$

Furthermore, applying equation(30) implies the usual representation of the spot rate

$$dr_c(t) = \beta[\theta(t) - r_c(t)]dt + \beta\sqrt{\eta}dW^*(t),$$

with

$$\theta(t) := h(t_0,t,t) - \frac{1}{\beta}\left.\frac{\partial^2 \ln H(t_0,t,\tau)}{\partial t \partial \tau}\right|_{\tau=t} + \frac{\beta\eta}{2}\left(1 - e^{-2\beta(t-t_0)}\right).$$

VI. Conclusion

The approach taken in this paper is the definition of a no-arbitrage model, based on futures prices, for the term structure of interest rates. At a first view this approach could seem to be mainly of theoretical interest. This intuition turns out to be wrong. Similar to option prices, also futures prices add information to the structure of the model. This additional information implies that the expected spot rate and the volatility structure of the futures prices are closely related to the initial futures prices. Two main consequences can be drawn.

First, the expected spot rate is through Proposition 1 related to the initial futures prices under the martingale measure. This opens a new and, to our knowledge, so far unconsidered way to value the accuracy of an assumed model specification for the term structure of interest rates. Since this property is independent of the volatility structure of the futures prices, it can be used to justify a specific volatility structure even if the structure can only be analyzed in numerical ways.

Second, the main result in Proposition 1 implies that the volatility structure itself is related to the initial futures prices. This is of importance in two ways. For short times to maturity an approximation of futures prices by a sufficiently smooth function can be used to fix the short end of the term structure of volatility. It means that, in addition to an estimation on the basis of a time series approach, the implied term structure of volatility can be expressed by using futures prices. Furthermore, a majority of the volatility structures applied in practice assume a deterministic specification. In this case our results imply that the initial futures prices can be used to completely determine this function. As an example this is shown for a multidimensional Vasicek model. In the same way this can be shown for other model specifications. This again allows us to relate a specific structure for the term structure of volatilities to market information. Furthermore, Example 1 seems to indicate that the functional form of the initial futures prices is related to the number of factors entering the term structure of interest rates. More precisely, suppose that the logarithm of initial futures prices can be approximated by a sum of basis functions. In this case the number of basis functions is equal to the number of factors and each basis function determines the structure of the corresponding projection of the volatility vector. In addition to these features, the paper adds a new measure transformation to the existing martingale results. Assuming sufficient regularity, the no-arbitrage condition implies that discounted spot and futures prices are martingales under the martingale measure, and that forward prices and forward rates are martingales under the appropriate forward risk adjusted measure. It is shown that nominal futures rates are martingales under the appropriate futures risk adjusted measure. As for the case of standard options this property is central to compute the no-arbitrage value of futures style options. Parallel to option pricing techniques for non-futures style options, this measure transformation can be used to efficiently compute the option value of futures style options.

To show this, we consider a LIBOR market structure. Again two results are derived. First, assuming log-normality yields the same structural problem already known in the LIBOR market situation. Under no-arbitrage it is not possible to impose log-normality on all nominal futures rate processes. In contrast to the LIBOR market this result is not derived by a simple duplication argument in connection with the instability of the log-normal distribution with respect to summation. In the futures market case this assumption is not compatible with the no-arbitrage restrictions on the volatility given by Proposition 1. On the other hand these restrictions do not prevent us from assuming log-normality for a limited number of processes, i.e. as in the LIBOR market case, the term structure of volatilities is not completely specified. In this case future style options on futures prices and futures rates can be calculated in closed form. In particular this yields a new pricing formula for a Eurodollar futures option. The interpretation of this formula is closely related to Blacks formula for the caplet and floorlet. Instead of the forward rate volatility the futures price volatility enters the formula. Furthermore, the margin system implies an additional option part which intuitively arises from the payment in advance property of a futures style option.

From an applied point of view the paper focuses on two aspects: The relationship between futures prices and the term structure of volatilities and the necessity to use observable data to further develop the no-arbitrage theory of the term structure of interest rates.

Appendix

A. **Regularity conditions on the volatility function.**

Assumption 1. *For any tuple (u,τ) with $u \le \tau$ we assume that the volatility process $\{\sigma_H(t,u,\tau)\}_{t\le u}$ of the futures price $H(t,u,\tau)$ satisfies the following conditions:*

- $\{\sigma_H(t,u,\tau)\}_{t\le u}$ *is a k-dimensional continuous and adapted stochastic process with $\sigma_H(t,\tau,\tau)=0 \quad \forall t\le\tau$.*
- *The processes of the partial derivatives $\left\{\frac{\partial\sigma_H(t,u,\tau)}{\partial u}\right\}_{t\le u}$ and $\left\{\frac{\partial\sigma_H(t,u,\tau)}{\partial \tau}\right\}_{t\le u}$ are k-dimensional continuous and adapted processes.*
- $E_{P^*}\left[\exp\left\{\frac{1}{2}\int_t^u ||\sigma_H(s,u,\tau)||^2 ds\right\} \;\middle|\; I\!\!F_t\right] < \infty \quad \forall t\le s\le u\le\tau.$
- $E_{P^*}\left[\left\|\frac{\partial\sigma_H(s,u,\tau)}{\partial u}\right\|^2 \;\middle|\; I\!\!F_t\right]$ *and* $E_{P^*}\left[\left\|\frac{\partial\sigma_H(s,u,\tau)}{\partial \tau}\right\|^2 \;\middle|\; I\!\!F_t\right]$ *are bounded on $t\le s\le u\le\tau$.*
- *There exists a predictable and bounded process $\{A(t,u,\tau)\}_t$ with $E_{P^*}[A(s,u,\tau)^2 \mid I\!\!F_t] < \infty \forall s\in[t,u]$ and $E_{P^*}[\int_t^u A(s,u,\tau)^2 ds \mid I\!\!F_t] < \infty$ such that*

$$\left\|\frac{\partial\sigma_H(s,u,\tau)}{\partial u} - \frac{\partial\sigma_H(s,u+\delta,\tau)}{\partial u}\right\| \le A(s,u,\tau)\cdot\delta \quad \forall s\le u\le\tau, \quad \forall\delta>0$$
$$\left\|\frac{\partial\sigma_H(s,u,\tau)}{\partial \tau} - \frac{\partial\sigma_H(s,u+\delta,\tau)}{\partial \tau}\right\| \le A(s,u,\tau)\cdot\delta \quad \forall s\le u\le\tau, \quad \forall\delta>0.$$

Furthermore, we assume that at any time t the futures price is continuously differentiable with respect to the delivery date and the maturity date of the underlying zero coupon bond, i.e. $\frac{\partial H(t,u,\tau)}{\partial u}$ and $\frac{\partial H(t,u,\tau)}{\partial \tau}$ exist.

B. **Proof of Proposition1.** The no-arbitrage condition (11) implies the following representation for the instantaneous spot rate process:

$$(31)\quad r_c(t) = \left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} = \frac{\partial \ln H(t_0,t,\tau)}{\partial t} - \frac{1}{2}\int_{t_0}^t \frac{\partial||\sigma_H(s,t,\tau)||^2}{\partial t}ds + \int_{t_0}^t \frac{\partial\sigma_H(s,t,\tau)}{\partial t}\cdot dW^*(s).$$

Taking in (31) the expectation given the information at time t_0, we find the following relationship between the expected spot rate under the martingale measure and the futures prices:

(32)
$$E_{P^*}\left[r_c(t) \mid \mathbb{F}_{t_0}\right] = \frac{\partial \ln H(t_0,t,\tau)}{\partial t} - E_{P^*}\left[\int_{t_0}^t \sigma_H(s,t,\tau)\cdot\frac{\partial\sigma_H(s,t,\tau)}{\partial t}ds \;\middle|\; \mathbb{F}_{t_0}\right].$$

This equation already shows that the expected spot rate and the initial futures prices are closely related. It still implies a dependency on the volatility structure of the model. Intuition at that point indicates that this model dependency is still too strong, i.e. initial futures prices should completely determine the expected spot rate. To see this we substitute the solution of the futures price in equation (9) into the definition of the spot rate given by equation (6). To simplify this second representation of the instantaneous spot rate, notice that, as $\sigma_H(s,t,t) = 0$, we have that

$$\frac{1}{2}\int_{t_0}^{t} \frac{\partial ||\sigma_H(s,t,\tau)||^2}{\partial \tau}\bigg|_{\tau=t} ds = \int_{t_0}^{t} \sigma_H(s,t,t)\cdot \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\bigg|_{\tau=t} ds = 0 \qquad P^* a.s.$$

With this remark a second representation of the instantaneous spot rate process is derived by

(33)

$$r_c(t) = \; -\frac{\partial \ln H(t,t,\tau)}{\partial \tau}\bigg|_{\tau=t} = -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\bigg|_{\tau=t} - \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\bigg|_{\tau=t} \cdot dW^*(s).$$

Taking again expectations, (33) yields for the expected spot rate a second relationship:

$$E_{P^*}\left[r_c(t)|\mathbb{F}_{t_0}\right] = -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\bigg|_{\tau=t} \tag{34}$$

Furthermore, combining the expressions (32) and (34) gives us a first no-arbitrage condition on the volatility of the term structure of futures prices. At this point of the analysis this condition is still weak, since it is based on the expectation under the martingale measure given the information at time t_0. More precisely any specification of the term structure of volatility for the futures prices has to satisfy the following no-arbitrage condition:

$$\begin{aligned} E_{P^*}\left[\int_{t_0}^{t} \sigma_H(s,t,\tau)\cdot \frac{\partial \sigma_H(s,t,\tau)}{\partial t} ds \;\bigg|\; \mathbb{F}_{t_0}\right] &= \frac{\partial \ln H(t_0,t,\tau)}{\partial t} + \frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\bigg|_{\tau=t} \\ &= \int_{t}^{\tau} \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s} ds \end{aligned} \tag{35}$$

In order to strengthen this condition, we derive the stochastic differential equations of the spot rate implied by the two representations (31) and (33). To facilitate this approach we use the following application of Fubini's Theorem

Lemma 2. *Suppose that for any $t \le \tau$ and $s \in [t_0,t]$ the drift $g(\cdot,t,\tau)$ and the k-dimensional volatility function $l(\cdot,t,\tau)$ of a stochastic process $\{Y(t)\}_t$ are given by*

$$g(s,t,\tau) = \int_{s}^{t} \frac{\partial g(s,u,\tau)}{\partial u} du + g(s,s,\tau), \tag{36}$$

$$l(s,t,\tau) = \int_{s}^{t} \frac{\partial l(s,u,\tau)}{\partial u} du + l(s,s,\tau). \tag{37}$$

By changing the order of integration the stochastic process $\{Y(t)\}_t$ with

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} g(s,t,\tau) ds + \int_{t_0}^{t} l(s,t,\tau)\cdot dW^*(s) \\ &= \int_{t_0}^{t} \left(\int_{s}^{t} \frac{\partial g(s,u,\tau)}{\partial u} du + g(s,s,\tau)\right) ds \\ &\quad + \int_{t_0}^{t} \left(\int_{s}^{t} \frac{\partial l(s,u,\tau)}{\partial u} du + l(s,s,\tau)\right)\cdot dW^*(s) \end{aligned}$$

is the solution to the stochastic differential equation

$$dY(t) = \left(\int_{t_0}^{t} \frac{\partial g(s,t,\tau)}{\partial t} ds + g(t,t,\tau) + \int_{t_0}^{t} \frac{\partial l(s,t,\tau)}{\partial t} dW^*(s) \right) dt$$
$$+ \, l(t,t,\tau) \cdot dW^*(t) \ .$$

Preparing for the application of the Lemma we define

$$l(t,u,\tau) := \frac{\partial \sigma_H(t,u,\tau)}{\partial u} \quad \text{and} \quad g(t,u,\tau) := \frac{\partial ||\sigma_H(t,u,\tau)||^2}{\partial u}.$$

Then, applying Lemma 2 we obtain through the no arbitrage condition (11):

$$dr_c(t) = \left(\frac{\partial^2 \ln H(t_0,t,\tau)}{\partial t^2} dt - \frac{1}{2} \int_{t_0}^{t} \frac{\partial g(s,t,\tau)}{\partial t} ds - \frac{1}{2} g(t,t,\tau) \right.$$
$$\left. + \int_{t_0}^{t} \frac{\partial l(s,t,\tau)}{\partial t} \cdot dW^*(s) \right) dt + l(t,t,\tau) \cdot dW^*(t).$$

With the definition

$$h(s,t) := \left. \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau} \right|_{\tau=t}$$

the same approach using equation (33) yields for the stochastic differential equation for the spot rate process

$$dr_c(t) = \left(-\frac{\partial}{\partial t} \left(\left. \frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} \right|_{\tau=t} \right) - \int_{t_0}^{t} \frac{\partial h(s,t)}{\partial t} \cdot dW^*(s) \right) dt - h(t,t) \cdot dW^*(t).$$

Equating the Wiener driven terms and the drift terms we find the following two restrictions on the volatility

$$-h(t,t) = l(t,t,\tau) \tag{38}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \ln H(t_0,t,\tau)}{\partial t} + \left. \frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} \right|_{\tau=t} \right) \tag{39}$$
$$= \frac{1}{2} \int_{t_0}^{t} \frac{\partial g(s,t,\tau)}{\partial t} ds + \frac{1}{2} g(t,t,\tau)$$
$$+ \int_{t_0}^{t} \left(\frac{\partial h(s,t)}{\partial t} + \frac{\partial l(s,t,\tau)}{\partial t} \right) \cdot dW^*(s)$$

respectively.

Observe, that the left hand side of (39) is $\mathbb{F}_{t_0}$ measurable and deterministic. This implies that the right hand side also has to be deterministic for all $t_0 \leq t \leq \tau$. As a consequence the process $V(t)$, defined by

(40)
$$V(t) = \frac{1}{2} \int_{t_0}^{t} \frac{\partial g(s,t,\tau)}{\partial t} ds + \frac{1}{2} g(t,t,\tau) + \int_{t_0}^{t} \left(\frac{\partial h(s,t,)}{\partial t} + \frac{\partial l(s,t,\tau)}{\partial t} \right) \cdot dW^*(s),$$

has to be a deterministic process. Applying again Lemma 2 to the right hand side of (40) we obtain the differential equation for $V(t)$. With only the Wiener terms explicitly specified we find that

$$dV(t) = (\cdot)\,dt - \left(\frac{\partial h(s,t)}{\partial t}\bigg|_{s=t} + \frac{\partial l(s,t,\tau)}{\partial t}\bigg|_{s=t} \right) \cdot dW^*(t). \tag{41}$$

The requirement that $V(t)$ should be a deterministic process therefore turns into the requirement that

$$0 = \frac{\partial h(s,t)}{\partial t}\bigg|_{s=t} + \frac{\partial l(s,t,\tau)}{\partial t}\bigg|_{s=t} \tag{42}$$

Transforming to the futures volatilities (38) and (42) appear as

$$0 = \frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\bigg|_{\tau=t} + \frac{\partial \sigma_H(t,u,\tau)}{\partial u}\bigg|_{u=t} \tag{43}$$

and

$$0 = \frac{\partial}{\partial t}\left(\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\bigg|_{\tau=t} \right)\bigg|_{s=t} + \frac{\partial}{\partial t}\left(\frac{\partial \sigma_H(s,t,\tau)}{\partial t} \right)\bigg|_{s=t} \tag{44}$$

respectively. This second no-arbitrage condition, equation (44), on the volatility of the futures prices is more general than the condition (35), i.e. taking the expectation in (44) implies the former condition.

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